# Optimal Portfolio Selection in a DC Pension with Multiple Contributors and the Impact of Stochastic Additional Voluntary Contribution on the Optimal Investment Strategy

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*Abstract*—In this paper, we studied the optimal portfolio selection in a defined contribution (DC) pension scheme with multiple contributors under constant elasticity of variance (CEV) model and the impact of stochastic additional voluntary contribution on the investment strategies. We assume that the voluntary contributions are stochastic and also consider investments in a risk free asset and a risky asset to increase the expected returns of the contributing members. We derived a stochastic differential equation which consists of the members' monthly contributions and the invested fund and obtained an optimized problem with the help of Hamilton Jacobi Bellman equation. Furthermore, we find an explicit solution for the optimal investment strategy with stochastic voluntary contribution using power transformation and change of variables method and the corresponding optimal fund size was obtained. We discussed the impact of the voluntary contribution on the optimal investment strategy with numerical simulations and observed that the voluntary contribution reduces the optimal investment strategy of the risky asset.

*Keywords*—DC pension fund, Hamilton-Jacobi-Bellman, optimal investment strategies, power transformation method, stochastic, voluntary contribution.

## I. INTRODUCTION

VOLUNTARY contributions and mandatory contributions in a DC pension scheme with multiple contributors are merge into one pension plan to study the impact of the voluntary contributions on the optimal investment strategy in a DC pension fund. Following the Nigerian pension reform act 2004, where members can increase their wealth in the pension scheme by remitting some proportion of their income different from the mandatory contributions. The voluntary contribution may be in constant proportion or may be stochastic. Reference [3] investigates stochastic optimal investment under inflammatory market with minimum guarantee, where members made extra contribution to amortize the pension fund. Reference [4] studied the effect of extra contribution on the optimal investment strategy in a DC pension with stochastic salary under affine interest model; in their model, the extra contribution was constant and they consider

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investment in three different assets. Reference [16] extended the work of [4] from that of constant extra contribution to stochastic extra contribution.

Currently there are two types of pension plan in operation in which members may be involve in; this include the defined benefit (DB) and defined contribution (DC) pension plan. In DB plan, benefits of members are predetermined in advance following certain criteria such as earning history, years of service, age etc. and this depends mostly on the contribution of the employers but in DC plan a certain proportion of the members income are remitted into the members' account and these funds are invested by the pension managers to increase the expected returns of the members as at the time of retirement. Despite the attractive nature of the DC pension plan, it requires members' understandings of investments principles in different assets since there are risks involved in it. Hence there is a need to study the optimal investment which basically is a way of investing in different assets to maximize profit with minimal risk. It simply explains what proportion of the members' wealth should be invested in different assets involved to yield optimal return with minimum risk. In financial market, there are different types of assets such as the risk free asset (cash), bond, risky asset (stock).

The optimal investment strategies in a DC pension plan have been studied by several authors. Reference [1] studied optimal portfolios for a DC pension plan under CEV model. Reference [5] studied optimal investment problems and volatility homogenization approximation. Reference [10] investigates stochastic optimal control of a DC pension fund. Reference [11] applied CEV model to derived two explicit solutions for the logarithm utility function in two different periods (before and after retirement) in a DC pension fund using Legendre transformation method. Reference [7] studied optimal investment strategies in the presence of a minimum guarantee. Reference [2] studied optimal investment strategies with bounded risk, general utilities and goal achieving. Reference [12] investigates stochastic strategies in a DC pension fund. Reference [14] investigates optimal investment strategies in a DC pension with multiple contributors using power transformation method. Reference [6] studied optimal investment strategies for a DC pension with multiple contributors under CEV model using Legendre transformation method and Dual theory. Some utility functions such as constant relative risk aversion (CRRA) and constant absolute

risk aversion (CARA) have been used to study the optimal investment strategies to maximize the expected utility of the members' terminal wealth. Different authors make use of different utility functions such as [1], [15] used only CARA utility function. References [7]-[10] used only CRRA utility function. References [1], [13] used both CARA and CRRA utility functions.

Our interest is to study the optimal portfolios in a DC pension with multiple contributors, also the impact of stochastic voluntary contributions on the investments strategy. We formalize the wealth of the pension fund together with voluntary contribution and obtain an optimized problem using HJB equation. The optimized problem is solved using power transformation and change of variable method to obtain the optimal investment strategy with stochastic voluntary contribution. The effect of the voluntary contributions on the optimal investment strategy is studied with numerical simulations.

## II. PRELIMINARIES

Starting with a complete and a less friction financial market which is continuously open over a fixed time interval $0 \leq t \leq T$, where $T$ is the retirement time of a given shareholder.

Let the market be of a risk free asset (cash) and a risky asset (stock). Suppose $(\Omega, F, P)$ is a complete probability space such that $\Omega$ is a real space and $P$ a probability measure, $\{W_0(t), W_1(t): t \geq 0\}$ is a standard two dimensional motion such that they orthogonal to each other. $F$ is the filtration and denotes the information generated by the Brownian motion $\{W_0(t), W_1(t)\}$.

Let $S_0(t)$ denote the price of the risk free asset, its model is given as

$$\frac{dS_0(t)}{S_0(t)} = rdt, \tag{1}$$

Let $S_t(t)$ denote the price of the risky asset and the price process as described by the CEV model in [14] as

$$\frac{dS_t(t)}{S_t(t)} = \omega dt + \gamma dW_0. \tag{2}$$

where $\omega$ an expected instantaneous rate of return of the risky asset and satisfies the general condition $\omega > r_0$. $\gamma$ is the instantaneous volatility,

In DC pension fund system with multiple contributors, we assume that pensions are paid to only retirees and continue till the death of the retirees after which they are automatically deleted from the system. With this, the payment process is stochastic and assumes the Brownian motion with drift as

$$dK(t) = k_0 dt - k_1 dW_1(t), \tag{3}$$

where $k_0$ and $k_1$ are positive constants and denote the amount given to the retired contributors and that which which is due death contributors which are out of the system.

We consider that in a DC pension fund system, members have the responsibility to remit a specific percentage of their income to the pension account every months; also based on the Nigerian Pension Reform Act of 2004 [17], members have the liberty to contribute additional percentage of their income to the pension account. Based on this, we consider a case where the rate of the additional contributions is stochastic. We assume that the number of contributors is constant and the contribution rate is modeled as follows

$$dC = cdt + c_0 dW_0 \tag{4}$$

where $c_0$ is the additional voluntary contribution and $c = (1 + \beta)k_0$ with safety loading $\beta > 0$. If there is no investment, the dynamics of the surplus is given by

$$dR(t) = dC - dK(t) = \beta k_0 dt + c_0 dW_0 + k_1 dW_1(t) \tag{5}$$

## III. MAIN RESULTS

### A. Hamilton-Jacobi-Bellman (HJB) Equation

Let $\tau$ be the strategy and we define the utility attained by the members from a given state z at time $t$ as

$$I_\tau(t, r, z) = E_\tau[U(Z(T)) \mid r(t) = r, Z(t) = z], \tag{6}$$

where $t$ is the time, $r$ is the short interest rate and $z$ is the wealth. The main aim of this section is to find the optimal value function and optimal strategy given as

$$I(t, r, z) = \sup_\tau I_\tau(t, r, z) \text{ and } \tau^* \tag{7}$$

Respectively such that

$$I_{\tau^*}(t, r, z) = I(t, r, z). \tag{8}$$

### B. Wealth Formulation with Stochastic Rate of Additional Voluntary Contribution

Let $Z(t)$ denote the wealth of pension fund at $t \in [0, T]$, let $\tau$ denote the proportion of the pension fund invested in the risky asset $S_t$ and $1 - \tau$, the proportion invested in risk free asset hence the dynamics of the pension wealth is given by

$$\begin{cases} dZ(t) = \tau Z(t) \frac{dS_t(t)}{S_t(t)} + (1 - \tau)Z(t) \frac{dS_0(t)}{S_0(t)} + dR(t) \\ \qquad\qquad Z(0) = z_0 \end{cases} \tag{9}$$

Substituting (1) and (2) into (5) we have

$$dZ(t) = [(\tau Z(t)(\omega - r)) + rZ(t) + \beta k_0]dt + (Z(t)\tau\gamma + c_o)dW_0(t) + k_1 dW_1(t) \tag{10}$$

The HJB equation associated with (46) is

$$I_t + \omega s I_s + (rz + \beta k_0)I_z + \frac{1}{2}\gamma^2 s^2 I_{ss} + \frac{1}{2}k_1{}^2 I_{zz} + \sup\left\{\frac{1}{2}(z\tau\gamma + c_0)^2 I_{zz} + \tau z(\omega - r)I_z + \tau z\gamma^2 s I_{zs}\right\} = 0. \tag{11}$$

Differentiating (11) with respect to $\tau$, we obtain the first order maximizing condition as

$$c_0\gamma I_{zz} + \tau z^2\gamma^2 I_{zz} + z\gamma^2 sI_{zs} + z(\omega - r)I_z = 0 \tag{12}$$

Solving (12) for $\tau$ we have

$$\tau^* = -\frac{[c_0\gamma I_{zz} + (\omega - r)I_z + \gamma^2 sI_{zs}]}{z\gamma^2 I_{zz}}. \tag{13}$$

Substituting (13) into (11), we have

$$I_t + \omega sI_s + (rz + \beta k_0)I_z + \frac{1}{2}\gamma^2 s^2 I_{ss} + \frac{1}{2}k_1^2 I_{zz} +$$
$$\sup\left\{\frac{1}{2}\left(-z\gamma\frac{[c_0\gamma I_{zz}+(\omega-r)I_z+\gamma^2 sI_{zs}]}{z\gamma^2 I_{zz}} + c_0\right)^2 I_{zz} -\right.$$
$$\left.\frac{[c_0\gamma I_{zz}+(\omega-r)I_z+\gamma^2 sI_{zs}]}{z\gamma^2 I_{zz}}z(\omega-r)I_z - \frac{[c_0\gamma I_{zz}+(\omega-r)I_z+\gamma^2 sI_{zs}]}{z\gamma^2 I_{zz}}z\gamma^2 sI_{zs}\right\} =$$
$$0. \tag{14}$$

So that

$$I_t + \omega sI_s + \left(rz + \beta k_0 - \frac{c_0(\omega-r)}{\gamma}\right)I_z + \frac{1}{2}\gamma^2 s^2\left[I_{ss} - \frac{I_{zs}^2}{I_{zz}}\right] +$$
$$\frac{1}{2}k_1^2 I_{zz} - \frac{(\omega-r)^2}{2\gamma^2}\frac{I_z^2}{I_{zz}} - (\omega-r)s\frac{I_zI_{zs}}{I_{zz}} = 0 \tag{15}$$

where $I(T, s, z) = U(z)$ and $U(z)$ is the marginal utility of the investor. Next, we consider solving (15) for $I$, then substitute it into (13) using power transformation and change of variable technique.

### C. Optimal Investment Strategy for a Member with Exponential Utility Function

Assume the member takes an exponential utility

$$U(z) = -\frac{1}{m}e^{-mz}, \quad m > 0. \tag{16}$$

The absolute risk aversion of a decision maker with the utility described in (16) is constant and is a CARA utility. Hence, we conjecture a solution to (15) with the form:

$$\begin{cases}I(t,s,z) = -\frac{1}{m}exp[-m[v(t)(z - b(t)) + l(t,s)]\\ \quad l(T,s) = 0, \ v(T) = 1, b(T) = 0\end{cases} \tag{17}$$

$$\left.\begin{array}{c}I_t = -mI[v_t(z - b(t)) - vd_t + l_t],\\ I_s = -mIl_s, I_z = -mvI,\\ I_{zz} = m^2v^2I, I_{ss} = (m^2l^2 - ml_{ss})I, I_{sz} = m^2vl_sI\end{array}\right\} \tag{18}$$

From (15), (17) and (18), we obtain

$$[v_t + vr]z + \left[\beta k_0 - \frac{c_0(\omega-r)}{\gamma} - \frac{v_t}{v}b - b_t - \frac{1}{2}k_1^2 mv\right]v + [l_t +$$
$$rsl_s + \frac{1}{2}\gamma^2 s^2 l_{ss} + \frac{(\omega-r)^2}{2m\gamma^2}] = 0 \tag{19}$$

From (19) we obtain the following equations

$$v_t + rv = 0 \tag{20}$$

$$\beta k_0 - \frac{c_0(\omega-r)}{\gamma} - \frac{v_t}{v}b - b_t - \frac{1}{2}mk_1^2 v = 0 \tag{21}$$

$$l_t + rsl_s + \frac{1}{2}\gamma^2 s^2 l_{ss} + \frac{(\omega-r)^2}{2m\gamma^2} = 0 \tag{22}$$

Solving (20), we obtain

$$v(t) = e^{r(T-t)} \tag{23}$$

From (20), (21) and (23) we have

$$b_t - rb = \beta k_0 - \frac{c_0(\omega-r)}{\gamma} - \frac{1}{2}mk_1^2 e^{r(T-t)} \tag{24}$$

From (24), we have

$$b(t) = \frac{(e^{-r(T-t)}-1)}{2r}\left(\beta k_0 - \frac{c_0(\omega-r)}{\gamma}\right) + \frac{mk_1^2(e^{r(T-t)}-1)}{4r} \tag{25}$$

Next we conjecture a solution to (22) in the following form

$$\begin{cases} l(t,s) = P(t) + Q(t)\gamma^2\\ P(T) = 0, \ Q(T) = 0, \gamma^2 = k^2 s^{2\varepsilon}\end{cases} \tag{26}$$

$$l_t = P_t + Q_t s^{2\varepsilon}, l_s = \frac{-2\varepsilon Qk^2}{s\gamma^2}, l_{ss} = \frac{2\varepsilon(2\varepsilon+1)Qk^2}{s^2\gamma^2}. \tag{27}$$

Substituting (27) in (22) we have

$$P_t + 2\varepsilon(2\varepsilon+1)Qk^2 + \frac{1}{\gamma^2}\left(Q_t - 2r\varepsilon Qk^2 + \frac{(\omega-r)^2}{2m}\right) = 0 \tag{28}$$

Decomposing (28) into two parts, we have

$$P_t + 2\varepsilon(2\varepsilon+1)Qk^2 = 0 \tag{29}$$

$$Q_t - 2r\varepsilon Qk^2 + \frac{(\omega-r)^2}{2m} = 0 \tag{30}$$

Solving (30), we have

$$Q(t) = \frac{(\omega-r)^2}{4r\varepsilon mk^2}\left[1 - e^{2r\varepsilon(t-T)}\right] \tag{31}$$

Substituting (31) into (29) and solving it, we have

$$P(t) = \frac{(2\varepsilon+1)(\omega-r)^2}{4rm}\left[\frac{1}{2r\varepsilon}\left(e^{2r\varepsilon(t-T)}-1\right) + (T-t)\right] \tag{32}$$

$$l(t,s) = P(t) + Q(t)\gamma^2 \tag{33}$$

$$l(t,s) = \frac{(2\varepsilon+1)(\omega-r)^2}{4rm}\left[\frac{1}{2r\varepsilon}\left(e^{2r\varepsilon(t-T)}-1\right) + (T-t)\right] +$$
$$\frac{(\omega-r)^2}{4r\varepsilon mk^2}\left[1 - e^{2r\varepsilon(t-T)}\right]\gamma^2 \tag{34}$$

**Result 1 .** The optimal investment strategy with stochastic additional contribution is given as

$$\tau^* = \frac{1}{zm\gamma^2}\left\{(\omega - r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} - \frac{c_0}{z\gamma}$$

**Proof.** Recall that $\tau^* = -\frac{[c_0\gamma I_{zz}+(\omega-r)I_z+\gamma^2 sI_{zs}]}{z\gamma^2 I_{zz}}$ and

$$l(t,s) = \frac{(2\varepsilon+1)(\omega-r)^2}{4rm}\left[\frac{1}{2r\varepsilon}\left(e^{2r\varepsilon(t-T)}-1\right) + (T-t)\right] +$$
$$\frac{(\omega-r)^2}{4r\varepsilon mk^2}\left[1 - e^{2r\varepsilon(t-T)}\right]\gamma^2$$

then $\tau^*$ and $l_s$ reduces to

$$\tau^* = -\frac{\left[c_0\gamma + \frac{(\omega-r)}{mv} + \frac{\gamma^2 sl_s}{v}\right]}{z\gamma^2}$$

and

$$l_s = \frac{-(\omega-r)^2}{2rms\gamma^2}\left[1 - e^{2r\varepsilon(t-T)}\right]$$

Substituting $l_s$ into $\tau^*$ , we have

$$\tau^* = \frac{-c_0\gamma + \frac{(\omega-r)e^{r(t-T)}}{m} + \frac{(\omega-r)^2}{2rm}e^{r(t-T)}\left[1-e^{2r\varepsilon(t-T)}\right]}{z\gamma^2}$$

Simplifying the equation above we have

$$\tau^* = \frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} - \frac{c_0}{z\gamma}$$

**Remark 1.** Assume that there is no voluntary contribution; i.e. $c_0 = 0$, then proposition 1 reduces to

$$\tau_0^* = \frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\}$$

**Result 2.** The optimal fund size $Z^{\tau^*}(t)$ corresponding to the optimal investment strategy $\tau^*$ is given as

$$Z^{\tau^*}(t) =$$
$$\left(\frac{(\omega-r)^2}{m\gamma^2} + \frac{(\omega-r)^3}{2mr\gamma^2}\right)te^{r(t-T)} + \frac{(\omega-r)^3}{4m\varepsilon r^2\gamma^2}e^{rt}e^{-r(2\varepsilon+1)T}(1 -$$
$$e^{2\varepsilon t}) + \left(\frac{c_0(\omega-r)}{r\gamma} - \frac{\beta k_0}{r}\right)(1 - e^{rt}) + z_0$$

**Proof.** From (10) and proposition 1, we have

$$dZ(t) = \left[\left(\tau Z(t)(\omega-r)\right) + rZ(t) + \beta k_0\right]dt + (Z(t)\tau\gamma + c_o)dW_0(t) + k_1 dW_1(t)$$

and
$$\tau^* = \frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} - \frac{c_0}{z\gamma}$$

Substituting $\tau^*$ into (10), we have

$$\frac{dZ(t)}{dt} - rZ(t) = \beta k_0 + (\omega-r)\left(\frac{1}{m\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} - \frac{c_0}{\gamma}\right)$$

$$\frac{d}{dt}[Ze^{-rt}] = (\omega-r)e^{-rt}\left(\frac{1}{m\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} - \frac{c_0}{\gamma}\right) + \beta k_0 e^{-rt}$$

Integrating both sides with respect to t, we have

$$Z(t) =$$
$$\left(\frac{(\omega-r)^2}{m\gamma^2} + \frac{(\omega-r)^3}{2rm\gamma^2}\right)te^{r(t-T)} - \left(\frac{(\omega-r)^3}{4r^2m\varepsilon\gamma^2}\right)e^{r(t-T)}e^{2\varepsilon r(t-T)} +$$
$$\frac{c_0(\omega-r)}{r\gamma} - \frac{\beta k_0}{r} + c_k e^{rt}$$

Using the condition $Z(0) = z_0$, we have

$$Z^{\tau^*}(t) =$$
$$\left(\frac{(\omega-r)^2}{m\gamma^2} + \frac{(\omega-r)^2}{2rm\gamma^2}\right)te^{r(t-T)} + \left(\frac{(\omega-r)^3}{4m\varepsilon r^2\gamma^2}\right)e^{rt}e^{-r(2\varepsilon+1)T}(1 -$$
$$e^{2\varepsilon t}) + \left(\frac{c_0(\omega-r)}{r\gamma} - \frac{\beta k_0}{r}\right)(1 - e^{rt}) + z_0$$

**Lemma 1.** Suppose $c_0 > 0$, $\omega > r$, $e^{r(t-T)} > 0$, $\left(1 - e^{2r\varepsilon(t-T)}\right) > 0$ then $\tau^* < \tau_0^*$.

**Proof.** Since $\omega > r$, $\left(1 - e^{2r\varepsilon(t-T)}\right) > 0$ and $z > 0, m > 0, \gamma > 0$ and $\frac{c_0}{z\gamma} > 0$ then

$$(\omega-r) > 0 \text{ and } e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right] > 0$$

and

$$\frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} > 0$$

This implies that

$$\frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} - \frac{c_0}{z\gamma} <$$
$$\frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\}$$

and

$$\tau^* = \frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} -$$
$$\frac{c_0}{z\gamma} < \frac{1}{zm\gamma^2}\left\{(\omega-r)e^{r(t-T)}\left[1 + \frac{(\omega-r)}{2r}\left(1 - e^{2r\varepsilon(t-T)}\right)\right]\right\} = \tau_0^*$$

Therefore $\tau^* < \tau_0^*$.

## IV.  NUMERICAL SIMULATIONS

In this section, numerical simulations for the optimal investment strategy are obtained via MATLAB programming language. The following parameters were used in the simulations; $\omega = 0.05$, m = 0.1, T = 40, r = 0.02, $\gamma = 1$, $\varepsilon$ = -1, $c_0 = 1$, $z_0 = 1$ $\beta = 0.01$, $k_0 = 0.01$ and t = 0:5:20.
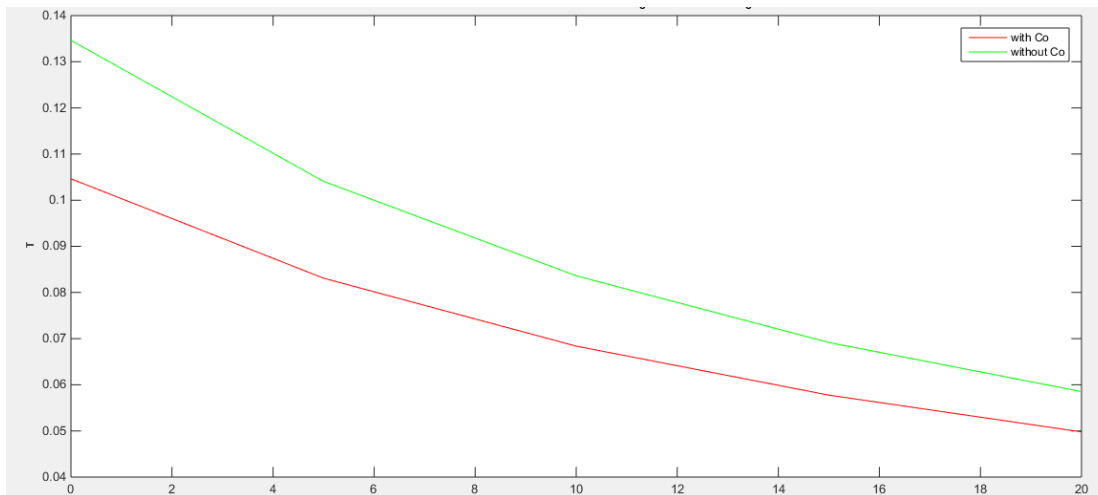
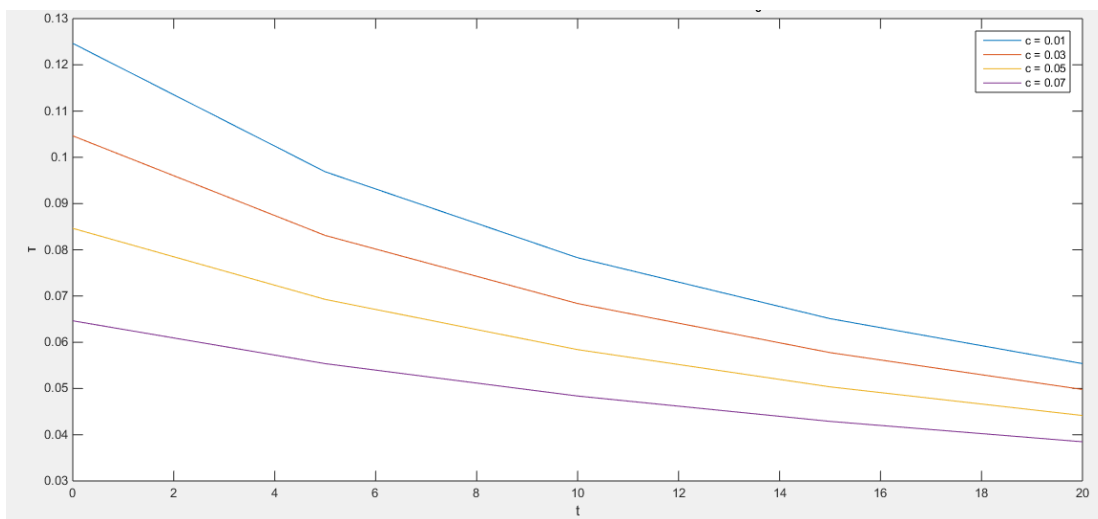Fig. 1 Evolution of optimal investment strategy with $c_0$ and without $c_0$ when $z = Z^{\tau^*}$



Fig. 2 Evolution of optimal investment strategy with different values of $c_0$ when $z = Z^{\tau^*}$
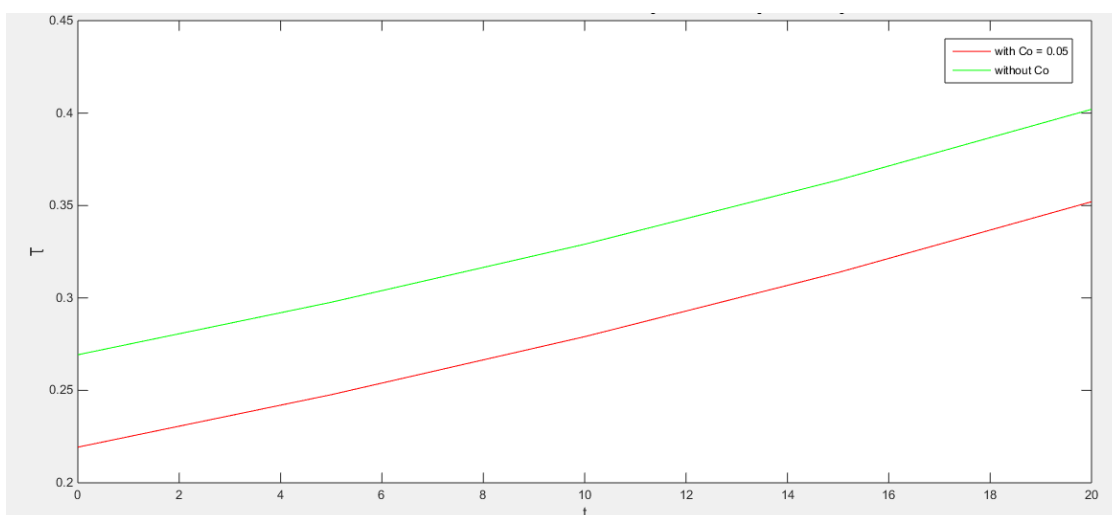


Fig. 3 Evolution of optimal investment strategy with and without $c_0$ when $z = z_0$

## V. Discussion

In Fig. 1, we observed that the optimal investment strategy with additional voluntary contributions is lower compared to the optimal investment strategy without additional voluntary contributions. This is because with the additional voluntary contributions, the overall pension wealth is increased and the pension manager will reduce the risk of investing more in risky asset. In Fig. 2, we observed that as the voluntary contributions increases, the optimal investment strategy decreases which implies that with more funds less risk is taken and if there are fewer funds the pension manager will increase the proportion of his wealth to be invested in risky asset. Also from Fig. 1, we observed that the optimal investment strategy decreases with time, this is because at the initial stage of investment, the optimal fund size which correspond to optimal investment strategy was used and with time the pension manager will reduce the proportion of his wealth invested in risky asset to avoid his members of losing what they have already and invest more in risk less asset as retirement age approaches. Similarly, in Fig. 3, we observed that the optimal investment strategy increases with time; this is because the pension manager started with the initial wealth and as retirement age approaches he is willing to take more risk to increase the expected return of his members.

## VI. Conclusion

This paper investigated the optimal portfolio selection in a DC pension fund with multiple contributors under CEV model and the impact of stochastic additional voluntary contributions on the investment strategies. We assume that the voluntary contributions are stochastic and consider investments in a risk free asset and a risky asset to increase the expected returns of the contributing members. Also a stochastic differential equation which consists of the members' monthly contributions and the invested fund are obtained. Furthermore, an optimized problem is obtained with the help of Hamilton Jacobi Bellman equation. We then used power transformation and change of variables method to obtain the optimal investment strategy with stochastic voluntary contribution and the corresponding optimal fund size was also obtained. We discussed the impact of the voluntary contribution on the optimal investment strategy with numerical simulations. We observed that the voluntary contribution reduces the optimal investment strategy of the risky asset and also that optimal investment strategy decreases with time if the optimal fund size is used at the initial stage.

## References

[1] J. Gao, Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model, Insurance, 45(1) (2009), 9–18.
[2] N. Dokuchaev, X. Yu Zhou, "Optimal investment strategies with bounded risks, general utilities, and goal achieving," Journal of Mathematical Economics, vol.35, no.2, pp.289–309, 2001
[3] B. Othusitse, X. Xiaoping, Stochastic Optimal Investment under Inflammatory Market with Minimum Guarantee for DC Pension Plans, Journal of Mathematics, 7(3) (2015).
[4] K.N. CNjoku., B. O. Osu., E. E. Akpanibah, R. N. Ujumadu. Effect of Extra Contribution on Stochastic Optimal Investment Strategies for DC Pension with Stochastic Salary under the Affine Interest Rate Model. Journal of Mathematical Finance, 7, (2017) 821-833.
[5] M. Jonsson, R. Sircar. Optimal investment problems and Volatility homogenization approximations, in Modern Methods in Scientific Computing and Applications, 75(2002), 255-281, Springer, Berlin, Germany
[6] B.O. Osu, E. E. Akpanibah, K.N.C Njoku,. On the Effect of Stochastic Extra Contribution on Optimal Investment Strategies for Stochastic Salary under the Affine Interest Rate Model in a DC pension Fund. General Letters in Mathematic, Vol. 2, No. 3, 2017, 138-149
[7] G. Deelstra, M. Grasselli, P. F. Koehl, Optimal investment strategies in the presence of a minimum guarantee, Insurance, 33(1) (2003), 189–207.
[8] J. F. Boulier, S. Huang, G. Taillard G, Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund, Insurance 28(2) (2001), 173–189.
[9] A. J. G. Cairns, D. Blake, K. Dowd, Stochastic lifestyling: optimal dynamic asset allocation for defined contribution pension plans, Journal of Economic Dynamics &Control 30(5) (2006) 843–877
[10] J. Gao, Stochastic optimal control of DC pension funds, Insurance, 42(3) (2008), 1159–1164.
[11] J. Xiao, Z. Hong, C. Qin, The constant elasticity of variance(CEV) model and the Legendre transform-dual solution for annuity contracts, Insurance, 40(2) (2007), 302–310.
[12] E. E. Akpanibah, S. K. Samaila, Stochastic strategies for optimal investment in a defined contribution (DC) pension fund, International Journal of Applied Science and Mathematical Theory, 3(3) (2017), 48-55.
[13] Z. Chubing, R. Ximing, Optimal investment strategies for DC pension with stochastic salary under affine interest rate model. Hindawi Publishing Corporationhttp://dx.doi.org/10.1155/2013/297875, (2013).
[14] G. Dawei, Z. Jingyi, Optimal investment strategies for defined contribution pension funds with multiple contributors", http://ssrn.com/abstract=2508109 (2014).
[15] P. Battocchio, F. Menoncin, Optimal pension management in a stochastic framework, Insurance 34(1) (2004) 79–95.
[16] B. O. Osu, E. E. Akpanibah, B I. Oruh, Optimal investment strategies for defined contribution (DC) pension fund with multiple contributors via legendre transform and dual theory, International journal of pure and applied researches, 2(2) (2017), 97-105.
[17] R. Abade, Pension Reforms Act 2004: What's in it for You? www.Newage-online.com (2004).