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Capture Zone of a Well Field in an Aquifer Bounded by Two Parallel Streams

S. Nagheli, N. Samani, D. A. Barry

Abstract**—**In this paper, the velocity potential and stream function of capture zone for a well field in an aquifer bounded by two parallel streams with or without a uniform regional flow of any directions are presented. The well field includes any number of extraction or injection wells or a combination of both types with any pumping rates. To delineate the capture envelope, the potential and streamlines equations are derived by conformal mapping method. This method can help us to release constrains of other methods. The equations can be applied as useful tools to design in-situ groundwater remediation systems, to evaluate the surface–subsurface water interaction and to manage the water resources.

Keywords**—**Complex potential, conformal mapping, groundwater remediation, image well theory, Laplace's equation, superposition principle.

I. Introduction

The capture zone of an aquifer is the region from which water is withdrawn by one or more pumping wells under steady state conditions [1]. After pumping is initiated, the capture zone grows and reaches its maximum size at steady state. The capture zone (also termed the capture envelope [2]) is a key factor in aquifer management, and provides basic understanding for different applications such as groundwater remediation projects (e.g., pump-and-treat, plume containment, bioremediation and chemical oxidation), surface-subsurface water interactions, well head protection, water rights, and transboundary aquifer management. For over-exploited aquifers (where extraction exceeds recharge), delineation of capture zones underpins optimal pumping plans to recover and sustain the depleted storage.

Capture zone determination for confined aquifers is long established in groundwater engineering [3]-[8]. For instance, a 2D pump-and-treat system was presented in [5], including type curves to determine the number and pumping rate of wells to contain a contamination plume. Capture zones in 3D [9] accounted for both horizontal drains and vertical wells in homogeneous, anisotropic aquifers. In [10], analytical and semi-analytical expressions for multiple wells and extraction rates and locations based on complex potential theory and superposition were presented. This was extended in [11] to the computation of stagnation point locations in a multi-well system. Despite the utility of these results, the abovementioned studies focused on simple aquifer geometries and did not account for different boundary configurations or unconfined flow. Efforts to extend analytical results include the case of a multi-well system in rectangular [2] and wedge-shaped confined and unconfined aquifers [12]. Capture zones for multi-well systems in peninsula-shaped aquifers are also available [13], also for confined and unconfined aquifers.

The approach most commonly used to obtain analytical results is the image well method in conjunction with the governing Laplace equation. For complex boundary conditions, the number of imaginary wells is numerous, and the method can become unwieldy. In this situation, however, conformal mapping can be used, since the number of imaginary wells is then limited. Conformal mapping was used in [14] to obtain analytical results for flow to a well between two parallel rivers or rivers that are at a fixed angle.

Our purpose here is to derive a general solution for the capture zone of a multi-well system in a layered aquifer with or without uniform regional flow. The well system includes any number of arbitrarily located extraction or injection wells. Conformal mapping and the Schwarz-Christoffel transformation are applied to determine the complex potential, complex velocity and streamline equations for determining the capture envelope. Based on the results, the effects of number, position and type of wells and regional flow rate on the capture envelope are investigated.

II. Conceptual Model

Fig. 1 (a) shows a plan view of the conceptual model of a confined aquifer with a fully penetrating well bounded by two long parallel boundaries that are a distance d apart. The aquifer is isotropic and homogeneous with uniform thickness. Steady, 2D flow is considered. In Fig. 1, the boundaries are fully penetrating streams with zero longitudinal gradient and ϕ_0 potential head, and that have no hydraulic resistance with the aquifer. An extraction or injection well is located at (x_0,y_0).

III. Mathematical Formulation

We consider a single well first, followed by multiple wells. The Schwarz-Christoffel transformation is applied. This transforms the conceptual model from the z-plane to the ζ-plane. For this conceptual model, a well is located between

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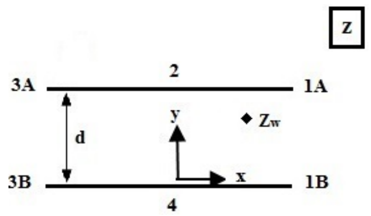
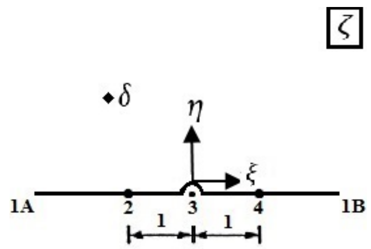
two parallel constant head boundaries. The origin is chosen in the z and ζ planes as shown in Fig. 1, with the x axis coinciding with one of the boundaries. Then the aquifer is transformed to the upper half ζ-plane. Points 1 and 3 are chosen at infinity and the origin of ζ-plane, respectively. After transformation, the coordinates of points 1-4 in the ζ-plane are: $\xi_{1A} = -\infty$, $\xi_2 = -1$, $\xi_3 = 0$, $\xi_4 = 1$ and $\xi_{1B} = \infty$. The values of πk_1 and πk_3 are both π. The magnitudes of πk_2 and πk_4 are zero. The equation of this state can be derived in the ζ–plane as described below.



(a)



(b)

Fig. 1 Solid line represents the constant-head boundary. Solid diamond indicates the locations of the real well.

Based on image well theory, the complex potential becomes:

$$\Omega = \frac{Q_w}{2\pi}\left[\ln\frac{(\zeta-\delta)}{(\zeta-\bar{\delta})}\right]+\phi_0 \tag{1}$$

where Ω is the complex potential in the ζ-plane, Q_w is the pumping rate, and $\bar{\delta}$ is the complex conjugate of δ.

Because of the linearity of Laplace's equation, the complex potential is obtained for a multi-well system by the principle of superposition:

$$\Omega = \phi_0 + \sum_{j=1}^{N} c_j \frac{Q_{wj}}{2\pi}\left[\ln\frac{(\zeta-\delta_j)}{(\zeta-\bar{\delta}_j)}\right] \tag{2}$$

where N is the number of wells, j is summation index, and δ_j is the spatial location of the j^{th} extraction or injection well.

Based on position of the points in the ζ–plane ($\xi_{1A} = -\infty$, $\xi_2 = -1$, $\xi_3 = 0$, $\xi_4 = 1$ and $\xi_{1B} = \infty$) and the values of πk ($\pi k_1 = \pi k_3 = \pi$ and $\pi k_2 = \pi k_4 = 0$), the Schwarz-Christoffel transformation leads to [14]:

$$z = C_1\int\frac{d\zeta}{\zeta} + C_2 \tag{3}$$

where $z = x + iy$, $\zeta = \xi + i\eta$ and C_1 and C_2 are constants. Given the position of the points in planes, the values of C_1 and C_2 can be calculated as d/π and 0, respectively. So,

$$z = \frac{d}{\pi}\ln\zeta \tag{4}$$

The transformation is readily inverted:

$$\zeta = e^{\pi z/d} \tag{5}$$

The position of well δ is:

$$\delta = e^{\pi z_w/d} \tag{6}$$

Therefore, the complex potential based on z is:

$$\Omega = \phi_0 - qze^{-i\beta} + \sum_{j=1}^{N} c_j \frac{Q_{wj}}{2\pi}\ln\left[\frac{e^{\pi z/d} - e^{\pi z_{wj}/d}}{e^{\pi z/d} - e^{\pi \bar{z}_{wj}/d}}\right] \tag{7}$$

where z_{wj} is the spatial location of the j^{th} extraction or injection well.

If there is a uniform flow that has the angle β with +x-axis, from the superposition principle (7) becomes:

$$\Omega = \phi_0 - qze^{-i\beta} + \sum_{j=1}^{N} c_j \frac{Q_{wj}}{2\pi}\ln\left[\frac{e^{\pi z/d} - e^{\pi z_{wj}/d}}{e^{\pi z/d} - e^{\pi \bar{z}_{wj}/d}}\right] \tag{8}$$

where β is measured in radians.

The solution is written in dimensionless form using:

$$z_D = \frac{z}{b},\ z_{wD} = \frac{z_w}{b},\ d_D = \frac{d}{b},\ Q_{wD} = \frac{Q_w}{b^2K},\ q_D = \frac{q}{bK},$$
$$\Omega_D = \frac{\Omega}{Kb},\ \phi_D = \frac{\phi}{Kb},\ \psi_D = \frac{\psi}{Kb} \tag{9}$$

where the subscript D denotes the dimensionless form of each respective parameter, b is the thickness of aquifer, and K is the hydraulic conductivity. Using (9), (8) becomes:

$$\Omega = \phi_{0D} - q_D z_D e^{-i\beta} + \sum_{j=1}^{N} c_j \frac{Q_{wDj}}{2\pi}\ln\left[\frac{e^{\pi z_D/d_D} - e^{\pi z_{wDj}/d_D}}{e^{\pi z_D/d_D} - e^{\pi \bar{z}_{wDj}/d_D}}\right] \tag{10}$$

Since $z = x + iy$ and $z_w = x_w + iy_w$, (10) becomes:

$$\Omega = \phi_{0D} - q_D(x_D + iy_D)e^{-i\beta} + \sum_{j=1}^{N} c_j \frac{Q_{wDj}}{2\pi}\ln\left[\frac{e^{\pi(x_D+iy_D)/d_D} - e^{\pi(x_{wDj}+iy_{wDj})/d_D}}{e^{\pi(x_D+iy_D)/d_D} - e^{\pi(x_{wDj}-iy_{wDj})/d_D}}\right] \tag{11}$$

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Further, bearing in mind that $e^{z}=e^{x}\left(\cos y+i\sin y\right)$, (11) can be written as:

$$\Omega=\phi_{0D}-q_D\left(x+iy\right)_D e^{-i\beta}+\sum_{j=1}^{N}c_j\frac{Q_{wDj}}{2\pi}\ln\left[\frac{\left(f_{1Dj}+if_{2Dj}\right)}{\left(f_{1Dj}+ig_{1Dj}\right)}\right] \quad (12)$$

where,

$$f_{1Dj}=e^{\pi x_D/d_D}\cos\pi y_D/d_D-e^{\pi x_{wDj}/d_D}\cos\pi y_{wDj}/d_D$$
(13)

$$f_{2Dj}=e^{\pi x_D/d_D}\sin\pi y_D/d_D-e^{\pi x_{wDj}/d_D}\sin\pi y_{wDj}/d_D \quad (14)$$

$$g_{1Dj}=e^{\pi x_D/d_D}\sin\pi y_D/d_D+e^{\pi x_{wDj}/d_D}\sin\pi y_{wDj}/d_D \quad (15)$$

The real part of Ω in (12) is the potential, ϕ, and the imaginary part of Ω is the stream function, ψ. The potential is:

$$\phi_D=\Re\left(\Omega\right)=\phi_{0D}-q_D(x_D\cos\beta+y_D\sin\beta)+\sum_{j=1}^{N}c_j\frac{Q_{wDj}}{4\pi}\ln\left[\frac{\left[\left(f_{1Dj}\right)^2+\left(f_{2Dj}\right)^2\right]}{\left[\left(f_{1Dj}\right)^2+\left(g_{2Dj}\right)^2\right]}\right] \quad (16)$$

while the stream function is:

$$\psi_D=\Im\left(\Omega\right)=-q_D\left(y_D\cos\beta-x_D\sin\beta\right)+\sum_{j=1}^{N}c_k\frac{Q_{wDj}}{2\pi}\left[\tan^{-1}\left(\frac{f_{2Dj}}{f_{1Dj}}\right)-\tan^{-1}\left(\frac{g_{2Dk}}{f_{1Dk}}\right)\right] \quad (17)$$

where, the all components are in dimensionless form.

In unconfined aquifers, the saturated thickness is not constant. However, this is accounted for by writing the velocity potential as:

$$\phi=\frac{1}{2}Kh^2 \quad (18)$$

where, h is the hydraulic head, and is equal to the saturated thickness of the aquifer. Consequently, (12) can be rewritten for unconfined aquifer via exchanging the product of Kb to the discharge per unit width (q_a) in the dimensionless parameters. So, they are redefined as:

$$Q_{wDj}=\frac{Q_{wj}}{bq_a},\ q_D=\frac{q}{q_a},\ \Omega_D=\frac{\Omega}{q_a},\ \phi_D=\frac{\phi}{q_a},\ \psi_D=\frac{\psi}{q_a} \quad (19)$$

In (8), the q replaced by q_a.

IV. Stagnation Points

Stagnation points are the locations where the flow velocity is zero, and is computed by setting the derivative of complex potential (10) to zero:

$$\frac{d\Omega}{dz_D}=0=-q_De^{-i\beta}+\sum_{j=1}^{N}c_j\frac{Q_{wDj}}{2\pi}\left[\frac{\pi e^{\pi z_D/d_D}}{d_D(e^{\pi z_D/d_D}-e^{\pi z_{wDj}/d_D})}-\frac{\pi e^{\pi z_D/d_D}}{d_D(e^{\pi z_D/d_D}-e^{\pi\bar{z}_{wDj}/d_D})}\right] \quad (20)$$

The roots of (20) give the positions of the stagnation points. The formulas for the roots are lengthy but can be calculated easily within the Maple software. For convenience, the roots from Maple were computed in MATLAB for further analysis and plotting.

V. Results and Discussion

In the following sections, we solve (10) and (11) for a strip-shaped aquifer with constant head-constant head boundaries as shown in Fig. 1. With the results, we can plot the well(s) capture zones to delineate their shape and properties. Due to variety of parameters involved (number, type, location and extraction/injection rates of wells, and magnitude and direction of regional flow), it is not feasible to present all possibilities. Cases showing major features of the model are presented below.

A. Increasing Pumping Rates

In these examples, we consider a strip-shaped aquifer bounded with constant head-constant head boundaries (Fig. 1a) in which the direction β and the rate $q_{0D}$of the uniform regional flow are $0\ rad$ and 0.001, respectively. Further, we assume that the aquifer thickness is 20 m, the hydraulic conductivity is 5 m/d and the effective porosity is 0.3. Equations (16), (17) and (20) are solved for different extraction rates and the results are plotted in Fig. 2. Figs. 2 (a)-(d) illustrate the potential, streamlines, stagnation points and capture envelopes for these situations. In Fig. 2 (a), the extraction well is positioned at $x_{wD}=5, y_{wD}=2.5$ and pumped at the rate of $Q_{wD}=0.001$. The extraction water is provided from the aquifer (i.e., regional flow) only. If the top of aquifer is contaminated near the point $[x_{cD}=7, y_{cD}=5]$, the contamination does not reach the extraction well.

In Fig. 2 (b), the well pumped at the rate of $Q_{wD1}=0.01$. The well gains water from the streams as well as the uniform regional flow. The capture envelope extends toward the constant head boundaries. In this situation, the contamination does not reach to the well due to the dividing streamline.

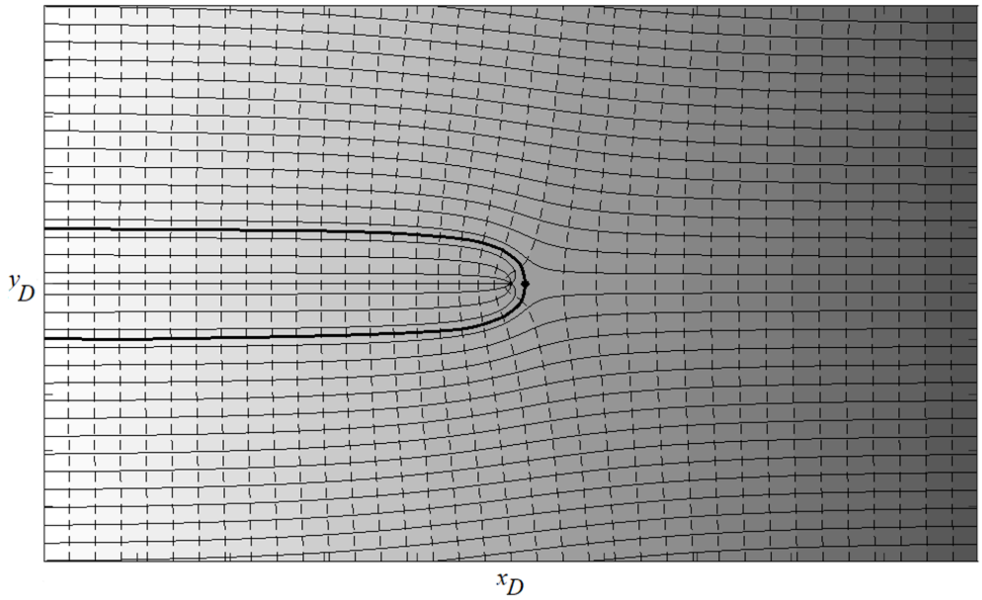
In Fig. 2 (c), the aquifer is pumped at the dimensionless rate of 0.1. The capture envelope develops more than the previous state and the contaminant reaches the well after a dimensionless time of 0.8 (3.2 d).

In Fig. 2 (d), the dimensionless extraction rate is considered as unity. The capture envelope is extensive and the contamination reaches the well after a dimensionless time of 0.51 (2.04 d).

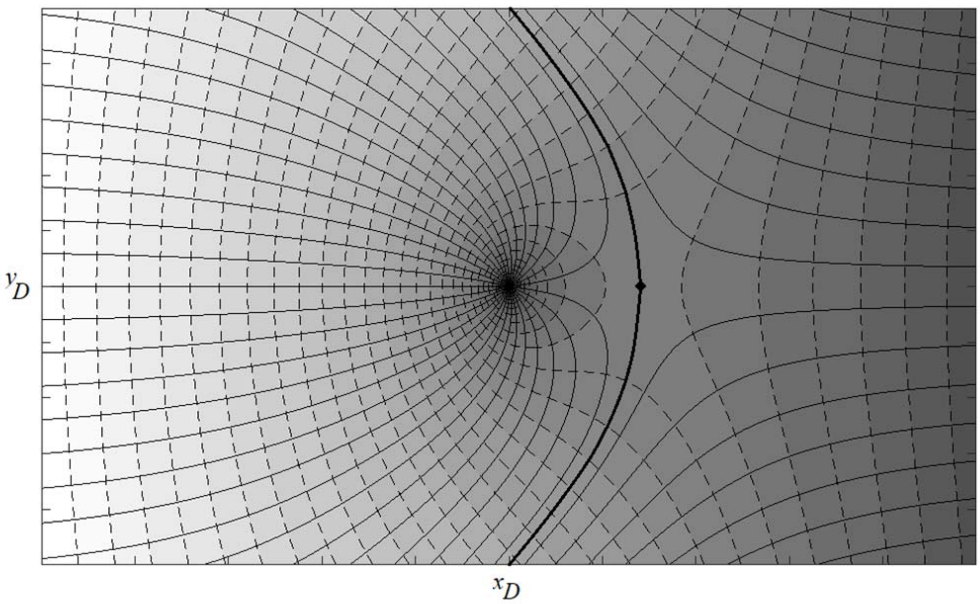
Overall, we see that, with an increasing extraction rate, the capture envelope is increases, and the time for contaminant to reach the well decreases.

B. Multiple Wells

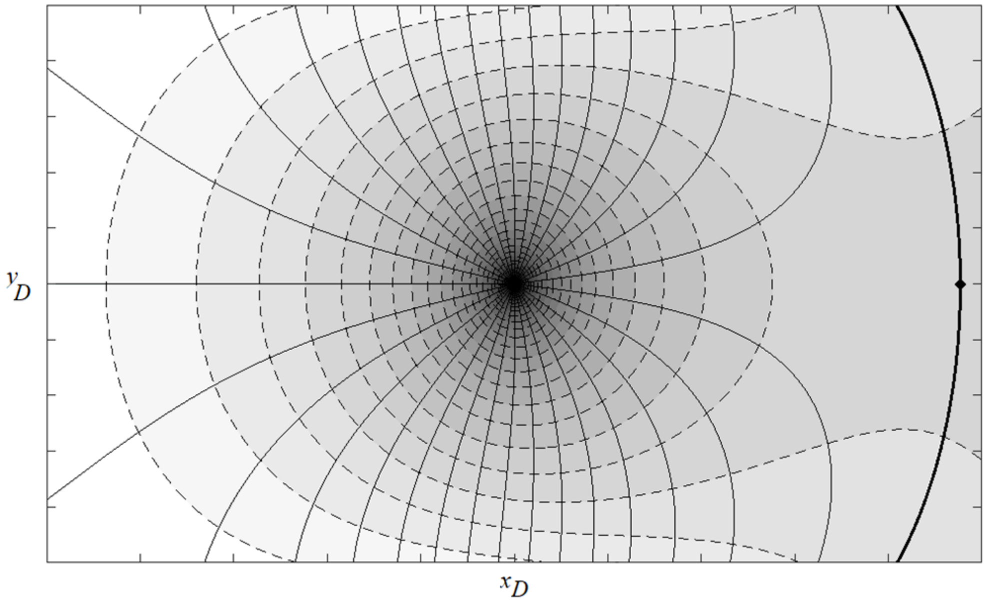
In these examples, we consider the same aquifer parameters as above (Fig. 1 (a)) but increase the number of wells. At first, the extraction well is positioned at $x_{wD} = 5, y_{wD} = 2.5$ and pumped at the rate of ($Q_{wD} = 0.1$). Equations (17), (18) and (21) are solved for this situation (Fig. 3 (a)). If contamination is considered like previously, the travel time for contamination is 0.8 (3.2 d).



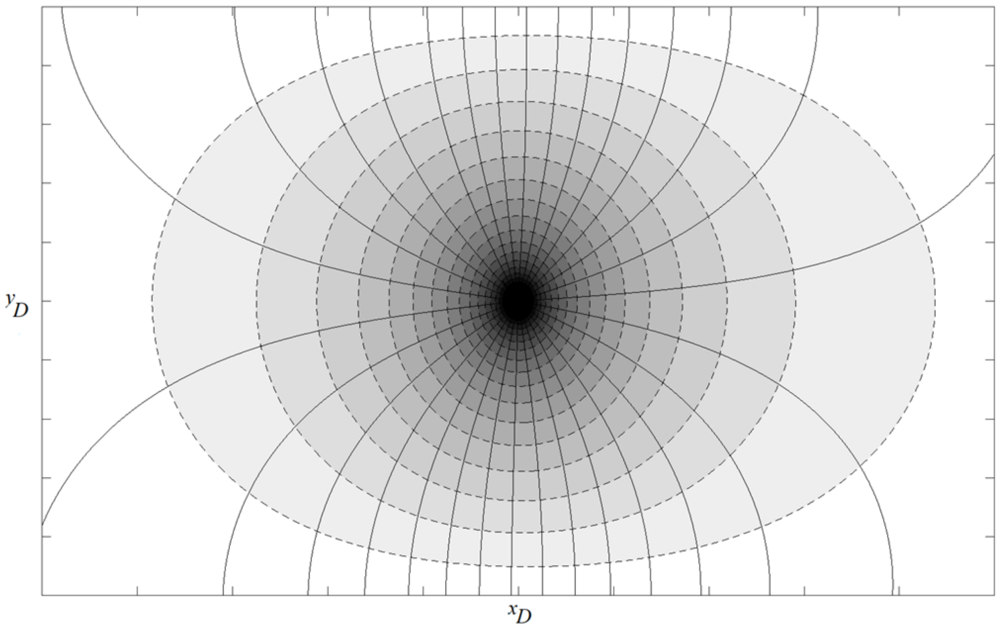
(a)



(b)



(c)



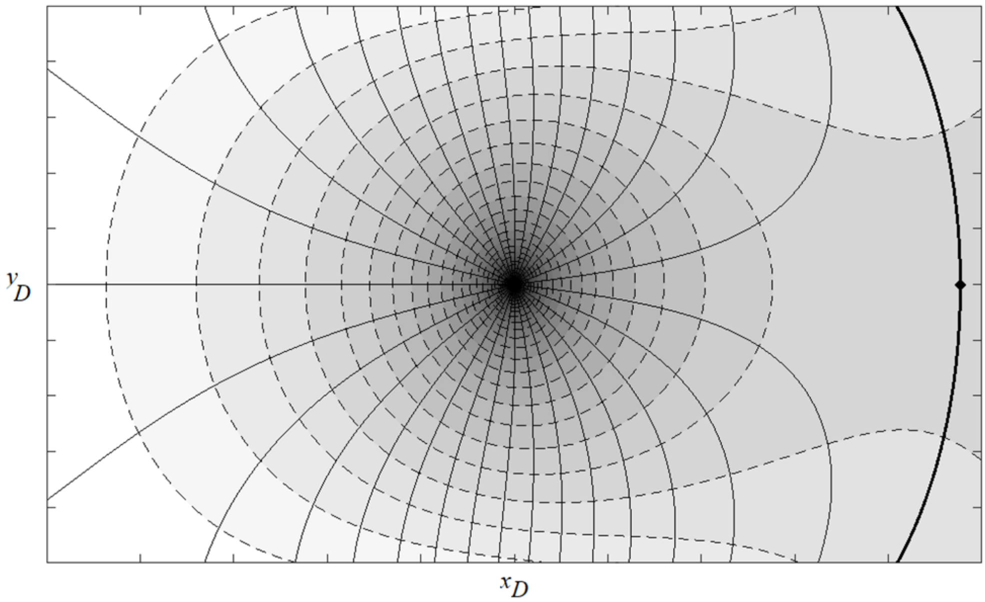
(d)

Fig. 2 Velocity potential and stream function for one well with extraction rate of 0.001 (a), 0.01 (b), 0.1 (c) and 1 (d) in a strip-shaped with constant head – constant head boundary ($\beta = 0$)

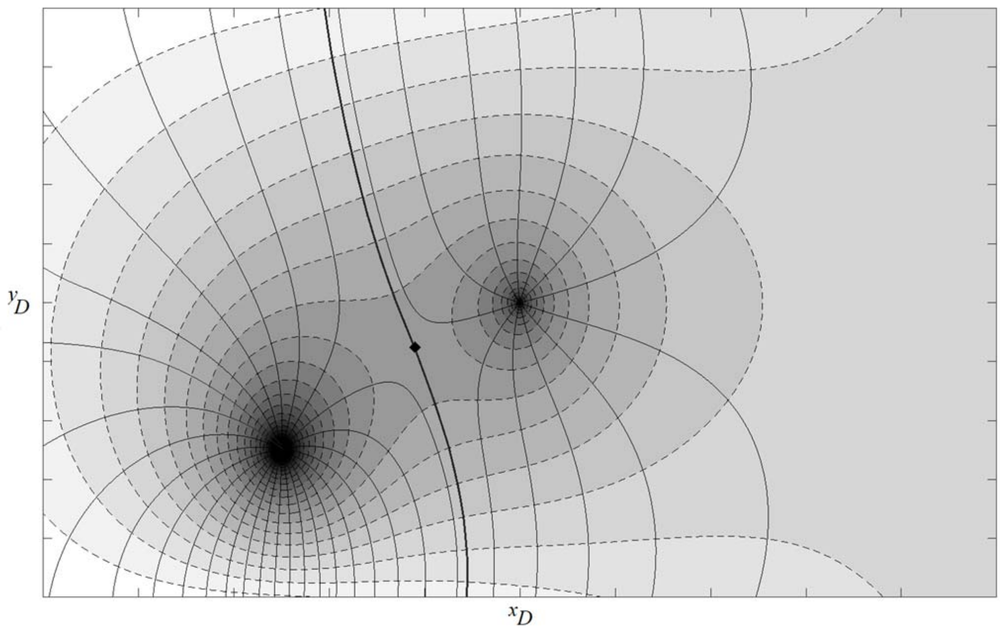
In Fig. 3 (b), we consider two wells at [$x_{wD1} = 5, y_{wD1} = 2.5$] and [$x_{wD2} = 2.5\,, y_{wD2} = 1.25$]. They are pumped at the rate of [$Q_{wD1} = 0$ and $Q_{wD2} = 0.2$], respectively. Both wells gain water from the streams as well as the uniform regional flow. A sharp water divide is established between the two wells so that the uniform regional flow has no contribution in providing any water to well (2); the water divide acts as a barrier boundary for well (2) on the east side and for well (1) on the west side. Streamlines run parallel along the divide. Therefore, the contaminant does not reach well (2).

In Fig. 3 (c), the aquifer is pumped by three wells that are positioned at [$x_{Dw1} = 5, y_{Dw1} = 1.25; x_{Dw2} = 7.5, y_{Dw2} = 3.75$ and $x_{Dw3} = 2.5, y_{Dw3} = 2.75$] and pumped at the rate of [$Q_{wD1} = 0.1$; $Q_{wD2} = 0.2$ and $Q_{wD3} = 0.3$], respectively. Well (3) has the maximum extraction rate, hence, it has the largest capture envelope. The extracted water from well (1) is provided from the south boundary while the other two wells gain water from both boundaries. A water divide is also formed that separates the capture zone of well (2) from those of the other wells. In the same position of contamination point, only well (2) will be contaminated after 0.69 (2.74 d).

In Fig. 3 (d), five wells are positioned at $[x_{Dw1} = 1, y_{Dw1} = 4.5; x_{Dw2} = 2, y_{Dw2} = 0.5; x_{Dw3} = 4, y_{Dw3} = 2.5;\ x_{Dw4} = 5, y_{Dw4} = 3.5$ and $x_{Dw5} = 7, y_{Dw3} = 1.5]$ and are pumped at the rates of $[Q_{wD1} = 0.1;\ Q_{wD2} = 0.2;\ Q_{wD3} = 0.3;\ Q_{wD4} = 0.1$ and $Q_{wD5} = 0.20.2]$, respectively. The regional flow value is assumed 0.001 and its direction is 0. Wells (1), (2), and (4) have individual capture zones and gain water from the north side and south side boundary, respectively. A water divide separates the capture zone of well (5) from those of the other wells while both boundaries as well as the uniform regional flow supply water to well (3). According the water divide, if ae contamination point is located at $[x_{cD} = 7, y_{cD} = 5]$, as in the other examples, the contamination will reach well (5) after the travel time 4.61 (18.45 d). Therefore, we can contain the contamination by designing the configuration and extraction rates of the wells.



(a)



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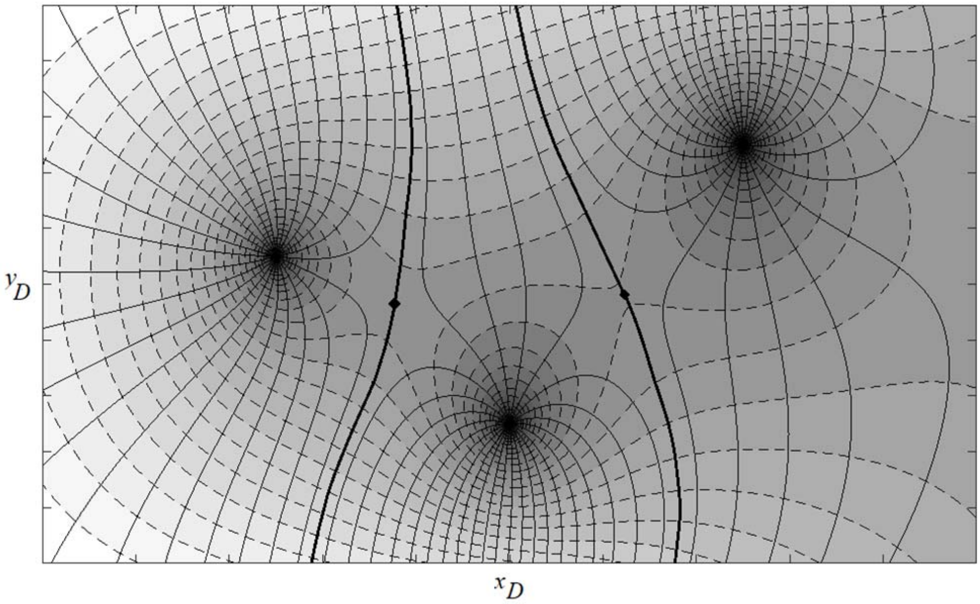

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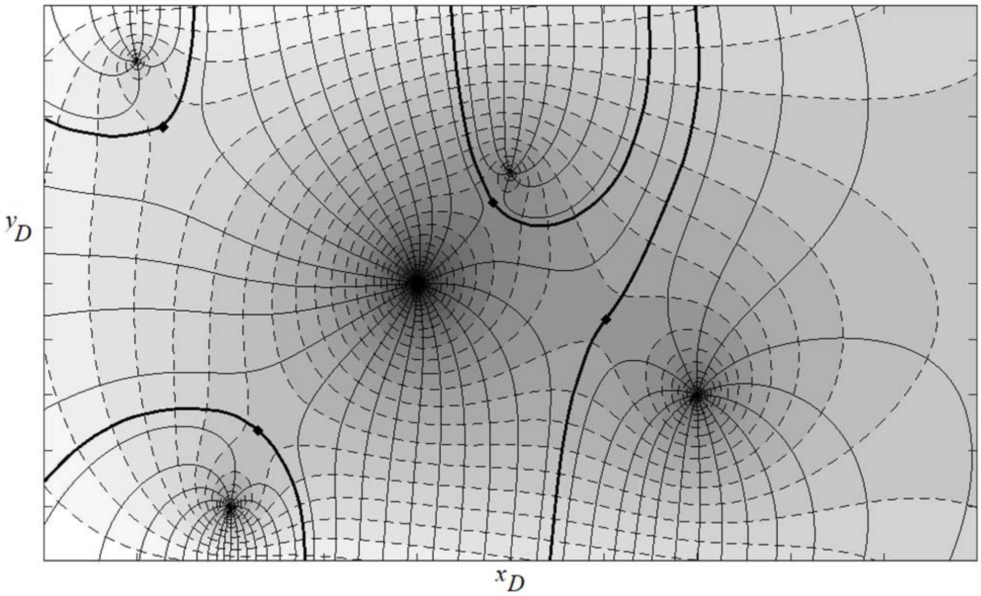

(d)

Fig. 3 Velocity potential and stream function for one (a), two (b), three (c) and five (d) wells in a strip-shaped with constant head – constant head boundary ($\beta = 0$)

C. Effect of Regional Flow Direction

In the above example (Fig. 3), the regional flow direction was 0 (left to right in the plots). Here, the flow direction is $\beta = 3\pi/2$ (top to bottom), with other parameters the same as Fig. 3 (a) (well at $x_0 = 5$, $y_0 = 2.5$, and the magnitude of regional flow is 0.001). In Fig. 4, the potential, streamline and stagnation points are shown. Here, the extracted water is from the regional flow and boundaries, as in Fig. 2 (c). However, the magnitude of receiving water from the northern boundary exceeds that from the southern boundary, due to the direction of regional flow. In addition, there are two stagnation points.

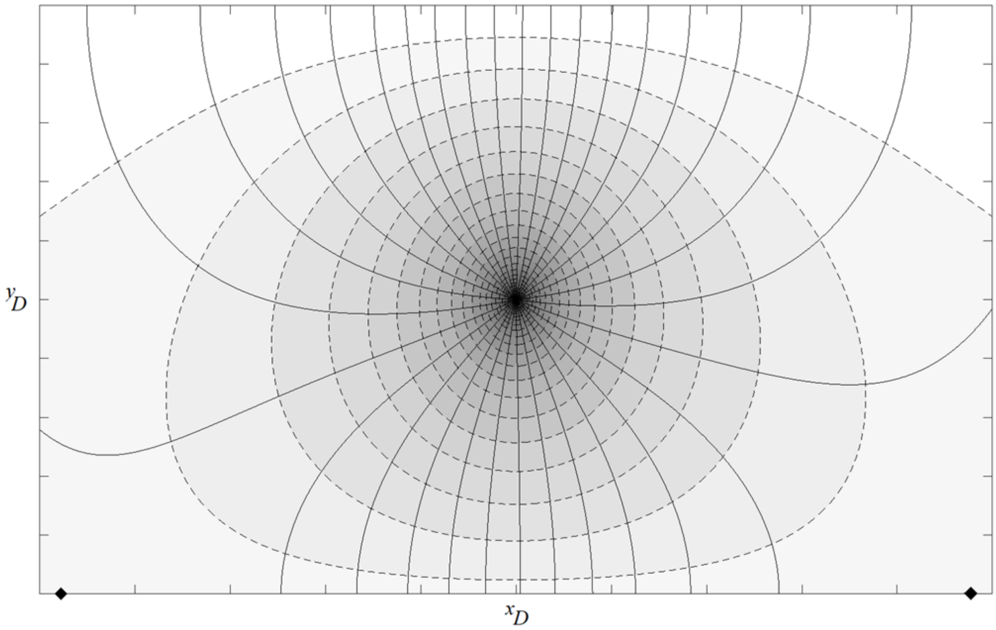

Fig. 4 Velocity potential and stream function for a well in a strip-shaped with constant head – constant head boundary ($\beta = 3\pi/2$).

VI. Conclusion

We presented a capture zone analysis that included calculation of the stream function, velocity potential and stagnation points for a multi-well system for both confined and unconfined strip-shaped aquifers. The governing model, based on Laplace's equation, was solved using conformal mapping and the Schwarz-Christoffel transformation. The results show that the capture zone(s) vary with well pumping rates. The numerical results quantify the influence on the capture envelope by individual wells in a multi-well system. The model is not computationally demanding, and offers a useful tool for the management of water resources such as groundwater and surface water interaction. The model is easily incorporated into an optimization scheme to find the best solution for well position and extraction rate for a target application, e.g., contaminant plume capture.

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