International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:11, No:3, 2017

Invariant Characters of Tolerance Class and Reduction under Homomorphism in IIS

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***Abstract*—**Some invariant properties of incomplete information systems homomorphism are studied in this paper. Demand conditions of tolerance class, attribute reduction, indispensable attribute and dispensable attribute being invariant under homomorphism in incomplete information system are revealed and discussed. The existing condition of endohomomorphism on an incomplete information system is also explored. It establishes some theoretical foundations for further investigations on incomplete information systems in rough set theory, like in information systems.

***Keywords*—**Attribute reduction, homomorphism, incomplete information system, rough set, tolerance relation.

I. Introduction

ROUGH set theory is a useful mathematics tool for analyzing data. It was first proposed by Pawlak in 1982 [7]. Because it can process inconsistent, imprecise and incomplete information, it is successively applied in many fields such as pattern recognition, machine learning, decision making and data mining [8]. It attracts wide interests by various national scholars from all over the world. But traditional rough set model is based on complete information systems, i.e. all attribute values of each object in the given study universe are known. However, due to the data measuring error or the limitation condition in acquiring data, incomplete information system (IIS) (i.e. possibly some attribute values of objects are unknown) is always in front of us.

Recently there are two main approaches to deal with IIS. One is called direct approach in which related concepts in complete information system in rough set theory are appropriately extended to the case of IIS. The other is called indirect approach in which domain experts fill in missing data in IIS by some values such as mean value or frequent appearing value of the related attribute. Compared with the indirect, the direct one avoids the interference from expert subjective factor and is more objective. It has already attracted many experts' interests from different study fields. There are two different semantic explanations about unknown attribute values in IIS. One is that the unknown attribute is missing but it really exists. The other is absent and is not allowed to be compared with other values.

In the direct approach area, such kind of researches has been done. Kryszkiewicz defines tolerance relation to process IIS based on the first semantic explanation. Based on the relation, knowledge reduction problem and etc. are deeply investigated by him [4]. Based on the second semantic explanation, Stefanowski suggests non-symmetric similarity relation [13]. Guoying proposes limited tolerant relation for the requiring condition of tolerance relation is relaxing [15]. With the granular points of view, Leung et al. introduce maximal consistent block technique for rule acquisition [5]. Wu et al. investigate information granules in general and complete covering [17]. Chen et al. discuss generalized model of rough set theory based on compatibility relation [1]. In order to deal with IIS under both semantic explanations for unknown attribute value simultaneously, Grzymala-Busse defines feature relation [2]. Many other scholars suggest some other methods. Anyway, building extended rough set models to study IIS has become a very important research topic.

The concept of homomorphism of complete information system is a powerful tool to study the relationship of complete information systems and is first put forward by Grzymala-Busse et al. [3]. In [3], the authors give the conditions of making an information system be selective. The endohomomorphism of complete information system is studied in [6]. An endohomomorphism complete information system based on attribute redundancy is also built in. Several meaningful results such as reduction preservation, core preservation etc. are obtained. Reference [12] discusses invariant characters of information systems under some homomorphisms. It reveals interdependence among object mapping, attribute mapping and value domain mapping. It also obtains some theorem results about invariant properties for upper and lower approximations in complete information system. Some other invariant properties are also explored in [14], [18]. Now experts even study multi-granular rough set models [9]-[11]. Acquiring knowledge from IIS from different granular views still remains as a hot topic [16], [19].

This paper studies some properties of IIS under the first semantic explanation, discusses conditions of invariant properties of tolerance class, attribute reduction, indispensable attribute and dispensable attribute in IIS under homomorphism. It also explores the condition of existing an endohomomorphism on an IIS. It lays a certain theoretical foundation of further studying IIS using rough set theory.

II. Basic Concepts

Let S=(U,A,V,f) be an IIS [3], where U is a finite non-empty

Sponsored and financially supported by a China National Science Foundation (61100116) and a Foundation from Educational Bureau of Jiangsu provincial government and a Foundation of Graduate Department of Jiangsu University of Science and Technology.

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set of objects, A is a finite non-empty set of condition attributes and decision attributes. For any $a\in AT$, $a:V\rightarrow V_a$, where V_a is called the value range of a. For each object, some attribute values are lost or missing, called null values. The set of all attribute values is denoted by $V=\cup V_a(a\in AT)$.

Definition 1. In IIS $S=(U,A,V,f)$, for any attribute subset $P\subseteq AT$, the tolerance relation on U is defined as $SIM(P)=\{(x,y)|\forall a\in P(f(x,a)=f(y,a)\vee f(x,a)=*\vee f(y,a)=*)\}$.

If P is a singleton set, for example $P=\{a\}$, then $SIM(P)=SIM(\{a\})$ is abbreviated to be $SIM(a)$. Obviously, $SIM(P)$ is reflexive and symmetry on U.

Definition 2. For any attribute subset $P\subseteq AT$, $S_P(x)=\{y|(x,y)\in SIM(P)\}$ is called a tolerance class generated by x as generator. If P is a singleton set, for example $P=\{a\}$, then $S_P(x)=S_{\{a\}}(x)$ is abbreviated by $S_a(x)$.

Definition 3. $U/SIM(P)=\{S_P(x)|x\in U\}$, U/P for short, a cover on U, is called a knowledge system.

Definition 4. For any $X\subseteq U$, $P\subseteq AT$, $P\_(X)=\{y|y\in U, S_P(y)\subseteq X\}$ is called the lower approximation of X; $P^-(X)=\{y|y\in U, S_P(y)\cap X\neq\varnothing\}$ is called the upper one.

Definition 5. Let $P\subseteq AT$, $P\neq\varnothing$. If $SIM(P)=SIM(P\text{-}\{a\})$ for $a\in P$, then a is called dispensable or redundant in P, otherwise a is called indispensable.

If each $a\in P$ is indispensable in P, then P is called independent, otherwise P is called dependent.

Theorem 1. If $a\in P$ is dispensable in P, $b\in P$ and for any $x\in U, S_a(x)=S_b(x)$, then a is also dispensable in P.

Proof. Since $a\in P$ is dispensable in P, we have $SIM(P)=SIM(P\text{-}\{a\})$. For any $x\in U$, $S_{P\text{-}\{a\}}(x)=S_P(x)$. $S_P(x)=S_c(x)(c\in P)=S_c(x)$ $(c\in P\text{-}\{a\})=S_c(x)$ $(c\in P\text{-}\{b\})=S_{P\text{-}\{b\}}(x)$. It follows that $SIM(P)=SIM(P\text{-}\{b\})$. So b is also dispensable in P.

Definition 6. Let $Q\subseteq P$. If Q is independent and $SIM(Q)=SIM(P)$, then Q is called a reduction of P.

Definition 7. The set consisted of all indispensable attribute in P is called the core of P, denoted by $core(P)$.

The relation of the core and reductions is $core(P)=\cap Q(Q\in red(P))$, where $red(P)$ is the collection of all reductions of P.

Theorem 2. Let $P\subseteq A$. If $a\in P$ and for any $x\in U$, $S_a(x)=\cup S_{P\text{-}\{a\}}(y)$ $(y\in S_a(x))$, then $SIM(P\text{-}\{a\})=SIM(P)$, i.e. a is dispensable in P.

Proof. Since $a\in P$, we must have $SIM(P)\subseteq SIM(P\text{-}\{a\})$. Therefore, we only need to prove $SIM(P\text{-}\{a\})\subseteq SIM(P)$ under given conditions. Because for any $(x,y)\in SIM(P\text{-}\{a\})$ we have $y\in S_{P\text{-}\{a\}}(y)\subseteq\cup S_{P\text{-}\{a\}}(y)$ $(y\in S_a(x))=S_a(x)$. Thus, $(x,y)\in SIM(a)$. It follows that $SIM(P\text{-}\{a\})\subseteq SIM(a)$. So $SIM(P)=SIM(P\text{-}\{a\})\cap SIM(a)\supseteq SIM(P\text{-}\{a\})$. That is $SIM(P\text{-}\{a\})\subseteq SIM(P)$. Thus, $SIM(P\text{-}\{a\})=SIM(P)$.

Theorem 3. Let $P\subseteq A$. If $a\in P$ is dispensable in P, that is $SIM(P\text{-}\{a\})=SIM(P)$, Then $\forall x\in U$, $S_a(x)\subseteq\cup S_{P\text{-}\{a\}}(y)$ $(y\in S_a(x))$.

Proof. For any $z\in S_a(x)$, since $z\in S_{P\text{-}\{a\}}(z)$, thus, $z\in\cup S_{P\text{-}\{a\}}(y)$ $(y\in S_a(x))$. Because $z\in S_a(x)$ is arbitrarily chosen, $S_a(x)\subseteq\cup S_{P\text{-}\{a\}}(y)$ $(y\in S_a(x))$ holds.

Theorem 4. Let $P\subseteq A$. $a\in P$ is dispensable in P if, and only if there exist $x,y\in U$ such that $y\notin S_a(x)$, $y\in S_{P\text{-}\{a\}}(x)$.

Proof. Since $a\in P$ is dispensable in P, $SIM(P\text{-}\{a\})\neq SIM(P)$. Because $P\text{-}\{a\}\subseteq P$, $SIM(P)\subseteq SIM(P\text{-}\{a\})$. Therefore, we only have $SIM(P)\subset SIM(P\text{-}\{a\})$. Thus, there exists $(x,y)\in SIM(P\text{-}\{a\})$, $(x,y)\notin SIM(P)$. It must have $(x,y)\notin SIM(a)$, otherwise, $(x,y)\in SIM(P)$. Furthermore, $y\notin S_a(x)$, $y\in S_{P\text{-}\{a\}}(x)$. Since $P\text{-}\{a\}\subseteq P$, $SIM(P)\subseteq SIM(P\text{-}\{a\})$. If $a\in P$ is dispensable in P, i.e. $SIM(P\text{-}\{a\})=SIM(P)$, then for $\forall(x,y)\in SIM(P\text{-}\{a\})=SIM(P)$, we have $(x,y)\in SIM(a)$, $(x,y)\in SIM(P\text{-}\{a\})$. Furthermore, $y\in S_a(x)$, $y\in S_{P\text{-}\{a\}}(x)$. It contradicts to the given condition. So $a\in P$ is indispensable.

Suppose $S=(U,A,V,f)$ and $S'=(U',A',V',f')$ are two IISs, $h_O:U\rightarrow U'$, $h_A: A\rightarrow A'$, $h_D: V\rightarrow V'$, then $h=(h_O, h_A, h_D):S\rightarrow S'$ is called a mapping from S to S'.

Definition 8. If for $\forall x\in U$, $\forall a\in A$, $h_D(f(x,a)=f'(h_O(x), h_A(a))$, then h is called a homomorphism between S and S'. If S=S', then h is called an endohomomorphism.

Definition 9. Let f: $U\rightarrow U$ be a mapping on U, $D\subseteq U$ be a subset of U. If $f(D)=D$, then D is called an invariant subset of f.

III. Invariant Properties

Lemma 1. Let $S=(U,A,V,f)$ and $S'=(U',A',V',f')$ be two IISs, $h=(h_O,h_A,h_D)$ be a homomorphism between IIS S and S'. If both S and S' contain attribute value *, and $h_D(*)=*$, then for any attribute subset $P\subseteq A$ and any object $x\in U$, we have $h_O(S_P(x))\subseteq S_m(h_O(x))$, where $m=h_A(P)$. Especially, if h_O is surjective, h_D is 1-to-1 corresponding injection and $h_D(*)=*$, then inverse conclusion also holds, i.e., $h_O(S_P(x))=S_m(h_O(x))$.

Proof. For $\forall y'\in h_O(S_P(x))$, there must have $\exists y\in S_P(x)$ such that $\forall$ $h_O(y)=y'$, and then $\forall a\in P$, $f(y,a)=f(x,a)\vee f(y,a)=*\vee f(x,a)=*$. From the definition of homomorphism, we have $y'=h_O(y)\in S_m(h_O(x))$.So $h_O(S_P(x))\subseteq S_m(h_O(x))$ for $y'\in h_O(S_P(x))$ is arbitrarily chosen.

Now we prove that when h_O is surjective, h_D is one to one corresponding injection and $h_D(*)=*$, $h_O(S_P(x))=S_m(h_O(x))$. Before $h_O(S_P(x))\subseteq S_m(h_O(x))$ is already proved, so now we need only to prove that $h_O(S_P(x))\supseteq S_m(h_O(x))$.

Take $y'\in S_m(h_O(x))$. Because h_O is surjective, there exists $y\in U$ such that $h_O(y)=y'$. By the definition of homomorphism, for any $a\in P$, we have $h_D(f(y,a))=f'(h_O(y), h_A(a))=f'(y', h_A(a)).f'(h_O(x), h_A(a))=h_D(f(x,a))$.

Since $y'\in S_m(h_O(x))$, we have $f'(y', h_A(a))=f'(h_O(x), h_A(a))$ or $f'(y', h_A(a))=*$ or $f'(h_O(x), h_A(a))=*$. Furthermore, we obtain $y'\in h_O(S_P(x))$. So we always have $S_m(h_O(x))\subseteq h_O(S_P(x))$. Therefore, under the condition that h_O is surjective, h_D is one to one corresponding injection and $h_D(*)=*$,we have $h_O(S_P(x))=S_m(h_O(x))$.

From the proof of Lemma 1, we can obtain the following conclusion: if h_O is surjective, h_D is one to one corresponding injection and $h_D(*)=*$, h_A is surjective, the number of tolerance classes in S is equal to that in S'.

Lemma 2. Let S and S' be two IIS, h_O: $U\rightarrow U'$, h_A: $A\rightarrow A'$,

h_D:$V \rightarrow V'$, h=(h_O, h_A, h_D):$S \rightarrow S'$ is called a mapping from S to S', and for h, h_O be surjective, h_D be 1- to-1 corresponding injection and $h_D(*)=*$ when both S and S' contain *. If $a \in A$ is redundant in A, A-{a}-$h^{-1}(h_A(a)) \neq \emptyset$, then $h^{-1}(h_A(a)) \subseteq A$ is also redundant in A.

Proof. Take $q' \in A'$ and $q'=h_A(a)$. For any $x \in U$ and any $b' \in h_A^{-1}(q')$, from Lemma 1, $h_O(S_a(x))= S_{q'}(h_O(x))= h_O(S_b(x))$. We assert that $S_a(x)= S_b(x)$. Otherwise, there exists an element $x_0 \in U$ such that $S_a(x_0) \neq S_b(x_0)$. Then there exists $x' \in S_a(x_0)$, $x' \notin S_b(x_0)$ or $x'' \notin S_a(x_0)$, $x'' \in S_b(x_0)$ In the first case, $f(x',b) \neq f(x_0,b) \wedge f(x',b) \neq * \wedge f(x_0,b) \neq *, f(x',a)=f(x_0,a) \vee f(x',a)=* \vee f(x_0,a)=*$.

Since h_D is one to one corresponding injection and $h_D(*)=*$, thus $* \neq h_D(f(x',b)) \neq h_D(f(x_0,b)) \neq *$. $* \neq f'(h_O(x'),q')=h_D(f(x',b)) \neq h_D(f(x_0,b))= f'(h_O(x_0),q') \neq *$. So, $* \neq h_D(f(x',b)) \neq f'(h_O(x_0),q') \neq *$. Therefore, from $f(x',a)=f(x_0,a) \vee f(x',a)=* \vee f(x_0,a)=*$, we obtain

1. If $f(x',a)=f(x_0,a) \neq *$, then we have $* \neq h_D(f(x',a))= h_D(f(x_0,a)) \neq *$, since h_D is an one-to-one corresponding injection and $h_D(*)=*$ when both S and S' contain attribute value *. Therefore, $f'(h_O(x'),q') =h_D(f(x',a)) =h_D(f(x_0,a))= f'(h_O(x_0),q') \neq *$. This contradicts to $f'(h_O(x'),q') \neq f'(h_O(x_0),q')$.
2. When $f(x',a)=*$, $f(x_0,a) \neq *$, since h_D is one-to-one corresponding injection and $h_D(*)=*$, then $f'(h_O(x'), q')= h_D(f(x',a))=h_D(*)=*$. This contradicts to $f'(h_O(x'), q') \neq *$.
3. When $f(x',a) \neq *$, $f(x_0,a)=*$, since h_D is one-to-one corresponding injection and $h_D(*)=*$, then $f'(h_O(x_0), q')= h_D(f(x_0,a))=h_D(*)=*$. This contradicts to $f'(h_O(x_0), q') \neq *$.
4. When $f(x',a)=*$, $f(x_0,a)=*$, for h_D is 1-to-1 corresponding injection, $h_D(*)=*$, then $f'(h_O(x'),q') =h_D(f(x',a))= h_D(*)=*$, $f'(h_O(x_0), q') =h_D(f(x_0,a))=h_D(*)=*$. These contradict to $f'(h_O(x'), q') \neq *$ and $f'(h_O(x_0),q') \neq *$ respectively.

That means $S_a(x)= S_b(x)$. In the other case, a contradiction can be also deduced. So the above assertion is true. Therefore, a is redundant in A, and if A-{a}-$h^{-1}(h_A(a)) \neq \emptyset$, b is also redundant in A. From that b is arbitrarily chosen, we obtain all attributes in $h^{-1}(q')=h^{-1}(h_A(a))$ are redundant in A.

Lemma 3. Let h=(h_O,h_A,h_D) be a homomorphism between S and S'. If for h we have: both h_O and h_A are surjective, h_D is an one to one corresponding injection and $h_D(*)=*$ when both S and S' contain *, then for any attribute subset $P \subseteq A$, all attributes in P are redundant in A, then if $h_A(P) \subset A'$, all attributes in $h_A(P)$ are redundant in S'.

Proof. From the definition of redundancy, we just only need to prove that $SIM(A')= SIM(A'-P')$, where $P' =h_A(P)$. Because $A'-P' \subseteq A'$, $SIM(A') \subseteq SIM(A'-P')$. Now we are to prove the inverse inclusion is held, i.e. $SIM(A'-P') \subseteq SIM(A')$. Let $x',y' \in U'$, satisfy $(x',y') \in SIM(A'-P')$, i.e., $y' \in S_{A'-P'}(x')$. Since h_A is surjective, $A'= h_A(A)$. For h_O is surjective, there exists $x \in U$ such that $h_O(x) = x'$. From Lemma 1, denote r= A-$h^{-1}_A(P')$,t= $h_A(r)$, we have $y' \in S_{A'-P'}(x')=S_t(x')= S_t(h_O(x))= h_O(S_r(x))$. So, there exists $y \in S_r(x)$ such that $h_O(y)=y'$ and $(y,x) \in SIM(r)$.

Because P is redundant in S, from Lemma 2, $h^{-1}_A(P')$ is also redundant in A, then $(x,y) \in SIM(A)$. Thus, $(h_O(x), h_O(y)) \in SIM(h_A(A))$, that is, $(x',y') \in SIM(A')$. This means $SIM(A'-P') \subseteq SIM(A')$.

Proposition. Let h=(h_O,h_A,h_D) be a homomorphism between S and S'. If for h, we have that both h_O and h_A are one to one corresponding injection and $h_D(*)=*$ when both S and S' contain *. If $P \subset A$ is redundant in S, $\forall a \in P$, A-{a}-$h^{-1}_A(h_A(a)) \neq \emptyset$, $h_A(P) \subset A'$, then $h_A(P)$ is also redundant in S'.

Proof. Let $A_1=h_A(A)$. Then $S_1=(U',A_1,V_1,f_1)$ is a subsystem of S', where $V_1=\cup V_{a'}(a' \in h_A(A))$, f_1 is the restriction of f' on $U' \times A_1$. Thus, h satisfies conditions given in Lemma 3 from S to S_1. So $h_A(P)$ is redundant in S_1. Since $A_1 \subseteq A'$, $h_A(P)$ is redundant in A', that is, $h_A(P)$ is redundant in S'.

Theorem 5. Let S=(U,A,V,f) be an IIS. Suppose $P \subseteq A$, A-P is redundant in A and for each $a \in A$-P there exists $a' \in P$ such that $SIM(a) \subseteq SIM(a')$. Use S' to denote IIS (U,A,V',f'), where $V'= V_a(a \in P)$, f' is the restriction of f on $U \times P$, then there exists a homomorphism h from S to S' and it is an endohomomorphism on S.

Proof. If P=A, then the theorem is obviously correct. So we assume that $P \subset A$. Define a mapping h=(h_O,h_A,h_D):$S \rightarrow S'$ as follows: $h_O=1_U$ is the identity mapping on U. $h_A(a)=a$, if $a \in P$; $h_A(a)=a'$, if $a \in A$-P, $SIM(a) =SIM(a')$.

For $\forall x \in U$, let $h_D(f(x,a))=f(x,a)$, if $a \in P$; $h_D(f(x,a)) =f(x,a')$,if $a \in A$-P, $a'= h_A(a)$. Next, we explain the rationality of the definition of h_D. For any fixed $a \in A$-P,if $f(x,a)=f(y,a)$ or $f(x,a))=*$ or $f(y,a)=*$ ($x,y \in U$), then $f(x,a') = f(y,a')$ or $f(x,a')=*$ or $f(y,a')=*$ (x, $y \in U$), here,$a'= h_A(a)$. In fact, notice that $SIM(a)= SIM(a')$ and $f(x,a)=f(y,a)$ or $f(x,a)=*$ or $f(y,a) =*$, then $(x,y) \in SIM(a)$ implies $(x, y) \in SIM(a')$. This means $f(x,a')=f(y,a')$ or $f(x,a') =*$ or $f(y,a')=*$. At this time, $h_D(f(x,a)) =f(x,a')= f(h_O(x),h_A(a))$ (because $h_O(x)$=x). Again for any $x \in U$ and any $a \in A$, if $a \in P$, then $h_D(f(x,a))= f(x,a')= f(h_O(x),h_A(a))$. Therefore, for any $x \in U$ and any $a \in A$, we have $h_D(f(x,a))= f(h_O(x),h_A(a))$. Thus, the definition of h_D is rational. h is a homomorphism from S to S' and it is an endohomomorphism on S.

Generally, note that an arbitrary reduction of an IIS may not be able to determine a homomorphism from the later to the former. See the following example.

Example. An IIS S is given in Table I.

TABLE I
AN INCOMPLETE INFORMATION SYSTEM

	a	b	c
x_1	0	1	0
x_2	1	*	*
x_3	*	2	*
x_4	2	0	1

Table II is IIS S', a reduction of S. However, obviously, there does not exist a homomorphism from S to S' or an endohomomorphism on S. This is because we cannot obtain an one-to-one corresponding mapping by specifying $h_A(0)=a$ or $h_A(0)=b$ where element 0 is in V_c, a, b are in V_a or V_b.

TABLE II
ANOTHER INCOMPLETE INFORMATION SYSTEM

	a	b
x_1	0	1
x_2	1	*
x_3	*	2
x_4	2	0

Theorem 6. Let S be an IIS, h be an endohomomorphism on S, h_D be one to one corresponding mapping and $h_D(*)=*$, and for any $x\in U$ and any $a\in A$, tolerance class $S_a(x)$ generated by x under attribute a satisfies $|f(S_a(x),a)-\{*\}|=1$. If $h_A(A)$ is a h_A -invariant subset, then $A-h_A(A)$ is redundant in A.

Proof. For convenience, let us denote $P=h_A(A)$. Take $a\in A-P$, as an any attribute from $A-P$. We assert that for any $x\in U$ we have $S_a(x)=\cup S_P(y)$ $(y\in S_a(x))$.

If not, i.e., $\exists\, x_0 \in U$, $S_a(x_0)\subset \cup S_P(y)$ $(y \in S_a(x_0))$, then $\exists\, y_0 \in S_a(x_0)$ such that $S_P(y_0) \not\subseteq S_a(x_0)$. In other words, $\exists\, y_1 \in S_P(y_0)$ but $y_1 \notin S_a(x_0)$. So, $f(x_0,a) \neq f(y_1,a) \wedge f(x_0,a) \neq * \wedge f(y_1,a) \neq *$. By the definition of endohomomorphism and h_D is one to one corresponding mapping and $h_D(*)=*$, we have $* \neq h_D(f(x_0,a)) \neq h_D(f(y_1,a)) \neq *$,i.e.,$* \neq f(h_O(x_0), h_A(a)) \neq f(h_O(y_1), h_A(a)) \neq *$. On the other hand, by Lemma 1 and P is h_A -invariant subset, i.e., $h_A(P)=P$, we have $h_O(S_P(y_0)) \subseteq S_P(h_O(y_0))$. Because $y_1\in S_P(y_0)$, thus $y_1, y_0\in S_P(y_0)$. Therefore, $h_O(y_0)$ and $h_O(y_1)$ both are in $S_P(h_O(y_0))$. Since for any $a\in P$ $h_A(a)\in P$, we have $f(h_O(y_0), h_A(a))=f(h_O(y_1), h_A(a))$ or $f(h_O(y_0), h_A(a))=*$ or $f(h_O(y_1), h_A(a))=*$. If $f(h_O(y_1),h_A(a))=*$, then it is contradict to the above assumption. Thus it must have $f(h_O(y_1), h_A(a))\neq *$. At this time

1. If $f(h_O(y_0), h_A(a))\neq *$, then
   i. If $f(h_O(x_0), h_A(a))=*$, then $h_D(f(x_0,a)) =*$, $f(x_0,a)=*$. It contradicts to $f(x_0,a)\neq *$.
   ii. If $f(h_O(x_0), h_A(a))\neq *$, then because $y_0 \in S_a(x_0)$, $f(x_0,a)= f(y_0,a) \vee f(x_0,a)=* \vee f(y_0,a)=*$. But $h_D(f(x_0,a))= f(h_O(x_0),h_A(a)) \neq * \Rightarrow$ implies $f(x_0,a) \neq *$; $h_D(f(y_0,a))= f(h_O(y_0),h_A(a)) \neq * \Rightarrow f(y_0,a)\neq *$. So it must have, $f(h_O(x_0), h_A(a))= f(h_O(y_0),h_A(a)) \neq *$. Thus $f(h_O(y_1),h_A(a)) = f(h_O(y_0),h_A(a))= f(h_O(x_0),h_A(a))\neq *$. It is a contradiction.
2. If $f(h_O(y_0),h_A(a))=*$, i.e., $h_D(f(y_0,a))= f(h_O(y_0), h_A(a))=* \Rightarrow f(y_0,a)=*$, then
   i. If $f(h_O(x_0),h_A(a))=*$, then $h_D(f(x_0,a))= f(h_O(x_0),h_A(a)) =* \Rightarrow f(x_0,a)=*$. This contradicts to the assumption $f(x_0,a) \neq *$ above.
   ii. If $f(h_O(x_0),h_A(a)) \neq *$, then $h_D(f(x_0,a))= f(h_O(x_0),h_A(a)) \neq * \Rightarrow f(x_0,a) \neq *$. Since for any $a\in A$ and any $x\in U$, $|f(S_a(x),a)-\{*\}|=1$, according to the condition and $x_0,y_0 \in S_P(x_0) \subseteq S_a(x_0), y_0,y_1 \in S_P(y_0) \subseteq S_a(y_0), h_A(a) \in P$, therefore $x_0,y_1 \in S_a(y_0)$. Thus $f(h_O(y_1), h_A(a))= \neq f(h_O(x_0),h_A(a))\neq *$. This also contradicts to the above conclusion. So, $S_a(x)=\cup S_P(y)$ $(y\in S_a(x))$.

By the former theorem, we know that each attribute in $A-P$ is redundant in A.

The essence of homomorphism is clustering. The condition that h_D is a one to one corresponding mapping and $h_D(*)=*$ when both S and S' contain * and $|f(S_a(x),a)-\{*\}| =1$ means that the classification grade of each attribute deduced by h does not decrease. In addition, if $h_A(A)$ is an h_A -invariant subset, then Theorem 6 ensures that $A-h_A(A)$ is also redundant in A.

Theorem 7. Let S and S' be two IISs, $h=(h_O,h_A,h_D)$ be a homomorphism between IIS S and S', satisfying that h_O and h_A are surjective, h_D is one to one corresponding mapping, $h_D(*)=*$ when both S and S' contain *, and for any $x\in U$ and any $a\in A$, tolerance class $S_a(x)$ generated by x under attribute a satisfies $|f(S_a(x),a)-\{*\}|=1$. If $P\subseteq A$ is a reduction of A in S, then $h_A(P)\subseteq A'$ is also a reduction of A' in S'.

Proof. Denote $h_A(P)$ by P'. Firstly, we prove $SIM(P')= SIM(A')$ It is obvious that we should only prove $SIM(P') \subseteq SIM(A')$.

Let $(x',y')\in SIM(P')$, i.e. $y'\in S_{P'}(x')$. Since h_O is surjective, There exists $x\in U$ such that $h_O(x)=x'$. Therefore according to Lemma 1, we have $S_{P'}(x')= S_{P'}(h_O(x))= h_O(S_P(x))$. Then there exists $y\in S_P(x)$ such that $h_O(y)=y'$. So we have $(x,y)\in SIM(P)$. Because P is a reduction of A in S, $(x,y)\in SIM(P)$ implies $(x,y)\in SIM(A)$. Thus, $(h_O(x), h_O(y))\in SIM(h_A(A))$. For h_A is surjective, we have $(x',y')\in SIM(A')$. Furthermore, we obtain $SIM(P')\subseteq SIM(A')$. So $SIM(P')=SIM(A')$.

By Lemma 2, the remainder work is to prove that P' does not contain redundant attribute again. Conversely, we assume that $a'\in P'$ is redundant in P'. Let B is the set of reverse image of attribute a', included in P, under mapping h_A, i.e., $B= h^{-1}{}_A(a')\cap P$.

Since $a'\in P'$ is redundant in P', for any $x',y'\in U'$ in U', if $y'\in S_{A'-\{a'\}}(x')$ then $y' \in S_{a'}(x')$.

Because B is not redundant in P, $SIM(P) \neq SIM(P-B)$. But for $P-B\subseteq P$, $SIM(P) \subset SIM(P-B)$. Thus, there exist $x,y\in U$ such that $(x,y) \in SIM(P-B)$, $(x,y) \notin SIM(P)$, i.e., $y \in S_{P-B}(x)$, $y \notin S_B(x)$. It follows that there exists $b \in B$ such that $* \neq f(y,b) \neq f(x,b) \neq *$. h_D is one to one corresponding mapping and $h_D(*)=*$, so $* \neq h_D(f(y,b)) \neq h_D(f(x,b)) \neq *$. That is to say, $* \neq f(h_O(y),a')) \neq f(h_O(x),a')) \neq *$, i.e., $h_O(y) \notin S_{a'}(h_O(x))$. However, from $y \in S_{P-B}(x)$ we have $h_O(y) \in h_O(S_{P-B}(x))= S_{P'-\{a'\}}(x)$, $h_O(S_B(x))= S_{a'}(h_O(x))$. This contradicts to that $a'\in P'$ is redundant in P' and for any $x',y'\in U'$ in U', if $y'\in S_{P'-\{a'\}}(x')$ then $y' \in S_{a'}(x')$. This contradiction shows that there is no redundant attribute in $h_A(P)$.

Theorem 8. Let S be an IIS, $h=(h_O,h_A,h_D)$ be an endohomomorphism from S to S' satisfying: h_O and h_A be surjective, h_D be one to one corresponding mapping and $h_D(*)=*$, and for any $x\in U$ and any $a\in A$, tolerance class $S_a(x)$ generated by x under attribute a satisfies $|f(S_a(x),a)-\{*\}|=1$. Then $core(S')=h_A(core(S))$.

Proof. Let $a\in A$ be indispensable in S. We are to prove $h_A(a)$ is indispensable in S'. Denote $a'=h_A(a)$. Because a is indispensable in S, there exist $x,y\in U$ such that $y\notin S_a(x)$, $y\in S_{A-\{a\}}(x)$. However, for $S_a(x)\subseteq \cup S_{A-\{a\}}(y)$ $(y\in S_a(x))$ and if $S_a(x)= \cup S_{A-\{a\}}(y)$ $(y \in S_a(x))$ then a is dispensable in S, therefore, it only follows that $S_a(x)\subset \cup S_{A-\{a\}}(y)(y\in S_a(x))$. Because $t=A-h^{-1}{}_A(a') \subseteq A-\{a\}$, thus $S_{A-\{a\}}(x) \subseteq S_t(x)$. But $y\in S_{A-\{a\}}(x)$, so $y\in S_t(x)$. Furthermore, $f(y,t)=f(x,t) \vee f(y,t)=* \vee f(x,t)=*$. Because $y\notin S_a(x)$, $* \neq f(x,a) \neq f(y,a) \neq *$.

Since h_D is a one-to-one corresponding mapping and

$h_D(*)=*$, it infers that $* \neq f(x,a) \neq f(y,a) \neq *$, $* \neq h_D(f(x,a)) \neq h_D(f(y,a)) \neq *$, $* \neq f(h_O(x), h_A(a)) \neq f(h_O(y), h_A(a)) \neq *$.

$$f(x,t) = f(y,t) \vee f(x,t)=* \vee f(y,t) =*$$
$$\Rightarrow h_D(f(x,t))=h_D(f(y,t)) \vee h_D(f(x,t))=* \vee h_D(f(y,t))=*$$
$$\Rightarrow f'(h_O(x), A'-\{a'\})=f'(h_O(y), A'-\{a'\}) \vee f'(h_O(x), A'-\{a'\}) =* \vee f'(h_O(y), A'-\{a'\})=*.$$

That is $h_O(y) \in S_t(h_O(x))$, $h_O(y) \notin S_{a'}(h_O(x))$. Thus, $a'=h_O(a)$ is indispensable in S'.

Because attribute in A' can be divided into two types, one is the image of indispensable attribute of A, another is the image of redundant attributes of A. According to Lemma 3, under the condition of the theorem, the image of redundant attribute is also redundant, so the indispensable attributes of S' are all images of indispensable attributes in S and therefore $core(S')=h_A(core(S)$.

IV. Conclusions

The present paper studies on some invariant properties in IISs. It finds out preserving invariance conditions respectively for tolerance classes in tolerance relation, attribute reduction, indispensable attribute and dispensable attribute under homomorphism in IISs. It also discusses the condition of endohomomorphism. It obtains several meaningful results. It lays a certain theoretic foundations for engaging researching IIS modeling under rough set theory.

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