

Generalized Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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Abstract—The aim of this paper is to introduce the concepts of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras with operators, generalized fuzzy subalgebras, generalized fuzzy ideals, generalized fuzzy quotient algebras.

I. INTRODUCTION

THE fuzzy set is a generalization of the classical set. After the introduction of fuzzy sets, there have been a number of generalizations of this fundamental concept, especially, in the branches of mathematics. Imai and Iseki [1], [2] introduced the concept of BCK/BCI-algebras, which are generalizations of BCK-algebras. In 1980, Ming et al. [13] introduced the neighbourhood structure of a fuzzy point.

In 1991, Xi [3] applied the fuzzy sets to BCK-algebras; fuzzy BCK/BCI-algebras have been widely researched. Meng et al. [4] introduced the concept of fuzzy implicative ideals of BCK-algebras in 1997. Liu and Meng [6], [7] introduced the notions of fuzzy positive implicative ideals and fuzzy implicative ideals of BCI-algebras. Zheng [5] defined operators in BCK-algebras and raised the concept of BCI-algebras with operators and gave some isomorphism theorems of it. In 2002, Liu [8] introduced the concept of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [9] introduced the (α, β) -fuzzy ideals of BCK/BCI-algebras and established the characterizations of $(\in, \in \vee q)$ -fuzzy ideals. In 2006, Liao et al. [11] introduced the $(\in, \in \vee q_{(\lambda, \mu)})$ -fuzzy normal subgroup.

In 2009, Jun et al. [12] introduced the concept of $(\in, \in \vee q)$ -ideals of BCI-algebras. In 2011, Liu and Sun [10] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigate some basic properties. In 2017, Hu et al. [14] introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with operators.

In this paper, we give the notions of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, in particular, discuss the basic properties of generalized fuzzy BCI-algebras

with operators and give several results about it.

II. PRELIMINARIES

We recall some definitions and propositions which may be needed.

An algebra $\langle X; *, 0 \rangle$ of type $(2, 0)$ is called a BCI-algebra, if for all $x, y, z \in X$, it satisfies the following conditions:

1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $(x * (x * y)) * y = 0$;
3. $x * x = 0$;
4. $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define $x * y = 0$ if and only if $x \leq y$, then the above conditions can be written as:

1. $(x * y) * (x * z) \leq z * y$;
2. $x * (x * y) \leq y$;
3. $x \leq x$;
4. $x \leq y$ and $y \leq x$ imply $x = y$.

If a BCI-algebra satisfies $0 * x = 0$, then it is called a BCK-algebra.

Definition 1. [5] $\langle X; *, 0 \rangle$ is a BCI-algebra, M is a non-empty set, if there exists a mapping $(m, x) \rightarrow mx$ from $M \times X$ to X which satisfies $m(x * y) = (mx) * (my)$, $\forall x, y \in X, m \in M$. then M is called a left operator of X , X is called a BCI-algebra with left operator M , or M -BCI-algebra for short.

Definition 2. [13] $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy subset A of X of the form

$$A(y) = \begin{cases} t (\neq 0), & y = x, \\ 0, & y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t , and is denoted by x_t .

Proposition 1. [10] Let $\langle X; *, 0 \rangle$ be a BCI-algebra, if A is a fuzzy generalized ideal of it, and $x * y \leq z$, then

$$A(x) \vee \lambda \geq A(y) \wedge A(z) \wedge \mu, x, y, z \in X.$$

Definition 3. [5] Let $\langle X; *, 0 \rangle$ and $\langle \bar{X}; *, 0 \rangle$ be two M -BCI-algebras, if f is a homomorphism from $\langle X; *, 0 \rangle$ to $\langle \bar{X}; *, 0 \rangle$, and $f(mx) = mf(x)$ for all $x \in X, m \in M$, then f is called a homomorphism with operators.

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Definition 4. If $\langle X; *, 0 \rangle$ is a BCI-algebra, A is a non-empty subset of X , and $mx \in A$ for all $x \in A, m \in M$, then $\langle A; *, 0 \rangle$ is called an M -subalgebra of $\langle X; *, 0 \rangle$.

In the following parts, X always means a M -BCI-algebra unless otherwise specified.

III. GENERALIZED FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS

Definition 5. $\langle X; *, 0 \rangle$ is a BCI-algebra, let A be a fuzzy subset of X , $t, \lambda, \mu \in [0, 1]$ and $\lambda < \mu$. if $A(x) \geq t$, we denoted $x_t \in A$; if $t > \lambda$ and $A(x) + t > 2\mu$, we denoted $x_t q_{(\lambda, \mu)} A$; if $x_t \in A$ or $x_t q_{(\lambda, \mu)} A$, we denoted $x_t \in \vee q_{(\lambda, \mu)} A$.

Definition 6. $\langle X; *, 0 \rangle$ is an M -BCI-algebra, let A be a fuzzy subset of X , if it satisfies:

1. $x_t \in A$ and $y_r \in A$ implies $(x * y)_{t \wedge r} \in \vee q_{(\lambda, \mu)} A, \forall x, y \in X, t, r \in [0, 1]$;
2. $x_t \in A$ implies $(mx)_t \in \vee q_{(\lambda, \mu)} A, \forall x \in X, t \in [0, 1]$.

Then A is called an M - $(\in, \in \vee q_{(\lambda, \mu)})$ -fuzzy subalgebra or a generalized M -fuzzy subalgebra for short.

Proposition 2. A fuzzy subset A of X is a generalized M -fuzzy subalgebra of X if and only if it satisfies:

1. $A(x * y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X$;
2. $A(mx) \vee \lambda \geq A(x) \wedge \mu, \forall x \in X$.

Proof. Suppose that A is a generalized M -fuzzy subalgebra of X . We first verify that

$$A(x * y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X.$$

Suppose there exists $x_0, y_0 \in X$ such that $A(x_0 * y_0) \vee \lambda < A(x_0) \wedge A(y_0) \wedge \mu$, choose t such that $A(x_0 * y_0) \vee \lambda < t < A(x_0) \wedge A(y_0) \wedge \mu$, then $A(x_0 * y_0) < t, \lambda < t < \mu, A(x_0) > t$ and $A(y_0) > t$, therefore $(x_0)_t \in A, (y_0)_t \in A$. Based on Definition 6, $(x_0 * y_0)_t \in \vee q_{(\lambda, \mu)} A$, but we have $A(x_0 * y_0) < t$, therefore $A(x_0 * y_0) + t \leq t + t < 2\mu$, this is a contradiction, therefore we have $A(x * y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \forall x, y \in X$. We shall now show that $A(mx) \vee \lambda \geq A(x) \wedge \mu, \forall x \in X$.

Suppose there exists $x_0 \in X$ such that $A(mx_0) \vee \lambda < A(x_0) \wedge \mu$, choose t such that $A(mx_0) \vee \lambda < t < A(x_0) \wedge \mu$, then $A(x_0) > t$, therefore $(x_0)_t \in A$. Based on Definition 6, $(mx_0)_t \in \vee q_{(\lambda, \mu)} A$, but we have $A(mx_0) < t$, therefore $A(mx_0) + t \leq t + t < 2\mu$, this is a contradiction, therefore we have $A(mx) \vee \lambda \geq A(x) \wedge \mu, \forall x \in X$. Conversely, assume that A satisfies condition 1, 2.

1). If $(x)_{t_1} \in A, (y)_{t_2} \in A, \forall x, y \in X, t_1, t_2 \in [0, 1]$, then $A(x) \geq t_1, A(y) \geq t_2$, choose $T = t_1 \wedge t_2$, since A is a generalized M -fuzzy subalgebra of X , we have

$$A(x * y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu > t_1 \wedge t_2 \wedge \mu,$$

if $T \leq \mu$, then $A(x * y) \geq T$, so we have $(x * y)_T \in A$, if $T > \mu$, then $A(x * y) \geq \mu$, thus $A(x * y) + T \geq \mu + T > 2\mu$, then $(x * y)_T q_{(\lambda, \mu)} A$, therefore we have $(x * y)_T \in \vee q_{(\lambda, \mu)} A$.

2). If $x_t \in A, \forall x \in X, t \in [0, 1]$, then $A(x) \geq t$, since A is a generalized M -fuzzy subalgebra of X , then $A(mx) \vee \lambda \geq A(x) \wedge \mu$, if $t \leq \mu$, then $A(mx) \vee \lambda \geq t$, since $\lambda < t$, so we have $A(mx) \geq t$, hence $(mx)_t \in A$, if $t > \mu$, then $A(mx) \vee \lambda \geq \mu$, since $\lambda < \mu$, so we have $A(mx) \geq \mu$, hence $A(mx) + t \geq \mu + t > 2\mu$, thus $(mx)_t q_{(\lambda, \mu)} A$, therefore we have $(mx)_t \in \vee q_{(\lambda, \mu)} A$. So A is a generalized M -fuzzy subalgebra of X .

Example 1. If A is a generalized M -fuzzy subalgebra of X , then X_A is a generalized M -fuzzy subalgebra of X , define X_A by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x * y \in A$, thus

$$X_A(x * y) \vee \lambda = 1 \geq X_A(x) \wedge X_A(y) \wedge \mu,$$

if there exists at least one which does not belong to A between x and y , for example $x \notin A$, thus

$$X_A(x * y) \vee \lambda \geq 0 = X_A(x) \wedge X_A(y) \wedge \mu.$$

(2) For all $x \in X, m \in M$, if $x \in A$, then $mx \in A$, therefore

$$X_A(mx) \vee \lambda = 1 \geq X_A(x) \wedge \mu,$$

if $x \notin A$, then $X_A(mx) \vee \lambda \geq 0 = X_A(x) \wedge \mu$, therefore X_A is a generalized M -fuzzy subalgebra of X .

Proposition 3. A is a generalized M -fuzzy subalgebra of X if only if A_t is a M -subalgebra of X , where A_t is a non-empty set, define X_{A_t} by $A_t = \{x | x \in X, A(x) \geq t\}, \forall t \in (\lambda, \mu]$.

Proof. Suppose A is a generalized M -fuzzy subalgebra of X , A_t is a non-empty set, $t \in (\lambda, \mu]$, then we have $A(x * y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$. If $x \in A_t, y \in A_t$, then $A(x) \geq t, A(y) \geq t$, thus $A(x * y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu \geq t$, thus we have $x * y \in A_t$.

For all $x \in X, m \in M$, if A is a generalized M -fuzzy subalgebra of X , hence $A(mx) \vee \lambda \geq A(x) \wedge \mu \geq t$, thus $mx \in A_t$, therefore A_t is an M -subalgebra of X . Conversely, suppose A_t is an M -subalgebra of X , then we have $x * y \in A_t$. Let

$A(x)=t$, then $A(x*y)\vee\lambda\geq t=A(x)\geq A(x)\wedge A(y)\wedge\mu$. For all $x\in X, m\in M$, if A_t is an M -subalgebra of X , then we have $A(mx)\vee\lambda\geq t=A(x)\geq A(x)\wedge\mu$, therefore A is a generalized M -fuzzy subalgebra of X .

Proposition 4. Suppose X, Y are M -BCI-algebras, f is a mapping from X to Y , if A is a generalized M -fuzzy subalgebra of the Y , then $f^{-1}(A)$ is a generalized M -fuzzy subalgebra of X .

Proof. Let $y\in Y$, suppose f is an epimorphism, then there exists x in X , we have $y=f(x)$. If A is a generalized M -fuzzy subalgebra of Y , then we have

$$A(x*y)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu; A(mx)\vee\lambda\geq A(x)\wedge\mu.$$

For all $x, y\in X, m\in M$, we have

$$(1) f^{-1}(A)(x*y)\vee\lambda = A(f(x)*f(y))\vee\lambda \geq A(f(x))\wedge A(f(y))\wedge\mu = f^{-1}(A)(x)\wedge f^{-1}(A)(y)\wedge\mu;$$

$$(2) f^{-1}(A)(mx)\vee\lambda = A(f(mx))\vee\lambda = A(mf(x))\vee\lambda \geq A(f(x))\wedge\mu = f^{-1}(A)(x)\wedge\mu.$$

Therefore $f^{-1}(A)$ is a generalized M -fuzzy subalgebra of X .

IV. GENERALIZED FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

Definition 7. $\langle X; *; 0 \rangle$ is an M -BCI-algebra, let A be a fuzzy subset of X , if it satisfies:

- $x_t \in A$ implies $0_t \in \vee q_{(\lambda, \mu)} A, \forall x \in X, t \in [0, 1]$;
- $(x*y)_t \in A$ and $y_r \in A$ implies $x_{t\wedge r} \in \vee q_{(\lambda, \mu)} A, \forall x, y \in X, t, r \in [0, 1]$;
- $x_t \in A$ implies $(mx)_t \in \vee q_{(\lambda, \mu)} A, \forall x \in X, t \in [0, 1]$.

Then A is called a M - $(\in, \in \vee q_{(\lambda, \mu)})$ -fuzzy subalgebra or a generalized M -fuzzy subalgebra for short.

Proposition 5. A fuzzy subset A of X is a generalized M -fuzzy ideal of X if and only if it satisfies:

- $A(0)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X$;
- $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu, \forall x, y\in X$;
- $A(mx)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X$.

Proof. Suppose that A is a generalized M -fuzzy ideal of X . We first verify that $A(0)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X$. Suppose there exists $x_0\in X$ such that $A(0)\vee\lambda < A(x_0)\wedge\mu$, choose t such that $A(0)\vee\lambda < t < A(x_0)\wedge\mu$, then $A(x_0) > t$ and $\lambda < t < \mu$, therefore $(x_0)_t \in A$. Based on Definition 7, $0_t \in \vee q_{(\lambda, \mu)} A$, but we have $A(0) < t \leq \mu$, therefore $A(0)+t \leq t+t \leq 2\mu$, this is a contradiction, therefore we have $A(0)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X$. We shall now show that

$$A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu, \forall x, y\in X.$$

Suppose there exists $x_0, y_0\in X$ such that $A(x_0)\vee\lambda < A(x_0*y_0)\wedge A(y_0)\wedge\mu$, choose t such that $A(x_0)\vee\lambda < t < A(x_0*y_0)\wedge A(y_0)\wedge\mu$, then $A(x_0) < t, \lambda < t < \mu, A(x_0*y_0) > t$ and $A(y_0) > t$, therefore $(x_0*y_0)_t \in A, (y_0)_t \in A$. Based on Definition 7, $(x_0)_t \in \vee q_{(\lambda, \mu)} A$, but we have $A(x_0) < t$, therefore $A(x_0)+t \leq t+t \leq 2\mu$, this is a contradiction, therefore we have $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu, \forall x, y\in X$.

Next, we shall show that $A(mx)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X$. Suppose there exists $x_0\in X$ such that $A(mx_0)\vee\lambda < A(x_0)\wedge\mu$, choose t such that $A(mx_0)\vee\lambda < t < A(x_0)\wedge\mu$, then $A(x_0) > t$, therefore $(x_0)_t \in A$. Based on Definition 7, $(mx_0)_t \in \vee q_{(\lambda, \mu)} A$, but we have $A(mx_0) < t$, therefore $A(mx_0)+t \leq t+t < 2\mu$, this is a contradiction, therefore we have $A(mx)\vee\lambda\geq A(x)\wedge\mu, \forall x\in X$. Conversely, assume that A satisfies condition 1, 2, 3.

1). If $x_t \in A, \forall x\in X, t\in(0, 1]$, then $A(x)\geq t$, since A is a generalized M -fuzzy ideal of X , we have $A(0)\vee\lambda\geq A(x)\wedge\mu\geq t\wedge\mu$, if $t\leq\mu$, then $A(0)\geq t$, so we have $0_t \in A$, if $t > \mu$, then $A(0)\geq\mu$, thus $A(0)+t\geq t+\mu > 2\mu$, then $0_t \in \vee q_{(\lambda, \mu)} A$, therefore we have $0_t \in \vee q_{(\lambda, \mu)} A$.

2). If $(x*y)_{t_1} \in A, y_{t_2} \in A, \forall x, y\in X, t_1, t_2 \in (\lambda, 1]$, then $A(x*y)\geq t_1, A(y)\geq t_2$, choose $T = t_1 \wedge t_2$, since A is a generalized M -fuzzy ideal of X . We have $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu > t_1 \wedge t_2 \wedge\mu$, if $T\leq\mu$, then $A(x)\geq T$, so we have $x_T \in A$, if $T > \mu$, then $A(x)\geq\mu$, thus $A(x)+T\geq\mu+T > 2\mu$, then $x_T \in \vee q_{(\lambda, \mu)} A$, therefore we have $x_T \in \vee q_{(\lambda, \mu)} A$.

3). If $x_t \in A, \forall x\in X, t\in(\lambda, 1]$, then $A(x)\geq t$, since A is a generalized M -fuzzy ideal of X . We have $A(mx)\vee\lambda\geq A(x)\wedge\mu$, if $t\leq\mu$, then $A(mx)\vee\lambda\geq t$, since $\lambda < t$, so we have $A(mx)\geq t$, hence $(mx)_t \in A$, if $t > \mu$, then $A(mx)\vee\lambda\geq\mu$, since $\lambda < \mu$, so we have $A(mx)\geq\mu$, hence $A(x)+t\geq\mu+t > 2\mu$, thus $(mx)_t \in \vee q_{(\lambda, \mu)} A$, therefore we have $(mx)_t \in \vee q_{(\lambda, \mu)} A$. So, A is a generalized M -fuzzy ideal of X .

Example 2. If A is a generalized M -fuzzy ideal of X , then X_A is a generalized M -fuzzy ideal of X , define X_A by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Proof. (1) For all $x, y\in X$, if $x, y\in A$, then $x*y\in A$, thus

$$X_A(0) \vee \lambda = 1 \geq X_A(x) \wedge \mu,$$

$$X_A(x) \vee \lambda = 1 \geq X_A(x * y) \wedge X_A(y) \wedge \mu,$$

if there exists at least one which does not belong to A between x and y , for example $x \notin A$, thus

$$X_A(0) \vee \lambda = 1 \geq X_A(x) \wedge \mu,$$

$$X_A(x) \vee \lambda \geq X_A(x * y) \wedge X_A(y) \wedge \mu = 0;$$

(2) For all $x \in X, m \in M$, if $x \in A$, then $mx \in A$, thus $X_A(mx) \vee \lambda = 1 \geq X_A(x) \wedge \mu$. If $x \notin A$, then

$X_A(mx) \vee \lambda \geq 0 = X_A(x) \wedge \mu$, therefore X_A is a generalized M -fuzzy ideal of X .

Proposition 6. A is a generalized M -fuzzy ideal of X if only if A_t is an M -ideal of X , where A_t is non-empty set, define A_t by $A_t = \{x | x \in X, A(x) \geq t\}, \forall t \in (\lambda, \mu]$.

Proof. Suppose A is a generalized M -fuzzy ideal of X , A_t is non-empty set, $t \in (\lambda, \mu]$, then we have $A(0) \vee \lambda \geq A(x) \wedge \mu \geq t$, thus $0 \in A_t$. If $x * y \in A_t, y \in A_t$, then $A(x * y) \geq t, A(y) \geq t$, thus $A(x) \vee \lambda \geq A(x * y) \wedge A(y) \wedge \mu \geq t$, thus we have $x \in A_t$. For all $x \in X, m \in M$, if A is a generalized M -fuzzy ideal of X , hence $A(mx) \vee \lambda \geq A(x) \wedge \mu \geq t$, thus $mx \in A_t$, therefore A_t is an M -ideal of X . Conversely, suppose A_t is an M -ideal of X , then we have $0 \in A_t, A(0) \geq t$. Let $A(x) = t$, thus $x \in A_t$, we have $A(0) \vee \lambda \geq t = A(x) \wedge \mu$, suppose there is no $A(x) \vee \lambda \geq A(x * y) \wedge A(y) \wedge \mu$, then there exist $x_0, y_0 \in X$, we have $A(x_0) \vee \lambda < A(x_0 * y_0) \wedge A(y_0) \wedge \mu$, let $t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge \mu$, then $A(x_0) \vee \lambda < t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge \mu$, if $x_0 * y_0 \in A_{t_0}, y_0 \in A_{t_0}$, then we have $x_0 \in A_{t_0}$, then $A(x_0) \geq t_0$, which is inconsistent with $A(x_0) \vee \lambda < t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge \mu$, then we have $A(x) \vee \lambda \geq A(x * y) \wedge A(y) \wedge \mu$. For all $x \in X, m \in M$, if A_t is an M -ideal of X , then we have $A(mx) \vee \lambda \geq t = t \wedge \mu = A(x) \wedge \mu$, therefore A is a generalized M -fuzzy ideal of X .

Proposition 7. Suppose X, Y are M -BCI-algebras, f is a mapping from X to Y , A is a generalized M -fuzzy ideal of Y , then $f^{-1}(A)$ is a generalized M -fuzzy ideal of X .

Proof. Let $y \in Y$, suppose f is an epimorphism, then there exists $x \in X$, we have $y = f(x)$. If A is a generalized M -fuzzy ideal of Y , then we have

$$A(0) \vee \lambda \geq A(y) \wedge \mu; A(x) \vee \lambda \geq A(x * y) \wedge A(y) \wedge \mu;$$

$$A(mx) \vee \lambda \geq A(x) \wedge \mu.$$

For all $x, y \in X, m \in M$, we have

$$(1) f^{-1}(A)(0) \vee \lambda = A(f(0)) \vee \lambda = A(0) \vee \lambda$$

$$\geq A(f(x)) \wedge \mu = f^{-1}(A)(x) \wedge \mu;$$

$$(2) f^{-1}(A)(x) \vee \lambda = A(f(x)) \vee \lambda \geq A(f(x) * f(y)) \wedge A(f(y)) \wedge \mu$$

$$= A(f(x * y)) \wedge A(f(y)) \wedge \mu = f^{-1}(A)(x * y) \wedge f^{-1}(A)(y) \wedge \mu;$$

$$(3) f^{-1}(A)(mx) \vee \lambda = A(f(mx)) \vee \lambda = A(mf(x)) \vee \lambda$$

$$\geq A(f(x)) \wedge \mu = f^{-1}(A)(x) \wedge \mu.$$

Therefore $f^{-1}(A)$ is a generalized M -fuzzy ideal of X .

V. GENERALIZED FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS

Definition 8. Let A be an M - $(\epsilon, \in \vee q_{(\lambda, \mu)})$ -fuzzy ideal of X , for all $a \in X$, fuzzy set A_a on X defined as: $A_a : X \rightarrow [0, 1]$ $A_a(x) = A(a * x) \wedge A(x * a) \wedge \mu, \forall x \in X$. Denote $X/A = \{A_a : a \in X\}; A(x) \geq \lambda$.

Proposition 8. Let $A_a, A_b \in X/A$, then $A_a = A_b$ if only if $A(a * b) \wedge A(b * a) \wedge \mu = A(0) \wedge \mu$.

Proof. Let $A_a = A_b$, then we have $A_a(b) = A_b(b)$, thus $A(a * b) \wedge A(b * a) \wedge \mu = A(b * b) \wedge A(b * b) \wedge \mu = A(0) \wedge \mu$, that is $A(a * b) \wedge A(b * a) \wedge \mu = A(0) \wedge \mu$. Conversely, suppose that $A(a * b) \wedge A(b * a) \wedge \mu = A(0) \wedge \mu$. For all $x \in X$, since $(a * x) * (b * x) \leq a * b, (x * a) * (x * b) \leq b * a$. It follows from Proposition 1 that

$$A(a * x) = A(a * x) \vee \lambda \geq A(b * x) \wedge A(a * b) \wedge \mu,$$

$$A(x * a) = A(x * a) \vee \lambda \geq A(x * b) \wedge A(b * a) \wedge \mu.$$

Hence

$$A_a(x) = A(a * x) \wedge A(x * a) \wedge \mu$$

$$\geq A(b * x) \wedge A(x * b) \wedge A(a * b) \wedge A(b * a) \wedge \mu$$

$$= A(b * x) \wedge A(x * b) \wedge A(0) \wedge \mu = A(b * x) \wedge A(x * b) \wedge \mu$$

$$= A_b(x),$$

that is $A_a \geq A_b$. Similarly, for all $x \in X$, since

$$(b * x) * A(a * x) \leq b * a, (x * b) * A(x * a) \leq a * b.$$

It follows from Proposition 1 that

$$A(b * x) = A(b * x) \vee \lambda \geq A(a * x) \wedge A(b * a) \wedge \mu,$$

$$A(x * b) = A(x * b) \vee \lambda \geq A(x * a) \wedge A(a * b) \wedge \mu.$$

Hence

$$A_b(x) = A(b * x) \wedge A(x * b) \wedge \mu$$

$$\geq A(a * x) \wedge A(x * a) \wedge A(b * a) \wedge A(a * b) \wedge \mu$$

$$= A(a * x) \wedge A(x * a) \wedge A(0) \wedge \mu$$

$$= A(a * x) \wedge A(x * a) \wedge \mu$$

$$= A_a(x),$$

that is $A_b \geq A_a$. Therefore, $A_a = A_b$. We complete the proof.

Proposition 9. Let $A_a = A_{a'}$, $A_b = A_{b'}$, then $A_{a*b} = A_{a'*b'}$.

Proof. Since

$$\begin{aligned} ((a*b)*(a'*b'))*(a*a') &= ((a*b)*(a*a'))*(a'*b') \\ &\leq (a'*b)*(a'*b') \leq b'*b, \\ ((a'*b)*(a*b))*(b*b') &= ((a'*b)*(b*b'))*(a*b) \\ &\leq (a'*b)*(a*b) \leq a'*a. \end{aligned}$$

Hence

$$\begin{aligned} A((a*b)*(a'*b')) &= A((a*b)*(a'*b')) \vee \lambda \\ &\geq A(a*a') \wedge A(b'*b) \wedge \mu, \\ A((a'*b)*(a*b)) &= A((a'*b)*(a*b)) \vee \lambda \\ &\geq A(b*b') \wedge A(a'*a) \wedge \mu. \end{aligned}$$

Therefore

$$\begin{aligned} &A((a*b)*(a'*b')) \wedge A((a'*b)*(a*b)) \wedge \mu \\ &= A(a*a') \wedge A(a'*a) \wedge \mu \wedge A(b*b') \wedge A(b'*b) \wedge \mu \wedge \mu \\ &= A(0) \wedge \mu, \end{aligned}$$

it follows from Proposition 8 that $A_{a*b} = A_{a'*b'}$, we completed the proof. Let A be a generalized M -fuzzy ideal of X , the operation "*" of R/A is defined as follows: $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$. By Proposition 8, the above operation is reasonable.

Proposition 10. Let A be a generalized M -fuzzy ideal of X , then $R/A = \{R/A; *, A_0\}$ is an M -BCI-algebra.

Proof. For all $A_x, A_y, A_z \in R/A$,

$$\begin{aligned} ((A_x * A_y) * (A_x * A_z)) * (A_z * A_y) &= A_{((x*y)*(x*z))*(z*y)} = A_0; \\ (A_x * (A_x * A_y)) * A_y &= A_{(x*(x*y))*y} = A_0; \\ A_x * A_x &= A_{x*x} = A_0; \end{aligned}$$

if $A_x * A_y = A_0, A_y * A_x = A_0$, then $A_{x*y} = A_0, A_{y*x} = A_0$, it follows from Proposition 8 that $A(x*y) = A(0), A(y*x) = A(0)$, hence $A(x*y) \wedge A(y*x) \wedge \mu = A(0) \wedge \mu$, then $A_x = A_y$. Therefore $R/A = \{R/A; *, A_0\}$ is a BCI-algebra. For all $A_x \in R/A, m \in M$, we define $mA_x = A_{mx}$. Firstly, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_y$, then we verify $mA_x = mA_y$, that is to verify $A_{mx} = A_{my}$. We have

$$A(mx * my) \wedge \mu = A(m(x * y)) \wedge \mu \geq A(x * y) \wedge \mu = A(0) \wedge \mu,$$

$$A(my * mx) \wedge \mu = A(m(y * x)) \wedge \mu \geq A(y * x) \wedge \mu = A(0) \wedge \mu,$$

so we have $A(mx * my) \wedge A(my * mx) \wedge \mu = A(0) \wedge \mu$, that is,

$A_{mx} = A_{my}$. In addition, for all $m \in M, A_x, A_y \in R/A$,

$$\begin{aligned} m(A_x * A_y) &= mA_{x*y} = A_{m(x*y)} = A_{(mx)*(my)} \\ &= A_{mx} * A_{my} = mA_x * mA_y. \end{aligned}$$

Therefore $R/A = \{R/A; *, A_0\}$ is an M -BCI-algebra.

Definition 9. Let μ be a generalized M -fuzzy subalgebra of X , and A be a generalized M -fuzzy ideal of X , we define a fuzzy set of X/A as follows:

$$\begin{aligned} \mu/A : X/A &\rightarrow [0, 1], \\ \mu/A(A_i) \vee \lambda &= \sup_{A_i=A_j} \mu(x) \wedge \mu, \forall A_i \in X/A. \end{aligned}$$

Proposition 11. μ/A is a generalized M -fuzzy subalgebra of X/A .

Proof. For all $A_x, A_y \in X/A$, we have

$$\begin{aligned} \mu/A(A_x * A_y) \vee \lambda &= \mu/A(A_{x*y}) \vee \lambda = \sup_{A_z=A_{x*y}} \mu(z) \wedge \mu \\ &\geq \sup_{A_i=A_x, A_j=A_y} \mu(s * t) \wedge \mu \geq \sup_{A_i=A_x, A_j=A_y} \mu(s) \wedge \mu(t) \wedge \mu \\ &= \sup_{A_i=A_x} \mu(s) \wedge \sup_{A_j=A_y} \mu(t) \wedge \mu = \mu/A(A_x) \wedge \mu/A(A_y) \wedge \mu. \end{aligned}$$

For all $m \in M, A_x \in R/A$, we have

$$\begin{aligned} \mu/A(A_{mx}) \vee \lambda &= \sup_{A_{mz}=A_{mx}} \mu(mz) \wedge \mu \\ &\geq \sup_{A_i=A_x} \mu(z) \wedge \mu = \mu/A(A_x) \wedge \mu. \end{aligned}$$

Therefore μ/A is a generalized M -fuzzy subalgebra of X/A .

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