

($\in, \in \vee q$)-Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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Abstract—The aim of this paper is to introduce the concepts of ($\in, \in \vee q$)-fuzzy subalgebras, ($\in, \in \vee q$)-fuzzy ideals and ($\in, \in \vee q$)-fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras with operators, ($\in, \in \vee q$)-fuzzy subalgebras, ($\in, \in \vee q$)-fuzzy ideals, ($\in, \in \vee q$)-fuzzy quotient algebras.

I. INTRODUCTION

THE fuzzy set is a generalization of the classical set and was used afterwards by several authors such as Imai [1], Iseki [2] and Xi [3], in various branches of mathematics. Particularly, in the area of fuzzy topology, after the introduction of fuzzy sets by Zadeh [15], much research has been carried out: the concept of fuzzy subalgebras and fuzzy ideals of BCK-algebras, and their some properties.

BCK-algebras and BCI-algebras are two important classes of logical algebras, which were introduced by Imai and Iseki [1], [2]. In 1991, Xi [3] applied the fuzzy sets to BCK-algebras and discussed some properties about fuzzy subalgebras and fuzzy ideals. From then on, fuzzy BCK/BCI-algebras have been widely investigated by some researchers. Jun et al. [4], [5] raised the notions of fuzzy positive implicative ideals and fuzzy commutative ideals of BCK-algebras. Ming and Ming [12] introduced the neighbourhood structure of a fuzzy point in 1980; Jun et al. [6] introduced the concept of ($\in, \in \vee q$)-ideals of BCI-algebras. In 1993, Zheng [7] defined operators in BCK-algebras and introduced the concept of BCI-algebras with operators and gave some isomorphism theorems of it. Then, Liu [9] introduced the university property of direct products of BCI-algebras. In 2002, Liu [8] introduced the notion of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [10] introduced the (α, β) -fuzzy ideals of BCK/BCI-algebras and established the characterizations of ($\in, \in \vee q$)-fuzzy ideals. Next, Pan [13] introduced fuzzy ideals of sub-algebra and fuzzy H-ideals of sub-algebra. In 2011, Liu and Sun [11] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigated some basic properties. In 2017, we [14] also introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with

operators.

In this paper, we introduce the concepts of ($\in, \in \vee q$)-fuzzy subalgebras, ($\in, \in \vee q$)-fuzzy ideals and ($\in, \in \vee q$)-fuzzy quotient algebras of BCI-algebras with operators. Moreover, the basic properties were discussed and several results have been obtained.

II. PRELIMINARIES

Some definitions and propositions were recalled which may be needed.

An algebra $\langle X; *, 0 \rangle$ of type (2,0) is called a BCI-algebra, if for all $x, y, z \in X$, it satisfies:

- (1) $((x * y) * (x * z)) * (z * y) = 0$;
- (2) $(x * (x * y)) * y = 0$;
- (3) $x * x = 0$;
- (4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define $x * y = 0$ if and only if $x \leq y$, and the above conditions can be written as:

- 1. $(x * y) * (x * z) \leq z * y$;
- 2. $x * (x * y) \leq y$;
- 3. $x \leq x$;
- 4. $x \leq y$ and $y \leq x$ imply $x = y$.

A BCI-algebra is called a BCK-algebra if it satisfies $0 * x = 0$.

Definition 1. [5] $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy subset A of X is called a fuzzy ideal of X if it satisfies:

- (1) $A(0) \geq A(x), \forall x \in X$,
- (2) $A(x) \geq A(x * y) \wedge A(y), \forall x, y \in X$.

Definition 2. [4] $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy subset A of X is called a fuzzy subalgebra of X if it satisfies:

$$A(x * y) \geq A(x) * A(y), \forall x, y \in X.$$

Definition 3. [12] $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy subset A of X of the form

$$A(y) = \begin{cases} t(\neq 0), & y = x, \\ 0, & y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t , and is

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denoted by x_t .

Definition 4. [12] If x_t is a fuzzy point, it is said to belong to (resp. be quasi-coincident with) a fuzzy subset A , written as $x_t \in A$ (resp. $x_t qA$) if $A(x) \geq t$ (resp. $A(x) + t > 1$). If $x_t \in A$ or $x_t qA$, then we write $x_t \in \vee qA$. The symbol $\overline{\in \vee q}$ (resp. $\overline{\in}$ or \overline{q}) means $\in \vee q$ (resp. \in or q) does not hold.

Definition 5. [10] $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy set A of X is called an $(\in, \in \vee q)$ -fuzzy ideal of X if for all $t, r \in (0, 1]$ and $x, y \in X$, it satisfies:

1. $x_t \in A \Rightarrow 0_r \in \vee qA$,
2. $(x * y)_t \in A$ and $y_r \in A \Rightarrow x_{t \wedge r} \in \vee qA$.

Definition 6. [10] A fuzzy set A is an $(\in, \in \vee q)$ -fuzzy ideal of X if and only if it satisfies:

- (1) $A(0) \geq A(x) \wedge 0.5, \forall x \in X$,
- (2) $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X$.

Definition 7. [7] $\langle X; *, 0 \rangle$ is a BCI-algebra, M is a non-empty set, if there exists a mapping $(m, x) \rightarrow mx$ from $M \times X$ to X which satisfies

$$m(x * y) = (mx) * (my), \forall x, y \in X, m \in M,$$

then M is called a left operator of X , X is called BCI-algebra with left operator M , or M -BCI-algebra for short.

Proposition 1. [6] Let $\langle X; *, 0 \rangle$ be a BCI-algebra, if A is an $(\in, \in \vee q)$ -fuzzy ideal of it, and $x * y \leq z$, then

$$A(x) \geq A(y) \wedge A(z) \wedge 0.5, \forall x, y, z \in X.$$

Definition 8. [13] Let A and B be fuzzy sets of set X , then the direct product $A \times B$ of A and B is a fuzzy subset of $X \times X$, define $A \times B$ by

$$A \times B(x, y) = A(x) \wedge B(y), \forall x, y \in X.$$

Definition 9. [7] Let $\langle X; *, 0 \rangle$ and $\langle \bar{X}; *, 0 \rangle$ be two M -BCI-algebras, if for all $x \in X, m \in M$, $f(mx) = mf(x)$, and f is a homomorphism from $\langle X; *, 0 \rangle$ to $\langle \bar{X}; *, 0 \rangle$, then f is called a homomorphism with operators.

Definition 10. [13] $\langle X; *, 0 \rangle$ is an M -BCI-algebra, let B be a fuzzy set of X , and A be a fuzzy relation of B , if it satisfies:

$$A_B(x, y) = B(x) \wedge B(y), \forall x, y \in X,$$

then A is called a strong fuzzy relation of B .

Definition 11. [14] If $\langle X; *, 0 \rangle$ is an M -BCI-algebra, A is a

non-empty subset of X , and $mx \in A$ for all $x \in A, m \in M$, then $\langle A; *, 0 \rangle$ is called a M -subalgebra of $\langle X; *, 0 \rangle$.

In this paper, X always means a M -BCI-algebra unless otherwise specified.

III. $(\in, \in \vee q)$ -FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS

Definition 12. $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy set A of X is called a $M - (\in, \in \vee q)$ -fuzzy subalgebra of X if for all $t, r \in (0, 1]$ and $x, y \in X$, it satisfies:

1. $x_t \in A$ and $y_r \in A \Rightarrow (x * y)_{t \wedge r} \in \vee qA$,
2. $x_t \in A \Rightarrow (mx)_t \in \vee qA$.

Proposition 2. $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy set A of X is an $M - (\in, \in \vee q)$ -fuzzy subalgebra of X if and only if it satisfies:

- (1) $A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, \forall x, y \in X$,
- (2) $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$.

Proof. Suppose that A is an $M - (\in, \in \vee q)$ -fuzzy subalgebra of X . (1) Let $x, y \in X$, suppose that $A(x) \wedge A(y) < 0.5$, then $A(x * y) \geq A(x) \wedge A(y)$, if not, then we have $A(x * y) < t < A(x) \wedge A(y), \exists t \in (0, 0.5)$; it follows that $x_t \in A$ and $y_t \in A$, but $(x * y)_{t \wedge t} = (x * y)_{t \wedge t} \overline{\in \vee qA}$, which is a contradiction, then whenever $A(x) \wedge A(y) < 0.5$. We have $A(x * y) \geq A(x) \wedge A(y)$. If $A(x) \wedge A(y) \geq 0.5$, then $(x)_{0.5} \in A$ and $y_{0.5} \in A$, which implies that $(x * y)_{0.5} = (x * y)_{0.5 \wedge 0.5} \in \vee qA$, therefore $A(x * y) \geq 0.5$, because if $A(x * y) < 0.5$, then $A(x * y) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, hence

$$A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, \forall x, y \in X.$$

(2) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(mx) < A(x)$, then we have $A(mx) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $(mx)_t \overline{\in A}$, since $A(mx) + t < 1$, we have $(mx)_t \overline{\in qA}$; it follows that $(mx)_t \overline{\in \vee qA}$, which is a contradiction, hence $A(mx) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, thus $(mx)_{0.5} \in \vee qA$, hence $A(mx) \geq 0.5$, otherwise $A(mx) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, consequently, $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$. Conversely, assume that A satisfies condition (1), (2).

(1) Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in A$ and $y_{t_2} \in A$, then $A(x) \geq t_1$ and $A(y) \geq t_2$. Suppose that $A(x * y) < t_1 \wedge t_2$, if $A(x) \wedge A(y) < 0.5$, then $A(x * y) \geq A(x) \wedge A(y) \wedge 0.5 = A(x) \wedge A(y) \geq t_1 \wedge t_2$, this is a contradiction, so we have $A(x) \wedge A(y) \geq 0.5$, it follows that

$$A(x * y) + t_1 \wedge t_2 > 2A(x * y) \geq 2(A(x) \wedge A(y) \wedge 0.5) = 1,$$

so that $(x * y)_{t_1 \wedge t_2} \in \vee q A$.

(2) Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in A$, then we have $A(x) \geq t$. Suppose that $A(mx) < t$, if $A(x) < 0.5$, then $A(mx) \geq A(x) \wedge 0.5 = A(x) \geq t$, this is a contradiction, hence we know that $A(x) \geq 0.5$, and we have

$$A(mx) + t > 2A(mx) \geq 2(A(x) \wedge 0.5) = 1,$$

then $(mx)_t \in \vee q A$. Consequently, A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra.

Example 1. If A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X , then X_A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X , define X_A by

$$X_A : X \rightarrow [0,1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x * y \in A$, then we have

$$X_A(x * y) = 1 \geq X_A(x) \wedge X_A(y) \wedge 0.5,$$

if there exists at least one which does not belong to A between x and y , for example $x \notin A$, thus

$$X_A(x * y) \geq 0 = X_A(x) \wedge X_A(y) \wedge 0.5.$$

(2) For all $x \in X, m \in M$, if $x \in A$, then $mx \in A$, therefore

$$X_A(mx) = 1 \geq X_A(x) \wedge 0.5,$$

if $x \notin A$, then $X_A(mx) \geq 0 = X_A(x) \wedge 0.5$, therefore X_A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X .

Proposition 3. A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X if and only if A_t is an M -subalgebra of X , where A_t is a non-empty set, define X_A by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 0.5].$$

Proof. Suppose A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X , A_t is a non-empty set, $t \in [0, 0.5]$, then we have $A(x * y) \geq A(x) \wedge A(y) \wedge 0.5$. If $x \in A_t, y \in A_r$, then $A(x) \geq t, A(y) \geq r$, thus

$$A(x * y) \geq A(x) \wedge A(y) \wedge 0.5 \geq t,$$

then we have $x * y \in A_t$. If A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X , then $A(mx) \geq A(x) \wedge 0.5 \geq t, \forall x \in X, m \in M$, then we have $mx \in A_t$. Therefore A_t is an M -subalgebra of X . Conversely, suppose A_t is an M -subalgebra of X , then we have $x * y \in A_t$. Let $A(x) = t$, then

$$A(x * y) \geq t = A(x) \geq A(x) \wedge A(y) \wedge 0.5.$$

If A_t is an M -subalgebra of X , then we have

$$A(mx) \geq t = A(x) \geq A(x) \wedge 0.5, \forall x \in X, m \in M,$$

therefore A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X .

Proposition 4. Suppose X, Y are M -BCI-algebras, f is a mapping from X to Y , if A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of the Y , then $f^{-1}(A)$ is a $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X .

Proof. Let $y \in Y$, suppose f is an epimorphism, and we have $y = f(x), \exists x \in X$. If A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of Y , then we have

$$A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, A(mx) \geq A(x) \wedge 0.5.$$

For all $x, y \in X, m \in M$, we have

$$\begin{aligned} (1) f^{-1}(A)(x * y) &= A(f(x) * f(y)) \geq A(f(x)) \wedge A(f(y)) \wedge 0.5 \\ &= f^{-1}(A)(x) \wedge f^{-1}(A)(y) \wedge 0.5; \\ (2) f^{-1}(A)(mx) &= A(f(mx)) = A(mf(x)) \\ &\geq A(f(x)) \wedge 0.5 = f^{-1}(A)(x) \wedge 0.5. \end{aligned}$$

Then $f^{-1}(A)$ is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy subalgebra of X .

IV. $(\epsilon, \epsilon \vee q)$ -FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

Definition 13. $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy set A of X is called an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X if for all $t, r \in (0,1]$ and $x, y \in X$, it satisfies:

1. $x_t \in A \Rightarrow 0_r \in \vee q A$,
2. $(x * y)_r \in A$ and $y_r \in A \Rightarrow x_{t \wedge r} \in \vee q A$,
3. $x_r \in A \Rightarrow (mx)_r \in \vee q A$.

Proposition 5. [13] $\langle X; *, 0 \rangle$ is a BCI-algebra, a fuzzy set A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X if and only if it satisfies:

- (1) $A(0) \geq A(x) \wedge 0.5, \forall x \in X$,
- (2) $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X$,
- (3) $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$.

Proof. Suppose that A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X .

(1) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(0) < A(x)$, then we have $A(0) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $0_{t \in \bar{A}}$, since $A(0) + t < 1$, we have $0_{t \in \bar{q}A}$, it follows that $0_{t \in \bar{q}A}$, which is a contradiction, then $A(0) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, then we have $0_{0.5} \in \vee qA$, hence $A(0) \geq 0.5$, otherwise, $A(0) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, consequently,

$$A(0) \geq A(x) \wedge 0.5, \forall x \in X.$$

(2) Let $x, y \in X$ and suppose that $A(x * y) \wedge A(y) < 0.5$, then $A(x) \geq A(x * y) \wedge A(y)$, if not, then we have $A(x) < t < A(x * y) \wedge A(y), \exists t \in (0, 0.5)$, it follows that $(x * y)_t \in A$ and $y_t \in A$, but $x_{t \wedge t} = x_t \in \bar{q}A$, which is a contradiction, hence whenever $A(x * y) \wedge A(y) < 0.5$, we have $A(x) \geq A(x * y) \wedge A(y)$. If $A(x * y) \wedge A(y) \geq 0.5$, then $(x * y)_{0.5} \in A$ and $y_{0.5} \in A$, which implies that $x_{0.5} = y_{0.5} \in \vee qA$, therefore $A(x) \geq 0.5$, because if $A(x) < 0.5$, then $A(x) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, then

$$A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X.$$

(3) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(mx) < A(x)$, then we have $A(mx) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $(mx)_{t \in \bar{A}}$, since $A(mx) + t < 1$, we have $(mx)_{t \in \bar{q}A}$, it follows that $(mx)_{t \in \bar{q}A}$, which is a contradiction, then $A(mx) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, thus $(mx)_{0.5} \in \vee qA$, hence $A(mx) \geq 0.5$, otherwise $A(mx) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, consequently, $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$. Conversely, suppose that A satisfies (1), (2), (3) of the Proposition 5, then we have

(1) Let $x \in X$ and $t \in (0, 1]$ be such that $x_t \in A$, then we have $A(x) > t$, suppose that $A(0) < t$, if $A(x) < 0.5$, then $A(0) \geq A(x) \wedge 0.5 = A(x) \geq t$, which is a contradiction, then we know that $A(x) \geq 0.5$, and we have $A(0) + t > 2A(0) \geq 2(A(x) \wedge 0.5) = 1$, thus $0_t \in \vee qA$.

(2) Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $(x * y)_{t_1} \in A$ and $y_{t_2} \in A$, then $A(x * y) \geq t_1$ and $A(y) \geq t_2$, suppose that $A(x) < t_1 \wedge t_2$, if $A(x * y) \wedge A(y) < 0.5$, then

$$A(x) \geq A(x * y) \wedge A(y) \wedge 0.5 = A(x * y) \wedge A(y) \geq t_1 \wedge t_2,$$

This is a contradiction, so we have $A(x * y) \wedge A(y) \geq 0.5$, it

follows that

$$A(x) + t_1 \wedge t_2 > 2A(x) \geq 2(A(x * y) \wedge A(y) \wedge 0.5) = 1,$$

so that $x_{t_1 \wedge t_2} \in \vee qA$.

(3) Let $x \in X$ and $t \in (0, 1]$ be such that $x_t \in A$, then $A(x) \geq t$, suppose that $A(mx) < t$, if $A(x) < 0.5$, then $A(mx) \geq A(x) \wedge 0.5 = A(x) \geq t$, which is a contradiction, then we know that $A(x) \geq 0.5$, and we have $A(mx) + t > 2A(mx) \geq 2(A(x) \wedge 0.5) = 1$, thus $(mx)_{t \in \bar{q}A}$. Consequently, A is an $M - (\in, \in \vee q)$ -fuzzy ideal.

Example 2. If A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X , then X_A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X , define X_A by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x * y \in A$, thus

$$\begin{aligned} X_A(0) &= 1 \geq X_A(x) \wedge 0.5, \\ X_A(x) &= 1 \geq X_A(x * y) \wedge X_A(y) \wedge 0.5, \end{aligned}$$

if there exists at least one between x and y which does not belong to A , for example $x \notin A$, thus

$$\begin{aligned} X_A(0) &= 1 \geq X_A(x) \wedge 0.5, \\ X_A(x) &\geq X_A(x * y) \wedge X_A(y) \wedge 0.5 = 0, \end{aligned}$$

therefore X_A is a $(\in, \in \vee q)$ -fuzzy ideal of X .

(2) For all $x \in X, m \in M$, if $x \in A$, then $mx \in A$, therefore $X_A(mx) = 1 \geq X_A(x) \wedge 0.5$. If $x \notin A$, then $X_A(mx) \geq 0 = X_A(x) \wedge 0.5$, therefore X_A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X .

Proposition 6. A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X if and only if A_t is an M -ideal of X , where A_t is non-empty set, define A_t by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 0.5].$$

Proof. Suppose A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X , A_t is non-empty set, $t \in [0, 0.5]$, then we have $A(0) \geq A(x) \wedge 0.5 \geq t$, then we have $0 \in A_t$. If $x * y \in A_t, y \in A_t$, then $A(x * y) \geq t, A(y) \geq t$, thus $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5 \geq t$, then we have $x \in A_t$. For all $x \in X, m \in M$, if A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X , hence $A(mx) \geq A(x) \wedge 0.5 \geq t$, thus $mx \in A_t$, therefore A_t is an M -ideal of X . Conversely, suppose A_t is

an M -ideal of X , then we have $0 \in A, A(0) \geq t$. Let $A(x) = t$, thus $x \in A$, we have $A(0) \geq t = A(x)$, suppose there is no $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5$, then there exist $x_0, y_0 \in X$, we have $A(x_0) < A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$, let $t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$, then $A(x_0) < t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$, if $x_0 * y_0 \in A_{t_0}, y_0 \in A_{t_0}$, then we have $x_0 \in A_{t_0}$, then $A(x_0) \geq t_0$, which is inconsistent with $A(x_0) < t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$, then we have $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5$. If A is an M -ideal of X , then we have $A(mx) \geq t = t \wedge 0.5 = A(x) \wedge 0.5, \forall x \in X, m \in M$, therefore A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X .

Proposition 7. Suppose X, Y are M -BCI-algebras, f is a mapping from X to Y , A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of Y , then $f^{-1}(A)$ is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X .

Proof. Let $y \in Y$, suppose f is an epimorphism, then we have $y = f(x), \exists x \in X$. If A is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of Y , then we have

$$\begin{aligned} A(0) &\geq A(x) \wedge 0.5, \\ A(x) &\geq A(x * y) \wedge A(y) \wedge 0.5, \\ A(mx) &\geq A(x) \wedge 0.5. \end{aligned}$$

For all $x, y \in X, m \in M$, we have

- (1) $f^{-1}(A)(0) = A(f(0)) = A(0) \geq A(f(x)) \wedge 0.5 = f^{-1}(A)(x) \wedge 0.5;$
- (2) $f^{-1}(A)(x) = A(f(x)) \geq A(f(x) * f(y)) \wedge A(f(y)) \wedge 0.5$
 $= A(f(x * y)) \wedge A(f(y)) \wedge 0.5 = f^{-1}(A)(x * y) \wedge f^{-1}(A)(y) \wedge 0.5;$
- (3) $f^{-1}(A)(mx) = A(f(mx)) = A(mf(x))$
 $\geq A(f(x)) \wedge 0.5 = f^{-1}(A)(x) \wedge 0.5.$

Therefore $f^{-1}(A)$ is an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X .

V. $(\epsilon, \epsilon \vee q)$ - FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS

Definition 14. Let A be an $M - (\epsilon, \epsilon \vee q)$ -fuzzy ideal of X , for all $a \in X$, fuzzy set A_a on X defined as: $A_a : X \rightarrow [0, 1]$

$$A_a(x) = A(a * x) \wedge A(x * a) \wedge 0.5, \forall x \in X.$$

Denote $X/A = \{A_a : a \in X\}$.

Proposition 8. Let $A_a, A_b \in X/A$, then $A_a = A_b$ if and only if $A(a * b) \wedge A(b * a) \wedge 0.5 = A(0) \wedge 0.5$.

Proof. Let $A_a = A_b$, then we have $A_a(b) = A_b(b)$, thus

$$A(a * b) \wedge A(b * a) \wedge 0.5 = A(b * b) \wedge A(b * b) \wedge 0.5 = A(0) \wedge 0.5,$$

that is $A(a * b) \wedge A(b * a) \wedge 0.5 = A(0) \wedge 0.5$. Conversely, suppose

that $A(a * b) \wedge A(b * a) \wedge 0.5 = A(0) \wedge 0.5$. For all $x \in X$, since

$$(a * x) * (b * x) \leq a * b, (x * a) * (x * b) \leq b * a.$$

It follows from Proposition 1 that

$$\begin{aligned} A(a * x) &\geq A(b * x) \wedge A(a * b) \wedge 0.5, \\ A(x * a) &\geq A(x * b) \wedge A(b * a) \wedge 0.5. \end{aligned}$$

Hence

$$\begin{aligned} A_a(x) &= A(a * x) \wedge A(x * a) \wedge 0.5 \\ &\geq A(b * x) \wedge A(x * b) \wedge A(a * b) \wedge A(b * a) \wedge 0.5 \\ &= A(b * x) \wedge A(x * b) \wedge A(0) \wedge 0.5 \\ &= A(b * x) \wedge A(x * b) \wedge 0.5 = A_b(x), \end{aligned}$$

that is $A_a \geq A_b$. Similarly, for all $x \in X$, since

$$(b * x) * A(a * x) \leq b * a, (x * b) * A(x * a) \leq a * b.$$

It follows from Proposition 1 that

$$\begin{aligned} A(b * x) &\geq A(a * x) \wedge A(b * a) \wedge 0.5, \\ A(x * b) &\geq A(x * a) \wedge A(a * b) \wedge 0.5. \end{aligned}$$

Hence

$$\begin{aligned} A_b(x) &= A(b * x) \wedge A(x * b) \wedge 0.5 \\ &\geq A(a * x) \wedge A(x * a) \wedge A(b * a) \wedge A(a * b) \wedge 0.5 \\ &= A(a * x) \wedge A(x * a) \wedge A(0) \wedge 0.5 \\ &= A(a * x) \wedge A(x * a) \wedge 0.5 = A_a(x), \end{aligned}$$

that is $A_b \geq A_a$. Therefore, $A_a = A_b$. We complete the proof.

Proposition 9. Let $A_a = A_{a'}, A_b = A_{b'}$, then $A_{a * b} = A_{a' * b'}$.

Proof. Since

$$\begin{aligned} ((a * b) * (a' * b')) * (a * a') &= ((a * b) * (a * a')) * (a' * b') \\ &\leq (a' * b) * (a' * b') \leq b' * b, \\ ((a' * b') * (a * b)) * (b * b') &= ((a' * b') * (b * b')) * (a * b) \\ &\leq (a' * b) * (a * b) \leq a' * a. \end{aligned}$$

Hence

$$\begin{aligned} A((a * b) * (a' * b')) &\geq A(a * a') \wedge A(b' * b) \wedge 0.5, \\ A((a' * b') * (a * b)) &\geq A(b * b') \wedge A(a' * a) \wedge 0.5. \end{aligned}$$

Therefore

$$\begin{aligned} A((a * b) * (a' * b')) &\wedge A((a' * b') * (a * b)) \wedge 0.5 \\ &= A(a * a') \wedge A(a' * a) \wedge 0.5 \wedge A(b * b') \wedge A(b' * b) \wedge 0.5 \wedge 0.5 \\ &= A(0) \wedge 0.5, \end{aligned}$$

it follows from Proposition 8. that $A_{a * b} = A_{a' * b'}$. We completed the proof.

Let A be an $M - (\in, \in \vee q)$ -fuzzy ideal of X . The operation "*" of R/A is defined as: $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$. By Proposition 8, the above operation is reasonable.

Proposition 10. A is an $M - (\in, \in \vee q)$ -fuzzy ideal of X , then $R/A = \{R/A; *, A_0\}$ is an M -BCI-algebra.

Proof. For all $A_x, A_y, A_z \in R/A$, we have

$$\begin{aligned} ((A_x * A_y) * (A_z * A_z)) * (A_z * A_y) &= A_{((x*y)*(z*z))*(z*y)} = A_0; \\ (A_x * (A_x * A_y)) * A_y &= A_{(x*(x*y))*y} = A_0; \\ A_x * A_x &= A_{x*x} = A_0; \end{aligned}$$

if $A_x * A_y = A_0, A_y * A_x = A_0$, then $A_{x*y} = A_0, A_{y*x} = A_0$, it follows from Proposition 8 that $A(x*y) = A(0), A(y*x) = A(0)$, hence $A(x*y) \wedge A(y*x) \wedge 0.5 = A(0) \wedge 0.5$, then we have $A_x = A_y$. Therefore $R/A = \{R/A; *, A_0\}$ is a BCI-algebra. For all $A_x \in R/A, m \in M$, we define $mA_x = A_{mx}$. Firstly, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_y$, then we verify $mA_x = mA_y$, that is to verify $A_{mx} = A_{my}$. We have

$$\begin{aligned} A(mx*my) \wedge 0.5 &= A(m(x*y)) \wedge 0.5 \geq A(x*y) \wedge 0.5, \\ A(my*mx) \wedge 0.5 &= A(m(y*x)) \wedge 0.5 \geq A(y*x) \wedge 0.5, \end{aligned}$$

so we have

$$A(mx*my) \wedge A(my*mx) \wedge 0.5 \geq A(x*y) \wedge A(y*x) \wedge 0.5 = A(0) \wedge 0.5,$$

then $A(mx*my) \wedge A(my*mx) \wedge 0.5 = A(0) \wedge 0.5$, that is $A_{mx} = A_{my}$.

In addition, for all $m \in M, A_x, A_y \in R/A$, we have

$$\begin{aligned} m(A_x * A_y) &= mA_{x*y} = A_{m(x*y)} \\ &= A_{(mx)*(my)} = A_{mx} * A_{my} = mA_x * mA_y. \end{aligned}$$

Therefore $R/A = \{R/A; *, A_0\}$ is an M -BCI-algebra.

Definition 15. Let μ be an $M - (\in, \in \vee q)$ -fuzzy subalgebra of X , and A be an $M - (\in, \in \vee q)$ -fuzzy ideal of X , we define a fuzzy set of X/A as follows:

$$\mu/A: X/A \rightarrow [0,1], \quad \mu/A(A_i) = \sup_{A_i=A_i} \mu(x) \wedge 0.5, \forall A_i \in X/A.$$

Proposition 11. μ/A is an $M - (\in, \in \vee q)$ -fuzzy subalgebra of X/A .

Proof. For all $A_x, A_y \in X/A$, we have

$$\begin{aligned} \mu/A(A_x * A_y) &= \mu/A(A_{x*y}) = \sup_{A_z=A_{x*y}} \mu(z) \wedge 0.5 \\ &\geq \sup_{A_z=A_x, A_t=A_y} \mu(s*t) \wedge 0.5 \geq \sup_{A_z=A_x, A_t=A_y} \mu(s) \wedge \mu(t) \wedge 0.5 \\ &= \sup_{A_z=A_x, A_t=A_y} \mu(s) \wedge \sup_{A_t=A_y} \mu(t) \wedge 0.5 \\ &= \mu/A(A_x) \wedge \mu/A(A_y) \wedge 0.5. \end{aligned}$$

For all $m \in M, A_x \in R/A$, we have

$$\begin{aligned} \mu/A(A_{mx}) &= \sup_{A_{mc}=A_{mx}} \mu(mc) \wedge 0.5 \\ &\geq \sup_{A_z=A_x} \mu(z) \wedge 0.5 = \mu/A(A_x) \wedge 0.5. \end{aligned}$$

Therefore μ/A is an $M - (\in, \in \vee q)$ -fuzzy subalgebra of X/A .

VI. DIRECT PRODUCTS OF $(\in, \in \vee q)$ -FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

Proposition 12. Suppose A and B are $M - (\in, \in \vee q)$ -fuzzy ideals of X , then $A \times B$ is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$.

Proof. (1) Let $(x, y) \in X \times X$, then

$$\begin{aligned} A \times B(0,0) &= A(0) \wedge B(0) \geq A(x) \wedge 0.5 \wedge B(y) \wedge 0.5 \\ &= A(x) \wedge B(y) \wedge 0.5 = A \times B(x, y) \wedge 0.5, \end{aligned}$$

then $A \times B(0,0) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X$;

(2) For all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$\begin{aligned} A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5 \\ &= A \times B(x_1 * y_1, x_2 * y_2) \wedge A \times B(y_1, y_2) \wedge 0.5 \\ &= (A(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge A(y_1) \wedge B(y_2) \wedge 0.5 \\ &= (A(x_1 * y_1) \wedge A(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)) \wedge 0.5 \\ &\leq A(x_1) \wedge B(x_2) = A \times B(x_1, x_2), \end{aligned}$$

then for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$A \times B(x_1, x_2) \geq A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5;$$

(3) For all $(x, y) \in X \times X$, we have

$$\begin{aligned} A \times B(m(x, y)) &= A \times B(mx, my) = A(mx) \wedge B(my) \\ &\geq A(x) \wedge 0.5 \wedge B(y) \wedge 0.5 = A(x) \wedge B(y) \wedge 0.5 \\ &= A \times B(x, y) \wedge 0.5, \end{aligned}$$

then we have

$$A \times B(m(x, y)) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X.$$

Therefore $A \times B$ is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$.

Proposition 13. Suppose A and B are fuzzy sets of X , if $A \times B$ is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$, then A or B is an $M - (\in, \in \vee q)$ -fuzzy ideal of X .

Proof. Suppose A and B are $M - (\in, \in \vee q)$ -fuzzy ideals of X , then for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$\begin{aligned} A \times B(x_1, x_2) &\geq A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5 \\ &= A \times B((x_1 * y_1), (x_2 * y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5, \end{aligned}$$

if $x_1 = y_1 = 0$, then

$$A \times B(0, x_2) \geq A \times B(0, x_2 * y_2) \wedge A \times B(0, y_2) \wedge 0.5,$$

then we have

$$A \times B(0, x) = A(0) \wedge B(x) = B(x),$$

thus $B(x_2) \geq B(x_2 * y_2) \wedge B(y_2) \wedge 0.5$. If $A \times B$ is an $M - (\in, \in \vee q)$ -fuzzy ideal of X , then

$$A \times B(m(x, y)) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X,$$

let $x = 0$, then

$$\begin{aligned} A \times B(m(x, y)) &= A \times B(mx, my) = A(mx) \wedge B(my) = B(my) \\ &\geq A(0) \wedge B(y) \wedge 0.5 = A(0) \wedge B(y) \wedge 0.5 \\ &= B(y) \wedge 0.5, \end{aligned}$$

then we have

$$B(my) \geq B(y) \wedge 0.5, \forall y \in X, m \in M.$$

Therefore B is an $M - (\in, \in \vee q)$ -fuzzy ideal of X .

Proposition 14. If B is a fuzzy set, A is a strong fuzzy relation A_B of B , then B is an $M - (\in, \in \vee q)$ -fuzzy ideal of X if and only if A_B is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$.

Proof. If B is an $M - (\in, \in \vee q)$ -fuzzy ideals of X , then for all $(x, y) \in X \times X$, we have

$$\begin{aligned} A_B(0, 0) &= B(0) \wedge B(0) \geq B(x) \wedge 0.5 \wedge B(y) \wedge 0.5 \\ &= A_B(x, y) \wedge 0.5; \end{aligned}$$

for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$\begin{aligned} A_B(x_1, x_2) &= B(x_1) \wedge B(x_2) \\ &\geq (B(x_1 * y_1) \wedge B(y_1) \wedge 0.5) \wedge (B(x_2 * y_2) \wedge B(y_2) \wedge 0.5) \\ &= (B(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge (B(y_1) \wedge B(y_2)) \wedge 0.5 \\ &= A_B(x_1 * y_1, x_2 * y_2) \wedge A_B(y_1, y_2) \wedge 0.5 \\ &= A_B((x_1, x_2) * (y_1, y_2)) \wedge A_B(y_1, y_2) \wedge 0.5; \end{aligned}$$

for all $(x, y) \in X \times X$, we have

$$\begin{aligned} A_B(m(x, y)) &= A_B(mx, my) = B(mx) \wedge B(my) \\ &\geq B(x) \wedge 0.5 \wedge B(y) \wedge 0.5 = A_B(x, y) \wedge 0.5. \end{aligned}$$

Therefore A_B is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$.

Conversely, suppose A_B is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$, for all $(x_1, x_2) \in X \times X$, we have

$$B(0) \wedge B(0) = A_B(0, 0) \geq A_B(x, x) \wedge 0.5 = B(x) \wedge B(x) \wedge 0.5,$$

for all $(x_1, x_2), (y_1, y_2) \in X \times X$, we have

$$\begin{aligned} B(x_1) \wedge B(x_2) &= A_B(x_1, x_2) \\ &\geq A_B((x_1, x_2) * (y_1, y_2)) \wedge A_B(y_1, y_2) \wedge 0.5 \\ &= A_B(x_1 * y_1, x_2 * y_2) \wedge A_B(y_1, y_2) \wedge 0.5 \\ &= (B(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge (B(y_1) \wedge B(y_2)) \wedge 0.5 \\ &= (B(x_1 * y_1) \wedge B(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)) \wedge 0.5, \end{aligned}$$

let $x_2 = y_2 = 0$, then

$$B(x_1) \wedge B(0) \geq (B(x_1 * y_1) \wedge B(y_1)) \wedge B(0) \wedge 0.5,$$

if A_B is an $M - (\in, \in \vee q)$ -fuzzy ideal of $X \times X$, then

$$A_B(m(x, y)) \geq A_B(x, y), \forall x, y \in X \times X, m \in M,$$

We have

$$B(mx) \wedge B(my) = A_B(mx, my) \geq A_B(x, y) \wedge 0.5 = B(x) \wedge B(y) \wedge 0.5,$$

if $x = 0$, then

$$B(0) \wedge B(my) = A_B(0, my) \geq A_B(0, y) \wedge 0.5 = B(0) \wedge B(y) \wedge 0.5,$$

namely, $B(my) \geq B(y) \wedge 0.5$. Therefore B is an $M - (\in, \in \vee q)$ -fuzzy ideal of X .

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