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Mathematical Modeling and Analysis of Forced Vibrations in Micro-Scale Microstretch Thermoelastic Simply Supported Beam

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***Abstract*—The present paper deals with the flexural vibrations of homogeneous, isotropic, generalized micropolar microstretch thermoelastic thin Euler-Bernoulli beam resonators, due to Exponential time varying load. Both the axial ends of the beam are assumed to be at simply supported conditions. The governing equations have been solved analytically by using Laplace transforms technique twice with respect to time and space variables respectively. The inversion of Laplace transform in time domain has been performed by using the calculus of residues to obtain deflection.The analytical results have been numerically analyzed with the help of MATLAB software for magnesium like material. The graphical representations and interpretations have been discussed for Deflection of beam under Simply Supported boundary condition and for distinct considered values of time and space as well. The obtained results are easy to implement for engineering analysis and designs of resonators (sensors), modulators, actuators.

***Keywords*—Microstretch, deflection, exponential load, Laplace transforms, Residue theorem, simply supported.

I. Introduction

ERINGEN [1] first, who proposed the theory of micropolar continua and discussed in detail the behaviour of materials possessing microstructure. This theory has been extended to include thermal effects by Eringen [2] and Nowacki [3], who also developed theory of thermo-microstretch elastic solids. Eringen [4] studied micro structural expansions and contractions due to thermal effects in microstretch elastic solids.

Micro-electro-mechanical system (MEMS) have been widely used as resonators for sensing, electrical filtering and communication application. Their light weight, small size, low-energy consumption, large deflection capacity and stability etc. made micro-electro-mechanical system components even more commercialization attractive. Micro beams have been widely studied by the MEMS community due to their application. For MEMS designers, it is important to understand the mechanical properties of flexible micro-components in order to predict the amount of deflection from an applied load and also to prevent cracking, improve performance and to increase the lifetime of MEMS devices.

Zener [5], many decays ago, derived an analytical solution which relate the energy dissipation and the material properties of a thin beam structure by assuming some mathematical and physical simplifications. Lifshitz and Roukes [6] improved upon Zener's work by developing exact and equivalent close-form expression for thermoelastic damping using the quasi 1D theory as their basic. Sun et al. [7] presented 2D analysis of frequency shifts by considering heat conduction along the beam thickness and beam span by taking sinusoidal temperature gradients across the thickness of the beam. Prabhakar and Vengallatore [8] developed a two dimensional theory for thermoelastic damping in Euler-Bernoulli beam resonators. Guo et al. [9] evaluated the effect of geometry on thermoelastic damping in micro beam resonators by using finite element method. Grover D [10]-[12] studied transverse vibrations in viscothermoelastic beam with fixed and linearly varying thickness and circular plate, respectively. Sharma and Grover D. [13] studied the thermoelastic damping and frequency shift in MEMS/NEMS transverse vibrations of a homogenous isotropic, thermoelastic thin beam with voids, based on Euler Bernoulli theory. Yanping and Yilong analyzed [14] the static deflections of micro-cantilever elastic beams under transverse loading by applying the neutral network method. Most of the previous researchers studied thermoelastic damping and frequency shifts instead of deflection variation during flexural vibrations.

The objective of this paper is to study deflection variation due to stretch forces and thermal variations in homogeneous isotropic, micropolar microstretch thermoelastic type micro beam resonator in the context of Lord and Shulman [15] model.

II. Governing Equations

Consider a homogeneous, isotropic, micropolar, microstretch thermally conductive media in Cartesian coordinate system $oxyz$ initially undeformed and at uniform temperature T_0. The basic governing equations of motion are given by

$$\sigma_{ji,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

where ρ is the density of medium, σ_{ij} are components of stress tensor, $\vec{u}(x, y, z, t) = (u_1, u_2, u_3)$ are components of displacement vector and t is time. The Constitutive equations and heat conduction equation for micropolar microstretch generalized thermoelasticity (LS model) media in the absence of body forces, stretch forces and heat sources are given by [4]

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$$(\lambda+2\mu+K)\nabla(\nabla\cdot\vec{u})-(\mu+K)\nabla\times\nabla\times\vec{u}+K\ \nabla\times\vec{\phi}+\lambda_0\nabla\phi^*-\beta_1\nabla T=\rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{2}$$

$$(\alpha+\beta+\gamma)\nabla(\nabla\cdot\vec{\phi})-\gamma\ \nabla\times(\nabla\times\vec{\phi})+K\ \nabla\times\vec{u}-2K\vec{\phi}=\rho j\frac{\partial^2\vec{\phi}}{\partial t^2} \tag{3}$$

$$(\alpha_0\nabla^2-\lambda_1)\phi^*-\lambda_0\nabla\cdot\vec{u}+\beta_2T=\frac{\rho j_0}{2}\frac{\partial^2\phi^*}{\partial t^2} \tag{4}$$

$$K^*\nabla^2T=\rho C_e(\frac{\partial T}{\partial t}+t_0\frac{\partial^2T}{\partial t^2})+\beta_1T_0(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2})\nabla\cdot\vec{u}+\beta_2T_0(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2})\phi^* \tag{5}$$

and the constitutive relations are given below:

$$\sigma_{ij}=(\lambda_0\phi^*+\lambda u_{k,k})\delta_{ij}+\mu(u_{i,j}+u_{j,i})+K(u_{j,i}-\varepsilon_{ijk}\phi_k)-\beta_1T\delta_{ij} \tag{6}$$

$$\begin{aligned}m_{ij}&=\alpha\phi_{k,k}\delta_{ij}+\beta\phi_{i,j}+\gamma\phi j,i+b_0\varepsilon_{lji}\ \phi^*_{,l},\\ \lambda^*_i&=\alpha_0\phi^*_{,i}+b_0\varepsilon_{ijl}\phi_{j,l}\end{aligned} \tag{7}$$

Here K^* is thermal conductivity, C_e is specific heat at constant strain, T is the temperature change, t_0 is thermal relaxation time, $\beta_1=(3\lambda+2\mu+K)\alpha_{t_1}$ and $\beta_2=(3\lambda+2\mu+K)\alpha_{t_2}$; α_{t_1}, α_{t_2} are coefficients of linear thermal expansion and $\lambda,\mu,\alpha,\beta,\gamma,K$ are material constants, $\alpha_0,b_0,\lambda_0,\lambda_1$ are microstretch constants, λ^*_i is the component of microstress, j is microinertia, j_0 is the microinertia of microelement, $\vec{\phi}=(\phi_1,\phi_2,\phi_3)$ is the microrotation vector, ϕ^* is the scalar microstretch, m_{ij} is component of couple stress tensor, δ_{ij} is Kronecker delta, ε_{ijk} is permutation tensor, the comma notation denotes spatial derivatives and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is Laplacian operator.

III. Formulation of the Problem

Now consider the flexural deflection of a homogeneous, isotropic, micropolar microstretch thermally conducting thin beam having dimensions of length $L(0\le x\le L)$, width $b(\frac{-b}{2}\le y\le\frac{b}{2})$ and thickness $h(\frac{-h}{2}\le z\le\frac{h}{2})$. Here, we define $x-axis$ along the axis of the beam, $y-axis$ and $z-axis$ correspond to the width and thickness direction of the beam, respectively. In equilibrium, the beam is unstrained, unstressed and also kept at uniform temperature T_0.

In this study, the usual Euler-Bernoulli assumption is adopted so that any plane cross section, initially perpendicular to axis of the beam, remains plane and perpendicular to the neutral surface during bending. Therefore, the displacements vector, microrotation vector $\vec{\phi}$, microstretch function ϕ^* and temperature distribution function T can be expressed as [20]

$$u_1=-z\frac{\partial w}{\partial x}\ ,\ u_2=0\ ,\ u_3(x,y,z,t)=w(x,t)\ ,\ \vec{\phi}=(0,\phi_2(x,z,t),0)\ ,\ \phi^*=\phi^*(x,z,t)\text{ and }T=T(x,z,t) \tag{8}$$

Now, substituting (8) into (2)-(5), the following system of equations are simplified as follows

$$(\lambda+2\mu+K)\left(-z\frac{\partial^3w}{\partial x^3}\right)-K\frac{\partial\phi_2}{\partial z}+\lambda_0\frac{\partial\phi^*}{\partial x}-\beta_1\frac{\partial T}{\partial x}=-\rho z\frac{\partial^3w}{\partial x\partial t^2} \tag{9}$$

$$-(\lambda-K)\frac{\partial^2w}{\partial x^2}+K\frac{\partial\phi_2}{\partial x}+\lambda_0\frac{\partial\phi^*}{\partial z}-\beta_1\frac{\partial T}{\partial z}=\rho\frac{\partial^2w}{\partial t^2} \tag{10}$$

$$\gamma\left(\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial z^2}\right)-2K\phi_2-2K\frac{\partial w}{\partial x}=\rho j\frac{\partial^2\phi_2}{\partial t^2} \tag{11}$$

$$\alpha_0\left(\frac{\partial^2\phi^*}{\partial x^2}+\frac{\partial^2\phi^*}{\partial z^2}\right)-\lambda_1\phi^*+\lambda_0z\frac{\partial^2w}{\partial x^2}+\beta_2T=\frac{1}{2}\rho j_0\frac{\partial^2\phi^*}{\partial t^2} \tag{12}$$

$$K^*\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial z^2}\right)=\rho C_e\left(\frac{\partial T}{\partial t}+t_0\frac{\partial^2T}{\partial t^2}\right)-\beta_1T_0\left(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\right)z\frac{\partial^2w}{\partial x^2}+\beta_2T_0\left(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\right)\phi^* \tag{13}$$

and the expression for stress tensor σ_{xx} from (6) with the help of (8) is given as follows

$$\sigma_{xx}=-(\lambda+2\mu+K)\left(z\frac{\partial^2w}{\partial x^2}\right)+\lambda_0\phi^*-\beta_1T \tag{14}$$

Also, the flexural moment of the cross-section of the beam is defined as follows [16]

$$M(x,t)=-\int_{\frac{-h}{2}}^{\frac{h}{2}}b\sigma_{xx}zdz=(\lambda+2\mu+K)Iw_{,xx}-\lambda_0M_{\phi^*}+\beta_1M_T \tag{15}$$

where $I=\frac{bh^3}{12}$ is the moment of inertia of the cross-section and $M_{\phi^*}=\int_{\frac{-h}{2}}^{\frac{h}{2}}b\phi^*zdz$ and $M_T=\int_{\frac{-h}{2}}^{\frac{h}{2}}bTzdz$, are defined as the moment of beam due to the presence of microstretch and thermal effects, respectively.

Now, the equation of transverse motion of the beam is given by

$$\frac{\partial^2M}{\partial x^2}+\rho A\frac{\partial^2w}{\partial t^2}=q_o(x,t) \tag{16}$$

where $A=bh$ is the area of cross section and $q_o(x,t)$ represents the load acting on the beam along the thickness direction.

Therefore, by substituting (15) into (16), the following equation of motion of the beam is obtained as

$$(\lambda+2\mu+K)I\frac{\partial^4 w}{\partial x^4}-\lambda_0\frac{\partial^2 M_{\phi^*}}{\partial x^2}+\beta_1\frac{\partial^2 M_T}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}=q_o(x,t) \tag{17}$$

To simplify the system, the following non-dimensional quantities are considered as

$$x'=\frac{x}{L}\ ,\ z'=\frac{z}{h}\ ,\ w'=\frac{w}{h}\ ,\ t'=\frac{c_1}{L}t\ ,\ t_0'=\frac{c_1}{L}t_0\ ,\ T'=\frac{T}{T_0},$$
$$\phi_2'=\frac{\rho c_1^2}{\beta_1 T_0}\phi_2\ ,\ \phi^{*'}=\frac{\rho c_1^2}{\beta_1 T_0}\phi^*\ ,\ M_{\phi^*}'=\frac{\rho c_1^2}{\beta_1 T_0 Ah}M_{\phi^*},$$
$$M_T'=\frac{M_T}{T_0 Ah}\ ,\ q_o'=\frac{A_R^2}{b\rho c_1^2}q_o \tag{18}$$

Using the above non-dimensional quantities, (9)-(13) and (17) can be rewritten as

$$-\frac{1}{A_R^2}\left(z\frac{\partial^3 w}{\partial x^3}\right)-p'\bar{\beta}A_R\frac{\partial\phi_2}{\partial z}+p_0\bar{\beta}\frac{\partial\phi^*}{\partial x}-\bar{\beta}\frac{\partial T}{\partial x}=-\frac{1}{A_R^2}\left(z\frac{\partial^3 w}{\partial x\partial t^2}\right) \tag{19}$$

$$-\frac{(1-2\delta^2)}{A_R}\frac{\partial^2 w}{\partial x^2}+p'\bar{\beta}\frac{\partial\phi_2}{\partial x}+p_0\bar{\beta}A_R\frac{\partial\phi^*}{\partial z}-\bar{\beta}A_R\frac{\partial T}{\partial z}=\frac{1}{A_R}\frac{\partial^2 w}{\partial t^2} \tag{20}$$

$$\left(\frac{\partial^2\phi_2}{\partial x^2}+A_R^2\frac{\partial^2\phi_2}{\partial z^2}\right)-2\delta_1^*\phi_2-\frac{2\delta_1^*}{\bar{\beta}A_R}\frac{\partial w}{\partial x}=\frac{1}{\delta_1^2}\frac{\partial^2\phi_2}{\partial t^2} \tag{21}$$

$$\left(\frac{\partial^2\phi^*}{\partial x^2}+A_R^2\frac{\partial^2\phi^*}{\partial z^2}\right)-p_1\delta_2^*\phi^*+\frac{p_0\delta_2^*}{\bar{\beta}A_R^2}\left(z\frac{\partial^2 w}{\partial x^2}\right)+\bar{v}\delta_2^*T=\frac{1}{\delta_2^2}\frac{\partial^2\phi^*}{\partial t^2} \tag{22}$$

$$\left(\frac{\partial^2 T}{\partial x^2}+A_R^2\frac{\partial^2 T}{\partial z^2}\right)-\bar{c}\left(\frac{\partial T}{\partial t}+t_0\frac{\partial^2 T}{\partial t^2}\right)+\frac{\bar{\varepsilon}_T}{\bar{\beta}A_R^2}\left(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\right)z\frac{\partial^2 w}{\partial x^2}-\bar{v}\bar{\varepsilon}_T\left(\frac{\partial\phi^*}{\partial t}+t_0\frac{\partial^2\phi^*}{\partial t^2}\right)=0 \tag{23}$$

$$\frac{1}{12A_R^2}\frac{\partial^4 w}{\partial x^4}-p_0\bar{\beta}\frac{\partial^2 M_{\phi^*}}{\partial x^2}+\bar{\beta}\frac{\partial^2 M_T}{\partial x^2}+\frac{\partial^2 w}{\partial t^2}=q_o \tag{24}$$

(After dropping the superscript for conveniences) where

$$A_R=\frac{L}{h}\ ,\ c_1^2=\frac{\lambda+2\mu+K}{\rho}\ ,\ c_2^2=\frac{\mu+K}{\rho}\ ,\ c_3^2=\frac{\gamma}{\rho j},$$
$$c_4^2=\frac{2\alpha_0}{\rho j_0}\ ,\ \delta_1^*=\frac{KL^2}{\gamma}\ ,\ \delta_2^*=\frac{\rho c_1^2 L^2}{\alpha_0},\delta^2=\frac{c_2^2}{c_1^2},\delta_1^2=\frac{c_3^2}{c_1^2},$$
$$\delta_2^2=\frac{c_4^2}{c_1^2}\ ,\ p'=\frac{K}{\rho c_1^2}\ ,\ p_0=\frac{\lambda_0}{\rho c_1^2}\ ,\ p_1=\frac{\lambda_1}{\rho c_1^2}\ ,\ \bar{\beta}=\frac{\beta_1 T_0}{\rho c_1^2},$$
$$\bar{c}=\frac{\rho C_e c_1 L}{K^*}\ ,\ \bar{\varepsilon}_T=\frac{\beta_1^2 T_0 L}{\rho c_1 K^*},\varepsilon_T=\frac{\bar{\varepsilon}_T}{\bar{c}}\ ,\ \bar{v}=\frac{\beta_2}{\beta_1} \tag{25}$$

IV. Initial and Boundary Conditions

In order to solve the problem, both the initial and boundary conditions have been considered. The initial conditions of the problem are as mentioned below:

$$w(x,0)=\frac{\partial w(x,0)}{\partial t}=0\ ,\ T(x,z,0)=\frac{\partial T(x,z,0)}{\partial t}=0$$
$$\phi^*(x,z,0)=\frac{\partial\phi^*(x,z,0)}{\partial t}=0\ ,\ \phi_2(x,z,0)=\frac{\partial\phi_2(x,z,0)}{\partial t}=0 \tag{26}$$

It is assumed that both the axial ends $x=0$ and $x=1$ of the beam are held at simply supported conditions. Therefore we have following set of boundary condition [16].

For Simply Supported Beam (SS):

$$w(0,t)=\frac{\partial^2 w(0,t)}{\partial x^2}=0 \text{ and } w(1,t)=\frac{\partial^2 w(1,t)}{\partial x^2}=0 \tag{27}$$

Also, As there is no flow of heat, micropolar and microstretch parameters across the upper and lower surfaces of the beam i.e.

$$\frac{\partial T}{\partial z}(x,z,t)=\frac{\partial\phi_2}{\partial z}(x,z,t)=\frac{\partial\phi^*}{\partial z}(x,z,t)=0$$
$$\text{at } z=\pm\frac{1}{2} \text{ and } t=0. \tag{28}$$

V. Laplace Transforms Technique for Time Domain

Now, the Laplace transforms with respect to time 't' is defined as under [18]:

$$W(x,s)=\int_0^\infty e^{-st}w(x,t)dt\ ,\ \Phi_2(x,z,s)=\int_0^\infty e^{-st}\phi_2(x,z,t)dt \tag{29}$$

$$\Phi^*(x,z,s)=\int_0^\infty e^{-st}\phi^*(x,z,t)dt\ ,\ \Theta(x,z,s)=\int_0^\infty e^{-st}T(x,z,t)dt \tag{30}$$

where $s=\frac{LS}{c_1}$ is a non-dimensional complex variable having positive real part. Here 'S' is a dimensional Laplace transform parameter with respect to time variable 't'. Applying Laplace transform (30) in system of equations (21)-(24) with initial conditions (26), the following system of equations are obtained

$$\left(\frac{\partial^2\Phi_2}{\partial x^2}+A_R^2\frac{\partial^2\Phi_2}{\partial z^2}\right)-\left(2\delta_1^*+\frac{s^2}{\delta_1^2}\right)\Phi_2-\frac{2\delta_1^*}{\bar{\beta}A_R}\frac{\partial W}{\partial x}=0 \tag{31}$$

$$\left(\frac{\partial^2\Phi^*}{\partial x^2}+A_R^2\frac{\partial^2\Phi^*}{\partial z^2}\right)-(p_1\delta_2^*+\frac{s^2}{\delta_2^2})\Phi^*+\frac{p_0\delta_2^*}{\bar{\beta}A_R^2}z\frac{\partial^2 W}{\partial x^2}+\bar{v}\delta_2^*\Theta=0 \tag{32}$$

$$\left(\frac{\partial^2\Theta}{\partial x^2}+A_R^2\frac{\partial^2\Theta}{\partial z^2}\right)-s\tau_0\bar{c}\Theta+\frac{\bar{\varepsilon}_T}{\bar{\beta}A_R^2}s\tau_0 z\frac{\partial^2 W}{\partial x^2}-\bar{v}\bar{\varepsilon}_T s\tau_0\Phi^*=0 \tag{33}$$

$$\frac{1}{12A_R^2}\frac{\partial^4 W}{\partial x^4}-p_0\bar{\beta}\frac{\partial^2 M_{\Phi^*}}{\partial x^2}+\bar{\beta}\frac{\partial^2 M_\Theta}{\partial x^2}+s^2W=q_o(x,s) \tag{34}$$

where

$$M_{\Phi^*}=\int_{\frac{-1}{2}}^{\frac{1}{2}}\Phi^*(x,z,s)zdz\ ,\ M_\Theta=\int_{\frac{-1}{2}}^{\frac{1}{2}}\Theta(x,z,s)zdz\ ,\ \tau_0=1+st_0 \tag{35}$$

Because there is no flow of heat, micropolar and microstretch parameters on the upper and lower surfaces of the beam and to evaluate approximate solution under some assumptions and uncoupled system($\bar{v}=0$) of above equations

(32), (33), then the trial solution of resulting system of equations are given as follows

$$\Theta(x,z,s) = \frac{\varepsilon_T}{\bar{\beta}A_R^2}\left(z - \frac{sinpz}{pcos(\frac{p}{2})}\right)\frac{d^2W}{dx^2}$$
$$\Phi^*(x,z,s) = \frac{p_0\delta_2^*}{\bar{\beta}A_R^2(p_1\delta_2^* + \frac{s^2}{\delta_2^2})}\left(z - \frac{sinqz}{qcos(\frac{q}{2})}\right)\frac{d^2W}{dx^2} \quad (36)$$

It is mentioned here that the values of p and q are still subjected to modifications.

Now, differentiating solution (36) w.r.t z twice and then substituting for $\frac{\partial^2\Phi^*}{\partial z^2}$ and $\frac{\partial^2\Theta}{\partial z^2}$ in (32), (33), one can obtain

$$\frac{d^2\Phi^*}{dx^2} = -\left[\frac{p_0\delta_2^*q^*}{\bar{\beta}\left(p_1\delta_2^*+\frac{s^2}{\delta_2^2}\right)}\frac{sinqz}{qcos(\frac{q}{2})} + \frac{\bar{v}\delta_2^*\varepsilon_T}{\bar{\beta}A_R^2}\left(z - \frac{sinpz}{pcos(\frac{p}{2})}\right)\right]\frac{d^2W}{dx^2}$$

$$\frac{d^2\Theta}{dx^2} = \left[-\frac{\varepsilon_T p^*}{\beta}\frac{sinpz}{pcos(\frac{p}{2})} + \frac{\bar{v}\bar{\varepsilon}_T s\tau_0}{\bar{\beta}A_R^2}\left(\frac{p_0\delta_2^*}{\left(p_1\delta_2^*+\frac{s^2}{\delta_2^2}\right)}\right)\left(z - \frac{sinqz}{qcos(\frac{q}{2})}\right)\right]\frac{d^2W}{dx^2} \quad (37)$$

where

$$p^* = p^2 + \frac{s\tau_0\bar{c}}{A_R^2}, q^* = q^2 + \frac{\left(p_1\delta_2^* + \frac{s^2}{\delta_2^2}\right)}{A_R^2}$$

Now from (35)

$$M_{\Phi^*} = \int_{\frac{-1}{2}}^{\frac{1}{2}}\Phi^* zdz \text{ and } M_\Theta = \int_{\frac{-1}{2}}^{\frac{1}{2}}\Theta zdz$$

with the help of (37),

$$\frac{d^2M_{\Phi^*}}{dx^2} = \int_{\frac{-1}{2}}^{\frac{1}{2}}\frac{d^2\Phi^*}{dx^2}zdz = \left[\frac{p_0\delta_2^*q^*}{12\bar{\beta}\left(p_1\delta_2^*+\frac{s^2}{\delta_2^2}\right)}g(q) - \frac{\bar{v}\delta_2^*\varepsilon_T}{12\bar{\beta}A_R^2}(1+f(p))\right]\frac{d^2W}{dx^2}$$

$$\frac{d^2M_\Theta}{dx^2} = \int_{\frac{-1}{2}}^{\frac{1}{2}}\frac{d^2\Theta}{dx^2}zdz = \left[\frac{\varepsilon_T p^*}{12\beta}f(p) + \frac{\bar{v}\bar{\varepsilon}_T s\tau_0}{12\bar{\beta}A_R^2}\left(\frac{p_0\delta_2^*}{(p_1\delta_2^*+\frac{s^2}{\delta_2^2})}\right)(1+g(q))\right]\frac{d^2W}{dx^2} \quad (38)$$

where

$$f(p) = \frac{24}{p^3}\left(\frac{p}{2} - tan\frac{p}{2}\right), g(q) = \frac{24}{q^3}\left(\frac{q}{2} - tan\frac{q}{2}\right) \quad (39)$$

Now, by substituting (38) into (34), the following equation is obtained as

$$\frac{1}{12A_R^2}\frac{d^4W}{dx^4} + \frac{1}{12A_R^2}\left[F(p,q) + p_0G(p,q)\right]\frac{d^2W}{dx^2} + s^2W = q_o(x,s) \quad (40)$$

where the coefficients $F(p,q)$ and $G(p,q)$ are given by

$$F(p,q) = \varepsilon_T p^* A_R^2 f(p) + \bar{v}\bar{\varepsilon}_T s\tau_0\left(\frac{p_0\delta_2^*}{(p_1\delta_2^* + \frac{s^2}{\delta_2^2})}\right)(1+g(q))$$
$$G(p,q) = \bar{v}\delta_2^*\varepsilon_T(1+f(p)) - \left(\frac{p_0\delta_2^*q^*A_R^2}{(p_1\delta_2^* + \frac{s^2}{\delta_2^2})}\right)g(q) \quad (41)$$

On the other hand, by substituting (36) in (35), the following equations are obtained as

$$\frac{d^2M_{\Phi^*}}{dx^2} = \left(\frac{p_0\delta_2^*}{12\bar{\beta}A_R^2(p_1\delta_2^* + \frac{s^2}{\delta_2^2})}\right)[1+g(q)]\frac{d^2}{dx^2}\left(\frac{d^2W}{dx^2}\right)$$
$$\frac{d^2M_\Theta}{dx^2} = \frac{\varepsilon_T}{12\bar{\beta}A_R^2}[1+f(p)]\frac{d^2}{dx^2}\left(\frac{d^2W}{dx^2}\right) \quad (42)$$

Now, on comparing both equations of (38) to both equations of (42), one can obtain

$$F(p,q) \simeq \varepsilon_T[1+f(p)]\frac{d^2}{dx^2}$$
$$G(p,q) \simeq -\left(\frac{p_0\delta_2^*}{(p_1\delta_2^* + \frac{s^2}{\delta_2^2})}\right)[1+g(q)]\frac{d^2}{dx^2} \quad (43)$$

Therefore, by substituting (43) in (40), one can obtain

$$D_s\frac{d^4W}{dx^4} + s^2W = q_o(x,s) \quad (44)$$

where

$$D_s = \frac{1}{12A_R^2}\left[1 + \varepsilon_T(1+f(p)) - \varepsilon_\phi(1+g(q))\right] \text{ and}$$
$$\varepsilon_T = \frac{\beta_1^2T_0}{\rho^2C_ec_1^2} \text{ and } \varepsilon_\phi = \left(\frac{p_0^2\delta_2^*}{p_1\delta_2^* + \frac{s^2}{\delta_2^2}}\right) \quad (45)$$

Here ε_T and ε_ϕ are the thermo-mechanical and elasto-stretch coupling constants of the beam respectively. As the thermal gradient and the gradient of microstretch element are negligible small along perpendicular to the thickness direction of the beam. Therefore, under these assumptions, we have

$$p^2 = \frac{-s\tau_0\bar{c}}{A_R^2}\left[1 - \frac{\bar{v}p_0\delta_2^*}{p_1\delta_2^* + \frac{s^2}{\delta_2^2}}\right]$$
$$q^2 = -\frac{(p_1\delta_2^* + \frac{s^2}{\delta_2^2})}{A_R^2}\left[1 + \frac{\bar{v}\varepsilon_T}{p_0}\right] \quad (46)$$

Thus the solution given by (36) now represent the solution of coupled equations (32), (33) with modified values of p and q given by (46).

VI. Solution of the Problem

Now consider Exponential decaying time varying load acting vertically downward along the thickness direction of the beam. Therefore

$$q_o(x,t) = -q^*(1 - exp(-\Omega t)) \quad (47)$$

where q^* and Ω are the magnitude and excitation frequency of the applied load, respectively.

Now, by applying Laplace Transform w.r.t 't' on (47), we get

$$q_o(x,s) = -q^*(\frac{\Omega}{s(s+\Omega)}) \quad (48)$$

Therefore, (44) can be rewritten as follows

$$\left[\frac{d^4}{dx^4} - \eta^4\right]W = \frac{-q^*}{D_s}(\frac{\Omega}{s(s+\Omega)}) \text{ where } \eta^4 = -\frac{s^2}{D_s} \quad (49)$$

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Now, we again employ Laplace transforms with respect to 'x' as defined by

$$\bar{W}(\xi,s)=\int_0^\infty e^{-\xi x}W(x,s)dx \quad (50)$$

where $\xi = L\xi'$ is a non dimensional Laplace transform parameter which may be real or complex. Here 'ξ'' is a dimensional Laplace parameter with respect to space variable 'x'.

Applying Laplace transform (50) in (49), the following equation is obtained

$$[\xi^4\bar{W}(\xi,s)-\xi^3W(0,s)-\xi^2W'(0,s)-\xi W''(0,s)-W'''(0,s)]-\eta^4\bar{W}(\xi,s)=-\frac{q^*}{D_s}(\frac{\Omega}{s(s+\Omega)})\frac{1}{\xi} \quad (51)$$

Now, by applying boundary condition (27), one can obtain

$$\bar{W}(\xi,s)=c_1(\frac{\xi^2}{\xi^4-\eta^4})+c_2(\frac{1}{\xi^4-\eta^4})-\frac{q^*}{D_s}(\frac{\Omega}{s(s+\Omega)})(\frac{1}{\xi(\xi^4-\eta^4)}) \quad (52)$$

where $c_1 = W'(0,s) = \frac{\partial W(0,s)}{\partial x}$ and $c_2 = W'''(0,s) = \frac{\partial^3 W(0,s)}{\partial^3 x}$.

Now, by Taking Inverse Laplace transform of equations (52) with respect to x, the following equation is derived as follows

$$W(x,s)=\frac{c_1}{2\eta}\tilde{S}(\eta x)+\frac{c_2}{2\eta^3}S(\eta x)-\frac{q^*}{2\eta^4 D_s}(\frac{\Omega}{s(s+\Omega)})(\tilde{C}(\eta x)-2) \quad (53)$$

where $\tilde{C}(\eta x)=cosh\eta x+cos\eta x$, $S(\eta x)=sinh\eta x-sin\eta x$, $\tilde{S}(\eta x)=sinh\eta x+sin\eta x$.

Now, by Applying Boundary condition (27) at the axial end $x=1$ of the beam, we obtain a non-homogeneous system of linear algebraic equations in unknowns $c_i's$ $(i=1,2)$. This system of equations have infinite many solution if rank of the matrix is less than number of unknown's i.e. determinant of the coefficients of unknown's is equal to zero. Therefore, the following characteristic equations of the beam are obtained as follows

$$sin\eta \; sinh\eta = 0 \quad (54)$$

The corresponding characteristic roots of equations are given by [16]

$$\eta_1=3.1416, \eta_2=6.2832 \text{ and } \eta_k=k\pi, \text{ for } k\geq 3 \text{ and } k\in I \quad (55)$$

Therefore, Deflection $W(x,s)$ can be obtained as

$$W(x,s)=\frac{-q^*\Omega}{4s^3(s+\Omega)G_1(\eta)}[A_1\tilde{S}(\eta x)+B_1S(\eta x)-2G_1(\eta)(\tilde{C}(\eta x)-2)] \quad (56)$$

where

$$G_1(\eta)=sin\eta sinh\eta \; , \; A_1(\eta)=sin\eta(cosh\eta-1)+sinh\eta(cos\eta-1) \; , \; B_1(\eta)=sin\eta(cosh\eta-1)-sinh\eta(cos\eta-1)$$

Using Laplace Inversion formula with respect to time 't' as under [18]

$$w(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}W(x,s)ds \quad (57)$$

Now by using (56), the above (57) can be rewritten as:

$$w(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{-q^*\Omega e^{st}H_1(\eta,x)}{4s^3(s+\Omega)G_1(\eta)}ds, \quad (58)$$

where $H_1(\eta,x)=A_1\tilde{S}(\eta x)+B_1S(\eta x)-2G_1(\eta)(\tilde{C}(\eta x)-2)$.

Now, by using Cauchy Residues theorem [18], [19] we get that

$$w(x,t)=\sum \text{Residues at the poles of } e^{st}W(x,s) \quad (59)$$

A. For Simply Supported Beam

In this case, Singular points of $e^{st}W(x,s)$ are given poles of order one at [19]

$$s=0,-\Omega \text{ and } G_1(\eta)=0 \quad (60)$$

Therefore, Residue of $e^{st}W(x,s)$ at singular points $s=0,-\Omega$ are given by

$$Res[e^{st}W(x,s), at\ s=0]=\frac{-q^*}{2}A_R^2x(x^3-2x^2+1)[1+\varepsilon_{\phi_1}(1+g(q_1))] \quad (61)$$

where

$$\varepsilon_{\phi_1}=\frac{p_0^2}{p_1}, g(q_1)=\frac{-12}{q_1^2}+\frac{24}{q_1^3}tanh(\frac{q_1}{2}),$$

$$q_1=\frac{\sqrt{p_1\delta_2^*\left(1+\frac{\bar{v}\varepsilon_T}{p_0}\right)}}{A_R}$$

Residue at $s=-\Omega$, is given by

$$Res[e^{st}W(x,s), at\ s=-\Omega]=\frac{q^*e^{-\Omega t}}{4\Omega^2}\frac{H_1(\hat{\eta},x)}{G_1(\hat{\eta})} \quad (62)$$

where

$$\hat{\eta}=\sqrt[4]{3}(1+i)\sqrt{\Omega A_R}\left[1-\frac{\varepsilon_T}{4}(1+f(\hat{p}))+\frac{\hat{\varepsilon}_\phi}{4}(1+g(\hat{q}))\right],$$

$$\hat{\varepsilon}_\phi=\frac{p_0^2\delta_2^*}{p_1\delta_2^*+\frac{\Omega^2}{\delta_2^2}}, \hat{p}=\frac{1}{A_R}\sqrt{\bar{c}\Omega(1-\Omega t_0)\left(1-\frac{\bar{v}p_0\delta_2^*}{p_1\delta_2^*+\frac{\Omega^2}{\delta_2^2}}\right)},$$

$$\hat{q}=\frac{i}{A_R}\sqrt{\left(p_1\delta_2^*+\frac{\Omega^2}{\delta_2^2}\right)(1+\frac{\bar{v}\varepsilon_T}{p_0})}$$

Now, the roots of equation $G_1(\eta)=0$ are given in (55). Also, the relation $\eta^4=\frac{-s^2}{D_s}$ which gives us

$$s_n=i\eta_n^2\sqrt{D_s}=is_0\left[1+\frac{\varepsilon_T}{2}(1+f(\hat{\hat{p}}))-\frac{\hat{\hat{\varepsilon}}_\phi}{2}(1+g(\hat{\hat{q}}))\right] \quad (63)$$

where

$$s_0=\frac{\eta_n^2}{2\sqrt{3}A_R}, \hat{\hat{p}}=\frac{1}{A_R}\sqrt{\bar{c}s_0(1+t_0s_0)\left(1-\frac{\bar{v}p_0\delta_2^*}{p_1\delta_2^*+\frac{s_0^2}{\delta_2^2}}\right)},$$

$$\hat{\hat{q}}=\frac{1}{A_R}\sqrt{\left(p_1\delta_2^*+\frac{s_0^2}{\delta_2^2}\right)(1+\frac{\bar{v}\varepsilon_T}{p_0})}, f(\hat{\hat{p}})=\frac{-12}{\hat{\hat{p}}^2}+\frac{24}{\hat{\hat{p}}^3}tanh\left(\frac{\hat{\hat{p}}}{2}\right),$$

$$g(\hat{\hat{q}})=\frac{-12}{\hat{\hat{q}}^2}+\frac{24}{\hat{\hat{q}}^3}tanh\left(\frac{\hat{\hat{q}}}{2}\right), \hat{\hat{\varepsilon}}_\phi=\frac{p_0^2\delta_2^*}{p_1\delta_2^*+\frac{s_0^2}{\delta_2^2}}$$

Therefore Residues at $s=s_n$ are given by

$$Res[e^{st}W(x,s), at\ s=s_n]=\frac{-q^*\Omega e^{s_nt}}{4s_n^3(s_n+\Omega)}\frac{H_1(\eta_n,x)}{\frac{dG_1}{ds}_{|s=s_n}} \quad (64)$$

Therefore Deflection for Simply Supported Beam as mentioned in (59), by using (61), (62) and (64) can be written as

$$w(x,t) = \frac{-q^*}{2}[x(x^3-2x^2+1)A_R^2[1+\varepsilon_{\phi_1}(1+g(q_1))] - \frac{e^{-\Omega t}}{2\Omega^2}\frac{H_1(\hat{\eta},x)}{G_1(\hat{\eta})} +$$

$$\frac{e^{s_n t}\Omega}{2s_n^3(s_n+\Omega)}\frac{H_1(\eta_n,x)}{\frac{dG_1}{ds}_{|s=s_n}}] \qquad (65)$$

VII. Numerical Illustrations and Graphical Interpretations

Here, Consider magnesium like material beam (microstretch thermoelastic solid). The physical parameters are given as follows [2], [17]
$\rho = 1.74\times10^3 Kg/m^3$, $\lambda = 9.4\times10^{10}N/m^2$, $\mu = 4.0 \times10^{10}N/m^2$, $K = 1.0\times10^{10}N/m^2$, $j_0 = 0.185\times10^{-19}m^2$, $T_0 = 298^oK$, $\beta_1 = 2.68\times10^6 N/m^2deg$, $\beta_2 = 2.0\times10^6N/m^2$,
$K^* = 1.7\times10^6 J/\text{m sec deg}$, $Ce = 1.04\times10^3 J/\text{Kg deg}$
and the value of relevant stretch parameters are given as
$\lambda_0 = 0.5\times10^{10}N/m^2$, $\lambda_1 = 0.5\times10^{10}N/m^2$, $\alpha_0 = 0.779\times10^{-9}N$

Here dimension of the beam are taken as length $L = 60\mu m$, width $b = 3\mu m$ and thickness $h = 1\mu m$. The magnitude (q^*) of the load is 2×10^{-7} and frequency (Ω) is 0.1076 Hz. The non-dimensional values of the characteristic times in case of Simply Supported boundary condition are obtained from the relation $t_0 = s_0^{-1}$. Therefore value of relaxation time are given as $t_0 = 21.05, 5.26$ for first and second mode respectively. Non-dimensional deflection has been computed in (65).

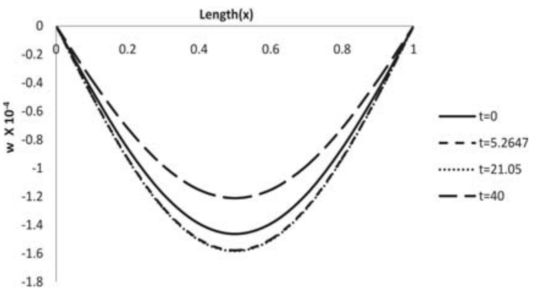

Fig. 1 Deflection (w) in Simply Supported Microstretch Thermoelastic Beam with length (x) for first mode at different considered times

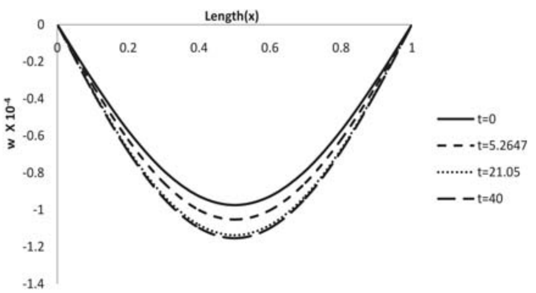

Fig. 2 Deflection (w) in Simply Supported Microstretch Thermoelastic Beam with length (x) for second mode at different considered times

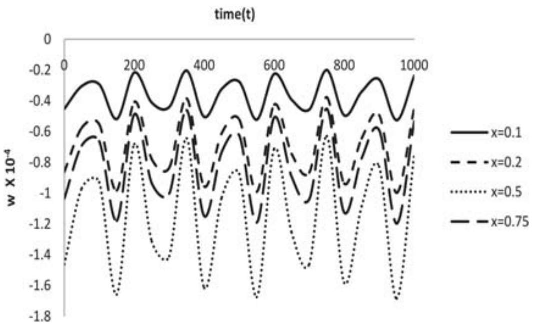

Fig. 3 Deflection (w) in Simply Supported Microstretch Thermoelastic Beam with time (t) for first mode at different considered lengths

Here Figs. 1 and 2 represent the variation of deflection for microstretch thermoelastic beam under exponential time varying load, for simply supported beam, versus length (x) for first and second modes respectively at different considered values of time (t). From Figs. 1 and 2, it is observed that deflection profiles are symmetrical about the mid point of the beam. Also, it is observed that magnitude of deflection decreases as mode increases from first to second. In Fig. 2, it is observed that magnitude of deflection increases as time increases.

Fig. 3 represents the variation of deflection for microstretch thermoelastic beam under exponential time varying load, for simply supported beam, versus time (t) for first mode at different considered values of length (x). From Fig. 3, it is observed that magnitude of deflection is maximum at the mid point of the beam and decreases as it moves away from this point on either side of the beam.

VIII. Conclusion

The flexural vibrations of homogeneous, isotropic, micropolar microstretch generalized thermoelastic thin beam resonators due to exponential time varying load have been investigated under Euler-Bernoulli hypothesis by using Laplace transforms technique twice. It is concluded that deflection profiles are symmetrical about the mid point of the beam in case of SS beam. It is observed that deflection has larger value in first mode as compare to second mode. The deflection for first two modes for microstretch thermoelastic beam represent significant results.

References

[1] A. C. Eringen, Linear theory of micropolar Elasticity, Journal of Mathematics and Mechanics, 909-923 (1966).
[2] A. C. Eringen, Microcontinuum Field Theories. I, Foundation and Solids, Springer -Verlag, New York (1999).
[3] W. Nowacki, Dynamic Problems of Thermoelasticity, Pwn-Polish Scientific Publishers, Warszawa, Poland (1975).
[4] A. C. Eringen, Theory of thermo-microstretch elastic solids, International Journal of Engineering Sciences, 28, 1291-1301 (1990).
[5] C. Zener, Internal friction in solids. I, Theory of internal friction in reeds, Physical Review, 52, 230-235 (1937).
[6] R. Lifshitz, M.L. Roukes, Thermoelastic damping in micro and nanomechanical systems, Physical Review B, 61, 5600-5609 (2000).
[7] Y. Sun, D. Fang, A.K. Soh, Thermoelastic damping in micro-beam resonators, International Journal of Solids Structure, 43, 3213-3229 (2006).

[8] S. Prabhakar, S. Vengallatore, Theory of thermoelastic damping in micromechanical resonators with two-dimensional heat conduction, Journal of Microelectro Mechanical Systems, 17, 494-502 (2008).
[9] X. Guo, Y.B. Yi, S. Pourkamali, A finite element anaylsis of thermoelastic damping invented MEMS beam resonators, International Journal of Mechanical Sciences, 74, 73-82 (2013).
[10] D. Grover, Transverse vibrations in micro-scale viscothermoelastic beam resonators, Archive of Applied Mechanics, 83(2), 303-314 (2013).
[11] D. Grover, Viscothermoelastic Vibrations in micro-scale beam resonators with linearly varying thickness, Canadian Journal of Physics, 90, 487-496 (2012).
[12] D. Grover, Damping in thin circular viscothermoelastic plate resonators, Canadian Journal of Physics, 93(12), 1597-1605 (2015).
[13] J. N. Sharma, D. Grover, Thermoelastic vibrations in micro-/nano-scale beam resonators with voids, Journal of Sound and Vibrations, 330, 2964-2977 (2011).
[14] B. Yanping, H. Yilong, Static deflection analysis of micro-cantilevers beam under transverse loading, Recent Researchers in Circuits, Systems, Electronics,Control and Signal Processing, 17-21 (2010).
[15] H. W. Lord, Y. Shulman, The generalized dynamical theory of thermoelasticity, Journal of the Mechanics and Physics of Solids, 15, 299-309 (1967).
[16] S. S. Rao, Vibration of Continuous Systems, John Wiley and Sons, New Jersey, USA (2007).
[17] R. S. Dhaliwal, A. Singh, Dynamic coupled thermoelasticity, Hindustan Publication Corporation, New Delhi, India (1980).
[18] R. V. Churchill, Operational Mathematics, McGraw Hill, New York, USA (1972).
[19] J. W. Brown, R. V. Churchill, Complex Variables and Applications, McGraw Hill, New York, USA (1996).
[20] G. Partap, N. Chugh, Deflection analysis of micro-scale microstretch thermoelastic beam resonators under harmonic loading, Applied Mathematical Modelling, 46, 16-27 (2017).

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