Generalized Rough Sets Applied to Graphs Related to Urban Problems

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Abstract—Branch of modern mathematics, graphs represent instruments for optimization and solving practical applications in various fields such as economic networks, engineering, network optimization, the geometry of social action, generally, complex systems including contemporary urban problems (path or transport efficiencies, biourbanism, & c.). In this paper is studied the interconnection of some urban network, which can lead to a simulation problem of a digraph through another digraph. The simulation is made univoc or more general multivoc. The concepts of fragment and atom are very useful in the study of connectivity in the digraph that is simulation - including an alternative evaluation of k- connectivity. Rough set approach in (bi)digraph which is proposed in premier in this paper contribute to improved significantly the evaluation of k-connectivity. This rough set approach is based on generalized rough sets - basic facts are presented in this paper.

Keywords—(Bi)digraphs, rough set theory, systems of interacting agents, complex systems.

I. INTRODUCTION

COMPLEX systems represent sets of elements (agents) which are not identical and connected through various interactions (networks). Biourbanism [36] is the science that focuses on the study of the concept of an urban organism (or city regarded as an urban organism), considering it as a hypercomplex system in relation to its internal and external dynamics, as well as their mutual interactions. Also, biourbanism aims to reformulate the epistemological foundation of architecture and urbanism, in line with the science of complex dynamic systems. In general, this approach therefore links biourbanism to the other sciences like life sciences (e.g., botany, biology, zoology, agriculture and food, microbiology, physiology, biochemistry, medical sciences) and integrated systems sciences (e.g., ecology, statistical mechanics, thermodynamics, operations research). Thus, the analysis of the evolutionary dynamics of a complex system can be described using network theory. From a mathematical point of view, network study (particular, urban networks) uses graph theory (one of the fundamental domains of discrete mathematics). In this respect, the networks are essential elements to understand the basic principles of other sciences, if these organizational principles are structured around of the mathematics of complexity, such as fractals and chaos theory. In terms of applicability in other areas can list some of them: biology, IT, economics, social sciences, urban planning. In some speciality works [33]–[36] has been demonstrated that the urban environment is an extremely

complex system that can be characterized by a large number of relations and interconnections that occur both between its components (agents) and between them and the external environment. For example, the network of streets and alleys whose interactions and connections determine the comfort's level of urban neighborhoods, as well as its overlapping with other networks (energy, informational, social, economic flows, ecological, etc.).

Graph theory [27], [33]–[35] is a tool for optimization and solving practical applications in all fields, such as representation and study of economic and social networks, engineering, optimization of networks (goods and information transport systems), social action geometry, complex general systems, including contemporary urban issues (analysis of transportation and distribution problems, biourbanism [36]), dynamic programming to determine the optimal policy, game theory, information theory (study of signs and codes). However, the critical behavior of classical graph theory [35] has been noted in solving existing problems (e.g., network security applications). In this respect, it was introduced the notion of bigraph (an extension of the graph [33]), improved subsequently by bigraph with sharing [34]. Bigraphs represent a mathematical model for interacting systems of agents (ubiquitous systems based on placing and linking) and having the ability to indicate the position in space, displacement, and the agents' interconnections. As an extrapolation of applications of (bi)digraphs is proposed rough set approach regarding to issues of possible uncertainty related to urban problems [1]–[4].

II. GENERALIZED ROUGH SETS: A NEW LOOK

Remark 1. *(A brief history)*

Rough sets (RS) - as an extension of classic (crisp) sets had an exponential development (by applications in various areas) in the quarter century between when opening [5] and testamentary time [6] - and continue today; this is illustrated by a *handbook* [7] and by series *LNCS Transactions in RS* [8] and *LNAI RS and Knowledge Technologies* [9] (a Pawlak dedication). Originally RS were defined in a classifying (partitioning) approximation space, i.e., in an equivalence relational structure, then - more general in a covering approximation space, i.e., a tolerance relational structure - up to a (homogeneous) relational approximation space [5], [11], [10], [12], but in [10] Pawlak speaks about an alternative - a topological approximation space which is an idea resumed in other paper [13], [14]. A category approach to RS is made in [15]. RS interfered and interferes with fuzzy sets (FS) theory

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that was initiated by Zadeh [18] and was then developed in seven handbooks edited by Dubois and Prade between 1998 - 2000 and in *IEEE Transactions on Fuzzy Systems* (for a synthesis see [16], [17]); the interference RS - FS consisted in hybridizations FS -RS (RFS), RS - FS (FRS) that was initiated by Dubois and Prade, see [19]. For other generalizations, respectively for *recent trends in applied RS* see [20], [21], respectively [22]–[26].

The paper [4] is an unification effort of some generalizations - with a weak alternative and with a new look of RS; in addition are proposed two impact applications. Basic facts are presented in this section.

Remark 2. *(Nonhomogeneous relational approximation space)* A (nonhomogeneous) relational approximation space $(U, V.R)$ is in fact a (nonhomogeneous binary) relational structure where $R \in \mathcal{R}el(U, V)$ - the set of (nonhomogeneous binary) relations between U, V; consequently are used - as a addenda relative to (binary) relations (co)kernel, restrictions and inducing [1], respectively sections [2]. In general R is in U, V, i.e., $U^* = U^R = \text{dom}(R) \subset U$, $V^* = {}^R V = \text{codom}(R) \subset V$; particularly, R is left-total if $U^* = U$, respectively R is right-total (or surjective) if $V^* = V$ and R is total if it is so left-total, right-total.

Let be $Y \in \mathcal{P}(V)$

Definition 1. ([right-] rough approximations) *The [right-] [rough] lower approximation of Y [with respect to R] - for short [R-right-] [r-]lower approximation of Y is*

$$r - low\, approx_R^{\rightarrow}(Y) = \mathring{Y}_R^{\rightarrow} =$$
$$= \cup \{R < u > |_Y | R < u >= R < u >, u \in U^*\}, \quad (1)$$

here r is informative and can omit.

The [right-] rough upper approximation of Y [with respect to R] - for short [R-right-] r-upper approximation of Y is

$$r - upp\, approx_R^{\rightarrow}(Y) = \mathring{\overline{Y}}_R^{\rightarrow} =$$
$$= \cup \{R < u > |_Y | R < u > \neq \varnothing, u \in U^*\}. \quad (2)$$

Definition 2. ([right-] rough boundary) *The [right-] rough boundary of Y [with respect to R] - for short [R-right-] r-boundary of Y is*

$$r - bn_R^{\rightarrow}(Y) = {}_*b_R^{\rightarrow}(Y) = \mathring{\overline{Y}}_R^{*\rightarrow} = \overline{Y} \setminus \mathring{Y} \quad (3)$$

(in both definitions [right], [→] are the default and [R] can omit if not any possibility of confusion).

Observation 1.

i) *(addenda)* Other expressions for $\mathring{Y}$ and $\overline{Y}$

$$\mathring{Y} = \{v \in V | \exists u \in U^*, v \in R < u >=_Y |R < u >\},$$
$$\overline{Y} = \{v \in V | \exists u \in U^*, v \in R < u >, _Y |R < u > \neq \varnothing\}. \quad (4)$$

In addition the [right-] rough outside of Y [with respect to R] - for short [R-right] r-outside of Y is

$$r - out_R^{\rightarrow}(Y) = {}_*o_R^{\rightarrow}(Y) = C_{V^*}\overline{Y}). \quad (5)$$

In general occur inequalities $\mathring{Y} \subseteq \overline{Y}$, $Y \subseteq \overline{Y}$ if $Y \in \mathcal{P}(V^*)$.

ii) *(sources)* The source of $\mathring{Y}_R^{\rightarrow}$, respectively $\overline{Y}_R^{*\rightarrow}$ is $\cup_{\mathring{Y}_R^{\rightarrow}} = \{u \in U^*|_Y | R < u >= R < u >\}$, respectively $\cup_{\overline{Y}_R^{*\rightarrow}} = \{u \in U^*|_Y | R < u > \neq \varnothing\}$, which means that $\mathring{Y}_R^{\rightarrow} = R\left(\cup_{\mathring{Y}_R^{\rightarrow}}\right)$ ([R−]$\cup_{\mathring{Y}_R^{\rightarrow}}$ - section), $\overline{Y}_R^{\rightarrow} = R\left(\cup_{\overline{Y}_R^{*\rightarrow}}\right)$ ([R−]$\cup_{\overline{Y}_R^{*\rightarrow}}$ - section), where again [→] is the default and [R] can omit, see [2].

Occur equalities

$$\mathring{Y} = R\left(\cup_{\mathring{Y}}\right) = \max_{\subseteq}\{R(U')|_Y | R(U') = R(U'), U' \in \mathcal{P}(U^*)\}$$
$$= \cup\{R(U')|_Y | R(U') = R(U'), U' \in \mathcal{P}(U^*)\},$$
$$\overline{Y} = R\left(\cup_{\overline{Y}}\right) = \max_{\subseteq}\{R(U')|_Y | R(U') \neq \varnothing, U' \in \mathcal{P}(U^*)\}$$
$$= \cup\{R(U')|_Y | R(U') \neq \varnothing, U' \in \mathcal{P}(U^*)\}. \quad (6)$$

iii) *(left-approximations, left-boundary)* Analog are defined the left - rough approximations $\mathring{X}_R^{\leftarrow} = \mathring{X}_{R^{-1}}$, $\overline{X}_R^{*\leftarrow} = \overline{X}_{R^{-1}}$, respectively the left - rough boundary $\mathring{X}_R^{\circ\leftarrow} = \mathring{\overline{X}}_{R^{-1}}$ and left-rough outside ${}_*o_R^{\leftarrow}(X) = {}_*o_{R^{-1}}^{\rightarrow}(X) = C_{U^*}(\overline{X})$ ([R] can omit) and respectively the sources.

iv) *(homogeneous cases)* In the case of a (homogeneous) relational approximation space (V, R) - which is isomorphic with the (homogeneous binary) relational structure (V, R), $R \in \mathcal{R}el(V)$ (see [1], [2]) the approximations and the boundaries are analogous. In particular if R is V^*-symmetric relation, then "right" and "left" coincide.

Definition 3. (*s*-surjective) $R \in \mathcal{R}el(U, V)$ is V^* *s-surjective* in $Y \in \mathcal{P}(V^*)$ if there is $X \in \mathcal{P}(U^*)$ so that $Y = R(X)$; R is V^* *s-surjective* if R is V^* *s-surjective* in any $Y \in \mathcal{P}(V^*)$.

Remark 3. *(Connection)* $Y = \mathring{Y}_R$ iff R is V^* s-surjective in Y, $Y \in \mathcal{P}(V^*)$ - and consequently $\overline{Y} = \overline{Y} \setminus Y$, see `Observation 1` (ii). In addition ${}_*b_R(Y) = \varnothing$ implies R is V^* s-surjective in Y, $Y \in \mathcal{P}(V^*)$ and in general, ${}_*b_R \equiv \varnothing$ implies R is V^* s-surjective (${}_*b_R$ is induced operator).

Definition 4. ([right-] rough set) *Set $Y \in \mathcal{P}(V^*)$ is [right-] rough [with respect to R] - for short [R-right-] rough if ${}_*b_R^{\rightarrow}(Y) = \overline{Y}_R^{*\rightarrow} \neq \varnothing$ (again [right], [→] are the default and [R] can omit).*

Observation 2.

i) *(representations)* The *classical* representation for $Y[R-]$ rough set is ${}_\circ\rho_R^{\rightarrow}(Y) = \left(\mathring{Y}_R^{\rightarrow}, \overline{Y}_R^{\rightarrow}\right)$. A new representation for $Y[R-]$ rough set is $\rho_R^{\rightarrow}(Y) = \left(\mathring{Y}_R^{\rightarrow}, \overline{Y}_R^{*\rightarrow}\right)$ according to bijection $\beta^{\rightarrow} : \mathcal{P}^2(V^*) \to \mathcal{P}^2(V^*)$, $(A, B) \overset{\beta}{\simeq} (A, B \setminus A)$, $A \subseteq B$ (${}_\circ\rho$, ρ are not injection or surjective).

ii) *(left-rough)* Analogously $X \in \mathcal{P}(U^*)$ is $[R-]$ left-rough set if ${}_*b_R^{\leftarrow}(X) = \overline{X} \neq \varnothing$ (with representations ${}_\circ\rho_R^{\leftarrow}$, $\rho_R^{\leftarrow}$).

iii) *(homogeneous cases)* For homogeneous cases, it is analogous, see `Observation 1` (iv).

iv) *(rough universe)* If is notes $\mathcal{RP}_R^{\rightarrow}(V^*)$, respectively $\mathcal{RP}_R^{\leftarrow}(U^*)$ the universe of [R-right-] rough sets of $\mathcal{P}(V^*)$, respectively of [R-left-] rough sets of $\mathcal{P}(U^*)$ (and analogous in the homogeneous cases $R \in \mathcal{R}el(V)$, respectively $R \in \mathcal{R}el(U)$).

v) (*rough sets type border*) In the special case of the [right-] rough sets type border (with $\mathring{Y}_R^{\rightarrow} = \varnothing$, $Y \in \mathcal{RP}_R^{\rightarrow}(V^*)$) in some applications is used n-conditioning $\mathring{\overset{*}{Y}}_R^{\rightarrow}/n$, i.e., $\mathring{\overset{*}{Y}}_R^{\rightarrow}/n = \mathring{\overset{*}{Y}}_R^{\rightarrow} \setminus Y$, $|\mathring{\overset{*}{Y}}_R^{\rightarrow}| - |Y| > n$, $n \in \mathbb{N}$ (and analogously at left).

Definition 5. (rough membership relations) *Roughly speaking rough membership relations is $_r \in = <\in, \in>^l$ (lexicographical). More specifically the [right-] rough membership relation [with respect to R] - for short [R-right-] rough membership relation $_r \in_R^{\rightarrow}$ is defined by $y_r \in_R^{\rightarrow} Y$ ($\in \mathcal{RP}(V^*)$) if $y \in \mathring{Y}_R^{\rightarrow}$ (interior point) or $y \notin \mathring{Y}_R^{\rightarrow}$, $y \in \mathring{\overset{*}{Y}}_R^{\rightarrow}$ (bounded point - again [right], [$\rightarrow$] are the default and [R] can omit). The [R-] left-rough membership relation $_r \in_R^{\leftarrow}$ is defined analogously.*

Remark 4. *(Pairs of primitive objects)* It has been obtained the pairs of primitive objects $\left(Y \in \mathcal{RP}_R^{\rightarrow}(V^*), _r\in_R^{\rightarrow}\right)$, respectively $\left(X \in \mathcal{RP}_R^{\leftarrow}(U^*), _r\in_R^{\leftarrow}\right)$ (nonhomogeneous and analogous homogeneous).

Example 1. *(induced classifying approximation spaces)* The relation $R \in \mathcal{R}el(U,V)$ induces two equivalence relations in V, U-$E_R^{\rightarrow}$ defined by $(v_1, v_2) \in E$ if $R^{-1} < v_1 >= R^{-1} < v_2 >$, $V^* = \text{dom}(E) = \text{codom}(E)$, respectively $E_R^{\leftarrow}$ defined by $(u_1, u_2) \in E^{\leftarrow}$ if $R < u_1 >= R < u_2 >$, $U^* = \text{dom}(E^{\leftarrow}) = \text{codom}(E^{\leftarrow})$. Are obtained E-rough sets, respectively $E^{\leftarrow}$-rough sets according to E-$(r$-$)$ approximations and E-$(r$-$)$ boundary, respectively to $E^{\leftarrow}$-$(r$-$)$ approximations and $E^{\leftarrow}$-$(r$-$)$ boundary.

III. GENERALIZED ROUGH SETS IN (BI)DIGRAPHS

Section III is developed in [4], but `Application 1` (*k*-vertex connectivity vs. roughness) is original.

Remark 5. *(Abbreviations)* A digraph $G = \left(V_{[G]}, E_{[G]}\right)$ (where $[G]$ can omit if not any possibility of confusion) - as a directed graph is bidigraph if his underlying graph is bipartite - with $\{P, Q\}$ a partition of V (see [27] - and for bigraphs "as a space and motion of communicating agents", see [33]).

Remark 6. *(Isomorphisms)* In *the bipartite case* the bidigraph (P, Q, E) is isomorphic with the (nonhomogeneous binary) relational structure (P, Q, R), where $R \in \mathcal{R}el(P, Q)$ and $(u, v) \in E$ iff $(u, v) \in R$. Further, the (nonhomogeneous binary) relational structure (P, Q, R) is isomorphic with the (nonhomogeneous) relational approximation space (P, Q, R), see `Remark 2`. The homogeneous case is analog, i.e., $(V, E) \simeq (V, R)$, (V, R) - as (homogeneous binary) relational structure, respectively as relational approximation space.

Definition 6. (right-rough set, left-rough set) *Set $Y \in \mathcal{P}(Q^*)$ is right-rough (with respect to R) - for short R-right-rough if*

$$_*b_R^{\rightarrow}(Y) = \mathring{\overset{*}{Y}}_R \neq \varnothing \quad, \text{ where } _*\text{b}(Y) = \overset{*}{\mathring{Y}} = \overset{*}{Y} \setminus \mathring{Y} \quad (7)$$

(including respectively $R - r$-boundary, $R - r$-upper approximation, R-lower approximation and $R - r$-outside of Y). Analogously $X \in \mathcal{P}(P^*)$ is R-left -rough set if $_*b_R^{\leftarrow}(X) = \mathring{\overset{*}{X}}_R^{\leftarrow} \neq \varnothing$ (again "right", "$\rightarrow$" are the default and "R" can omit, see `Definition 1`, `Definition 4`, respectively `Observation 2 (ii)` and `Observation 1 (iii)`).

Observation 3.

i) (*representation - at right and at left*) The representations are $\rho_R^{\rightarrow}(Y) = \left(\mathring{Y}_R^{\rightarrow}, \mathring{\overset{*}{Y}}_R^{\rightarrow}\right)$, $Y \in \mathcal{P}(Q^*)$, respectively $\rho_R^{\leftarrow}(X) = \left(\mathring{X}_R^{\leftarrow}, \mathring{\overset{*}{X}}_R^{\leftarrow}\right)$, $X \in \mathcal{P}^*(P)$ according to the isomorphism $(A, B) \overset{\beta}{\simeq} (A, B \setminus A)$, $A \subseteq B$ ("$\rightarrow$" is the default and "R" can omit, see `Observation 2 (i,ii)`).

ii) (*pair of primitive objects - at right and at left*) It is notes $\mathcal{RP}_R^{\rightarrow}(Q^*)$ the universe of R-right-rough sets of $\mathcal{P}(Q^*)$; the R-right-rough membership relation $_r\in_R^{\rightarrow}$ is defined by $y_r \in_R^{\rightarrow} Y \left(\in \mathcal{RP}_R^{\rightarrow}(Q^*)\right)$ if $y \in \mathring{Y}_R^{\rightarrow}$ (interior vertex) or $y \notin \mathring{Y}_R^{\rightarrow}$, $y \in \mathring{\overset{*}{Y}}_R^{\rightarrow}$ (bounded vertex). It has been obtained the pair of primitive objects at right $\left\{Y \in \mathcal{RP}_R^{\rightarrow}(Q^*), _r\in_R^{\rightarrow}\right\}$. Analogously it obtains the primitive object at left $\left\{X \in \mathcal{RP}_R^{\leftarrow}(P^*), _r\in_R^{\leftarrow}\right\}$ (again "right", "$\rightarrow$" are the default and "R" can omit - see `Definition 4`, `Observation 2 (iii,iv)`, `Definition 5`, `Remark 4`).

iii) (*homogeneous cases*) It is analogous, see `Observations 1 (iv)`, `2 (iii)` and `Remark 4`.

Example 2.

a. (*induced classifying approximation spaces in a bidigraph $G = (V = \{P, Q\}, E) \simeq (P, Q, R)$*) According to `Example 1` are obtained $\left(Q, E_Q\right)$, respectively $\left(P, E_P^{\leftarrow}\right)$ as induced classifying approximations spaces where $(v_1, v_2) \in E_Q$ if $R^{-1} < v_1 >= R^{-1} < v_2 >$, respectively $(u_1, u_2) \in E_P^{\leftarrow}$ if $R < u_1 >= R < u_2 >$. Analogously are obtained $\left(V, E_V\right)$, respectively $\left(V, E_V^{\leftarrow}\right)$ as induced classifying approximations spaces in a digraph $G = (V, E)$ (homogeneous case)

b. (*another classifying approximation space in a digraph $G = (V, E)$*) Let be $A_{[G]} = \{a | a : V \rightarrow C_a\}$ a set of vertex attributes; A induces a equivalence relation in V - E_A defined by $(u, v) \in E_A$ if $a(u) = a(v)$, $a \in A$ - and so is obtained (V, E_A) as induced classifying approximation space. Still can be considered operations with E_V, $E_V^{\leftarrow}$, E_A - but this is another problem, see [3], [4].

Definition 7. (induced rough subgraphs - in a digraph) *In a digraph $G = \left(V_{[G]}, E_{[G]}\right) \simeq (V, R)$ the induced R-right-rough subgraph on a R-right-rough set $W \in \mathcal{P}(V^*)$ is a graph $H_R^{\rightarrow} = \left(W_R^{\rightarrow}, E_R^{\rightarrow}\right)$, where $W_R^{\rightarrow} = W$ and the end-points of edge $e \in E_R^{\rightarrow}$ are in $W_R^{\rightarrow}$. Analogously is defined an induced R-left-rough subgraph on W as $H_R^{\leftarrow} = \left(W_R^{\leftarrow}, E_R^{\leftarrow}\right)$ (again "right" is the default and "R" can omit, see `Definition 6`).*

Observation 4.

i) ("*incompleteness*") An induced rough subgraph is an "incomplete" induced subgraph.

ii) (*induced rough subgraphs in a bidigraph*) Analogous are defined the induced rough subgraphs in a bidigraph $G = (V = \{P, Q\}, E) \simeq (P, Q, R)$.

Application 1. *(k-vertex connectivity vs. roughness)* Let be a (simple) digraph $G = (V, E) \simeq (V, R)$. The results of the following are contained in [29]–[32] (see and [27]).

Definition 8. (boundaries) *The positive, respectively negative boundary of $W \in \mathcal{P}(V^*)$ denoted by $\partial^+ W$, respectively $\partial^- W$ is the set of vertices that are adjacent from, respectively to W.*

Definition 9. (fragments) *Let G be a k-strongly connected digraph; $F \in \mathcal{P}(V^*)$ is a positive fragment of G if $|\partial^+ F| = k$ and $V^* \setminus \{F \cup \partial^+ F\} \neq \varnothing$. Analogously, F is a negative fragment of G if $|\partial^- F| = k$ and $V^* \setminus \{F \cup \partial^- F\} \neq \varnothing$.*

Definition 10. (atom) *An atom A is a (positive or negative) fragment of minimum cardinality.*

Theorem 1.

T_1. (alternative definition)

$$k = \min \left\{ \left|\partial^+ F\right| \,|\, F \in \mathcal{P}(V^*), \, F \cup \partial^+ F \neq V^* \text{ or } |\mathrm{F}| = 1 \right\}. \tag{8}$$

T_2. (distinct-disjoint) *For a connected G any two distinct atoms are disjoints.*

T_3. (k-disconnecting set) *For a k-connected G, T a k-disconnecting set ($|T| = k$) and A an atom are occur $A \subset T$ or $A \cap T = \varnothing$.*

An analysis in short is next:

A_1. (*sections*) In fact boundaries are sections, i.e., $\partial^+ F = R(W)$, $\partial^- F = R^{-1}(W)$.

A_2. (*replacements*) For $W \in \mathcal{RP}_R^{\rightarrow}(V^*)$, respectively $W \in \mathcal{RP}_R^{\leftarrow}(V^*)$ the replacements $R(W) - \mathring{\mathring{W}}_R^{\rightarrow}$, respectively $R^{-1}(W) - \mathring{\mathring{W}}_R^{\leftarrow}$ are justified topological (see and Observation 2 (v)).

A_3. (*recover*) It recover the Definitions 8–10 (according to A_2).

A_4. (*truth*) It is investigate if T_1-T_3 of Theorem 1 remain true (according to A_3).

Example 3. *(classic and new)*
Let be $G_{++} = (V, E_{++}) \simeq (V, R_{++})$, where

$$\begin{aligned} V &= V_1 \cup V_2 \\ V_1 &= \{v_1, v_3, v_5, v_7\} \\ V_2 &= \{v_2, v_4, v_6, v_8\}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} E_{++} &= E_+^1 \cup E_+^2 \cup E^3 \\ E_+^1 &= \{(v_1, v_3), (v_3, v_5), (v_5, v_7), (v_7, v_1)\} \\ E_+^2 &= \{(v_2, v_4), (v_4, v_6), (v_6, v_8), (v_8, v_2)\} \\ E^3 &= \{(v_1, v_2), (v_2, v_1), (v_3, v_4), (v_4, v_3), \\ &\quad (v_5, v_6), (v_6, v_5), (v_7, v_8), (v_8, v_7)\}, \; k = 2. \end{aligned} \tag{10}$$

1) Classic
For $A_i = \{v_i\}$, $i = \overline{1,8}$, $\partial^+ A_1 = F_{23}$, $\partial^- A_1 = F_{27}$ (see F_{ij}), $|\partial^+ A_1| = |\partial^- A_1| = 2$, A_1 is an atom, etc. For $F_{ij} = \{v_i, v_j\}$, $i, j = \overline{1,8}$, $i \neq j$, $\partial^+ F_{12} = F_{34}$, $\partial^- F_{12} = F_{78}$, $|\partial^+ F_{12}| = |\partial^- F_{12}| = 2$, respectively $\partial^+ F_{13} = F_{24}$, $\partial^- F_{13} = F_{47}$, $|\partial^+ F_{13}|$, $|\partial^- F_{13}| = 2$, etc.; F_{ij} are (positive, negative) fragments.

2) New
For A_i, $i = \overline{1,8}$, $\mathring{A}_1^{\rightarrow} = \mathring{A}_1^{\leftarrow} = \varnothing$, $\overset{*}{\mathring{A}}_1^{\rightarrow} = \overset{*}{\mathring{A}}_1^{\rightarrow} =$

$\overset{*}{A}_1^{\leftarrow} = \overset{*}{\mathring{A}}_1^{\leftarrow} = A_1 \cup F_{48}$, $\rho^{\rightarrow}(A_1) = \rho^{\leftarrow}(A_1) = (\varnothing, A_1 \cup F_{48})$, $|\overset{*}{\mathring{A}}_1^{\rightarrow}/1| = |\overset{*}{\mathring{A}}_1^{\leftarrow}/1| = |3 - 1| = 2$ (see Observation 2 (v)), A_1 remains atom, etc.
For F_{ij}, $i, j = \overline{1,8}$, $i \neq j$, $\mathring{F}_{12}^{\rightarrow} = \mathring{F}_{12}^{\leftarrow} = \varnothing$, $\overset{*}{F}_{12}^{\rightarrow} = \overset{*}{\mathring{F}}_{12}^{\rightarrow} = \overset{*}{F}_{12}^{\leftarrow} = \overset{*}{\mathring{F}}_{12}^{\leftarrow} = F_{12} \cup F_{34} \cup F_{78} = V \setminus F_{56}$, $\rho^{\rightarrow}(F_{12}) = \rho^{\leftarrow}(F_{12}) = (\varnothing, V \setminus F_{56})$, $|\overset{*}{\mathring{F}}_{12}^{\rightarrow}/1| = |\overset{*}{\mathring{F}}_{12}^{\leftarrow}/1| = |6 - 2| = 4$ (see again Observation 2 (v)), F_{12} is removed, respectively $\mathring{F}_{13}^{\rightarrow} = \mathring{F}_{13}^{\leftarrow} = \varnothing$, $\overset{*}{F}_{13}^{\rightarrow} = \overset{*}{\mathring{F}}_{13}^{\rightarrow} = \overset{*}{F}_{13}^{\leftarrow} = \overset{*}{\mathring{F}}_{13}^{\leftarrow} = F_{13} \cup F_{34} \cup F_{68} = V \setminus F_{57}$, $\rho^{\rightarrow}(F_{13}) = \rho^{\leftarrow}(F_{13}) = (\varnothing, V \setminus F_{57})$, $|\overset{*}{\mathring{F}}_{13}^{\rightarrow}/1| = |\overset{*}{\mathring{F}}_{13}^{\leftarrow}/1| = |6 - 2| = 4$ (see again Observation 2 (v)), F_{13} is removed, etc.
For point T_1 of Theorem 1 are obtained fewer fragments - at right and at left (remain only atoms) and

$$k = \min \left\{ |\overset{*}{\mathring{F}}^{\rightarrow}/1| \,|\, F \in \mathcal{RP}_R^{\rightarrow}(V^*), \, {}_* o_R^{\rightarrow}(F) \neq \varnothing. \right\} \tag{11}$$

The case digraph $G_{+-} = (V, E_{+-})$, where

$$\begin{aligned} E_{+-} &= E_+^1 \cup E_-^2 \cup E^3 \\ E_-^2 &= \{(v_4, v_2), (v_2, v_8), (v_8, v_6), (v_6, v_4)\} \end{aligned} \tag{12}$$

is left for reader.

IV. Urban Interconnection Network

In the following is presented the idea of simulating an urban interconnection network based on interconnection network for an adapted model of parallel machine [27], [38] and finally, which can lead to a simulation problem of a digraph through another digraph, see Sections II-III and [1]–[4].

Remark 7. *(Parallel Machine Scheduling)*
Problem formulation: In the parallel machine scheduling [37] there is a number M of machines that can process all tasks in different or same speeds. Scheduling in parallel machines can be considered as a two step process, i.e.

S_1 - how to efficiently assign the tasks to each machines: which tasks to which machines

S_2 - which is the sequence of the tasks allocated to each machine.

In principle, an interconnection network in a parallel machine transfers information from any source vertex (source node) to any destination vertex (destination node) which is desired - e.g., in parallel computing, a collection of processors which are linked between them. Thus, the tasks should be fulfilled with as small response time as possible which would allow that a large number of such transfers to take place concurrently; moreover, the process cost it should be inexpensive as compared to the rest. A network consists of links and switches (help to send the information from the source node S to the destination node D) and is specified by its topology, routing algorithm, switching strategy, and flow control mechanism. Interconnection networks are composed of switching elements. Its topology is the pattern to connect the individual switches to other elements. A network allows exchange of data between processors in the parallel

system - *direct connection networks* which have point-to-point connections between neighboring nodes (that are fixed which means that these networks are static - e.g., rings, meshes and cubes) and *indirect connection networks* which have no fixed neighbors nodes and can subdivided into three parts (bus networks, multistage networks, crossbar switches) - in this case, the communication topology can be changed dynamically based on the application demands. From the point of view of evaluating design trade-offs in network topology, if the main concern is the routing distance, then the dimension has to be maximized and this problem is reduced at the hypercube case.

The hypercube, denoted Q_n, is a graph of remarkable properties and numerous applications in coding, computer science, and other areas of mathematics. By definition, a n-dimensional hypercube graph Q_n has 2^n vertices, $n \cdot 2^{n-1}$ edges and is a n-regular graph witch means that every vertex is of degree n.

Remark 8. It can construct the n-dimensional hypercube Q_n recursively

$$Q_n = \begin{cases} K_1 & , n = 0 \\ K_2 & , n = 1 \\ Q_{n-1} \times K_2 & , n \geq 2 \,, \end{cases} \quad (13)$$

where K_1, K_2 are complete.

Theorem 2. (Hypercube characterization theorem [27]) *Let consider a graph G. If its vertices are the binary sequences of length n and two vertices are adjacent if their sequences differ in exactly one place, then it said that G is isomorphic to Q_n.*

Observation 5.

i) (*bipartiteness*) Hypercube Q_n is also bipartite, i.e. the vertex set of the graph can be partitioned into two subsets, where, within each set no vertices are adjacent. Furthermore, for the n-dimensional hypercube, the cardinalities of these sets are equal, i.e. each set has 2^{n-1} vertices. Thus, a n-dimensional hypercube Q_n can be drawn in two ways, one of them emphasizing the bipartition.

ii) (*vertex transitivity*) Hypercubes are vertex-transitive graphs, i.e. given any two vertices in a n-hypercube Q_n, there is an automorphism mapping one vertex to the other while maintaining vertex adjacency .

iii) (*adjacency matrix of a hypercube*)

$$A_{Q_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \,, \quad A_{Q_2} = \begin{bmatrix} A_{Q_1} & I_2 \\ I_2 & A_{Q_1} \end{bmatrix} \,, \quad (14)$$

and, recursively

$$A_{Q_n} = \begin{bmatrix} A_{Q_{n-1}} & I_{2^{n-1}} \\ I_{2^{n-1}} & A_{Q_{n-1}} \end{bmatrix}, \forall\, n \geq 3 \,. \quad (15)$$

Remark 9. *(the bandwidth concept)* Basic idea is a problem of optimal assignment of number to vertices of a n-hypercube Q_n, see [28]. In fact, it is determined the minimum value of $\sum \Delta_{ij}$ over all possible assignments, where $\Delta_{ij} = |i - j|$ and i, j are assigned to adjacent vertices. The problem is precisely that of determining the bandwidth of a hypercube Q_n and this concept is presented in the following.

Observation 6. Let consider the graph G. According to [27] are defined the notions

i) *Proper numbering of G:* Is a bijection $f : V \to \{1, 2, \ldots, n\}$.

ii) *Bandwidth of f denoted $B_f(G)$:* If f is a proper numbering of a graph G, then $B_f(G) = \max\{|f(x) - f(y)| | xy \in E\}$, where xy represents the edge with endpoints x and y.

iii) *Bandwidth of G denoted $B(G)$:* $B(G) = \min\{B_f(G)| f$ is a proper numbering of $G\}$.

iv) *Bandwidth numbering of G:* Is a proper numbering f such that $B(G) = B_f(G)$.

Remark 10. One of important application of bandwidth (related to coding theory) is interconnection networks problem - can be modeled by a graph G, where the vertices represent the processors and edges correspond to the links, see [27].

1) It is simulate the network represented by G on a second network modeled by graph H.

2) This can be done by a one-to-one mapping $f : V(G) \to V(H)$ - the processor x in G is simulated by processor $f(x)$ in H, respectively the link xy in G is simulated by a shortest path between $f(x)$ and $f(y)$ in H.

3) If it denoted with t the communication time for link xy in G, then dt is represented the corresponding time in H, where $d = |f(x) - f(y)|$ (distance between $f(x)$ and $f(y)$) in H. If it considered $t = 1$ and H a path, then the greatest possible delay in the simulation is $B(G)$.

4) Everywhere the (bipartite) graph can be a (bi)digraph.

5) (multivoc variants - (bi)simulation) Let consider $(U, E_U) \simeq (U, R_U)$, respectively $(V, E_V) \simeq (V, R_V)$. A simulation $S \in \mathcal{R}el(U, V)$ between the homogeneous relational structures (U, R_U), (V, R_V) is defined by a non-banal existential variant of the compatibility condition with R_U, R_V - for each $u, u' \in U$, $v \in V$, with $(u, v) \in S$, $(u, u') \in R_U$ implies there exits $v' \in V$, $(u', v') \in S$, $(v.v') \in R_V$ - which is equivalently with the usual condition "for each $u \in U$, $v \in V$, with $(u, v) \in S$ and for each $u' \in U$, $(u, u') \in R_U$ implies there exists $v' \in V$, $(u', v') \in S$, $(v, v') \in R_V$". A (bi)simulation is a pair of simulation (S, S^{-1}), see [1].

V. Conclusions

This paper proposed in premier a rough set approach in (bi)digraphs which contribute to improve significantly the evaluation of k-connectivity. This is relative to a simulation problem of a digraph through another digraph and is related to the interconnection of some urban network. The simulation problem can be solved also in bigraphs - as space and motion extension of graphs, but this is another problem.

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