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Mathematical Properties of the Viscous Rotating Stratified Fluid Counting with Salinity and Heat Transfer in a Layer

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***Abstract*—**A model of the mathematical fluid dynamics which describes the motion of a three-dimensional viscous rotating fluid in a homogeneous gravitational field with the consideration of the salinity and heat transfer is considered in a vertical finite layer. The model is a generalization of the linearized Navier-Stokes system with the addition of the Coriolis parameter and the equations for changeable density, salinity, and heat transfer. An explicit solution is constructed and the proof of the existence and uniqueness theorems is given. The localization and the structure of the spectrum of inner waves is also investigated. The results may be used, in particular, for constructing stable numerical algorithms for solutions of the considered models of fluid dynamics of the Atmosphere and the Ocean.

***Keywords*—**Fourier transform, generalized solutions, Navier-Stokes equations, stratified fluid.

I. Introduction

LET us consider a bounded domain $\Omega \subset R^3$ and the following system of fluid dynamics

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 - \nu \Delta u_1 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 - \nu \Delta u_2 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} - \nu \Delta u_3 + \dfrac{\partial p}{\partial x_3} - \alpha_1 \rho + \alpha_2 W = 0 \\ \operatorname{div} \vec{u} = 0 \\ \dfrac{\partial \rho}{\partial t} + \alpha_3 u_3 = 0 \\ \dfrac{\partial W}{\partial t} - \nu \Delta W + \alpha_4 u_3 = 0 \qquad x \in \Omega, \quad t \geq 0. \end{cases}$$

Here $\vec{u} = (u_1, u_2, u_3)$ is a velocity field, $p(x,t)$ is the scalar field of the dynamic pressure, $\rho(x,t)$ is the dynamic density of the fluid, $W(x,t)$ is either dynamic salinity or dynamic temperature, $\omega = \text{Const}$ is the Coriolis parameter, and α_i , $i = 1,...4$ are constant nonzero stratification parameters.

For the kinematic viscosity coefficient ν ,we assume $\nu > 0$.

The considered equations are deduced, for example, in [1]. The study of mathematical properties of different systems of fluid dynamics of rotating fluid was started in [2]-[4]. Various problems involving the spectrum of normal vibrations for stratified and rotating fluid were considered in [5]-[10]. For non-linear model considered in bounded domains, but without the equations for salinity and heat transfer, the solution of similar systems was studied in [11]. We can observe that, for some problems of Ocean and Atmosphere dynamics, particularly for the cases when the horizontal dimensions are considerably larger than vertical dimensions, the third equation of the previous system does not contain the terms $\dfrac{\partial u_3}{\partial t}$ and Δu_3 (see, for example, [12]). Therefore, we will consider the system

$$\begin{cases} \dfrac{\partial v_1}{\partial t} - \omega v_2 - \nu \Delta v_1 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial v_2}{\partial t} + \omega v_1 - \nu \Delta v_2 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial p}{\partial x_3} - \alpha_1 v_4 + \alpha_2 v_5 = 0 \\ \operatorname{div} \vec{v} = 0 \\ \dfrac{\partial v_4}{\partial t} + \alpha_3 v_3 = 0 \\ \dfrac{\partial v_5}{\partial t} - \nu \Delta v_5 + \alpha_4 v_3 = 0 \qquad x \in \Omega, \quad t > 0 \end{cases} \tag{1}$$

in the domain

$$Q = \Omega \times \{t > 0\}, \; \Omega = \{x = (x', x_3) = (x_1, x_2, x_3), \; x' \in R^2, \; 0 < x_3 < h\}.$$

We will consider the initial conditions

$$v_i\big|_{t=0} = v_i^0(x) \; , \; i = 1,2,4,5 \tag{2}$$

and boundary value conditions

$$\left.\frac{\partial v_i}{\partial x_3}\right|_{\substack{x_3=0 \\ x_3=h}} = 0 \; , \; i = 1,2 \; ; \; v_i\big|_{\substack{x_3=0 \\ x_3=h}} = 0 \; , \; i = 3,4,5 \; . \tag{3}$$

II. Problem Formulation

Our primary aim is to construct the solution of the problem

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(1)-(3). The general idea of construction of such solution in a layer is taken from [14].

We will use the Laplace transform with respect to t, the Fourier transform with respect to x' and finite integral transforms with respect to x_3. We apply the Cosine-Fourier transform to the first, the second and the fourth equations of (1), and the Sine-Fourier transform to the rest of the equations. For that purpose, we multiply the first, the second and the fourth equations by $\cos\lambda_n x_3$, the rest of the equations we multiply by $\sin\lambda_n x_3$, and integrate with respect to x_3 on the interval $0 < x_3 < h$. Let us introduce the following notations:

$$\lambda_n = \pi n / h,$$
$$(\hat{v}_i, \hat{p})(x', n, t) = \int_0^h (v_i, p)(x', x_3, t)\cos\lambda_n x_3 dx_3,\ i = 1,2,$$
$$(\hat{v}_3, \hat{v}_4, \hat{v}_5)(x', n, t) = \int_0^h (v_3, v_4, v_5)(x', x_3, t)\sin\lambda_n x_3 dx_3,$$
$$(\hat{v}_i, \hat{v}_4, \hat{v}_5)(x', n, t)\big|_{t=0} = (\hat{v}_i^0, \hat{v}_4^0, \hat{v}_5^0)(x', n,)\ ,\ i = 1,2.$$

Using the boundary value conditions (3), we transform the problem (1)-(3) into the following:

$$\begin{cases} \dfrac{\partial \hat{v}_1}{\partial t} - \omega\hat{v}_2 - \nu\Delta_2\hat{v}_1 + \nu\lambda_n^2\hat{v}_1 + \dfrac{\partial\hat{p}}{\partial x_1} = 0 \\ \dfrac{\partial \hat{v}_2}{\partial t} + \omega\hat{v}_1 - \nu\Delta_2\hat{v}_2 + \nu\lambda_n^2\hat{v}_2 + \dfrac{\partial\hat{p}}{\partial x_2} = 0 \\ -\lambda_n\hat{p} = \alpha_1\hat{v}_4 + \alpha_2\hat{v}_5 \\ \dfrac{\partial \hat{v}_1}{\partial x_1} + \dfrac{\partial \hat{v}_2}{\partial x_2} + \lambda_n\hat{v}_3 = 0 \\ \dfrac{\partial \hat{v}_4}{\partial t} + \alpha_3\hat{v}_3 = 0 \\ \dfrac{\partial \hat{v}_5}{\partial t} - \nu\Delta_2\hat{v}_5 + \nu\lambda_n^2\hat{v}_5 + \alpha_4\hat{v}_3 = 0, \end{cases} \quad (4)$$

$$(\hat{v}_i, \hat{v}_4, \hat{v}_5)(x', n, t)\big|_{t=0} = (\hat{v}_i^0, \hat{v}_4^0, \hat{v}_5^0)(x', n,)\ ,\ i = 1,2. \quad (5)$$

To solve the problem (4), (5), we assume that the initial conditions are sufficiently smooth and rapidly decreasing functions for $|x'| \to \infty$, which allows us to apply the Fourier transform in x' and Laplace transform in t.

After introducing the notations

$$F_{x'\to\xi'}L_{t\to\lambda}\left[\hat{v}, \hat{p}, \hat{v}_4, \hat{v}_5\right](x', n, t) = L_{t\to\lambda}\left[\overline{v}, \overline{p}, \overline{v}_4, \overline{v}_5\right](\xi', n, t) =$$
$$= (\tilde{v}, \tilde{p}, \tilde{v}_4, \tilde{v}_5)(\xi', n, \lambda),$$
$$F_{x'\to\xi'}\left[\hat{v}_i^0, \hat{v}_4^0, \hat{v}_5^0\right](x', n) = (\tilde{v}_i^0, \tilde{v}_4^0, \tilde{v}_5^0)(\xi', n)\ ,\ i = 1,2,$$

we obtain the system of algebraic equations

$$\begin{aligned} &\left(\lambda + \nu|\xi'|^2 + \nu\lambda_n^2\right)\tilde{v}_1 - \omega\tilde{v}_2 + i\xi_1\tilde{p} = \tilde{v}_1^0 \\ &\left(\lambda + \nu|\xi'|^2 + \nu\lambda_n^2\right)\tilde{v}_2 + \omega\tilde{v}_1 + i\xi_2\tilde{p} = \tilde{v}_2^0 \\ &\lambda_n\tilde{p} + \alpha_1\tilde{v}_4 + \alpha_2\tilde{v}_5 = 0 \\ &i\xi_1\tilde{v}_1 + i\xi_2\tilde{v}_2 + \lambda_n\tilde{v}_3 = 0 \\ &\lambda\tilde{v}_4 + \alpha_3\tilde{v}_3 = \tilde{v}_4^0 \\ &\left(\lambda + \nu|\xi'|^2 + \nu\lambda_n^2\right)\tilde{v}_5 + \alpha_4\tilde{v}_3 = \tilde{v}_5^0. \end{aligned} \quad (6)$$

Let us introduce the functions

$$\overline{\Psi}_i(\xi', n, \lambda) = \frac{R^i}{\Delta}\ ,\ i = 0,1,2, \quad (7)$$

where

$$R = \lambda + \nu|\xi'|^2 + \nu\lambda_n^2,$$
$$\Delta = R\left(\lambda_n^2 R^2 + \omega^2\lambda_n^2 + \gamma|\xi'|^2\right),$$
$$\gamma = \alpha_1\alpha_3 + \alpha_2\alpha_4.$$

From (7), we can represent the inverse Laplace transform for the functions $\overline{\Psi}_i$ as follows.

$$\Psi_0(\xi', n, t) = \frac{2e^{-\nu\left(|\xi'|^2 + \lambda^2{}_n\right)t}}{\Lambda^n}\sin^2\left(\frac{\Lambda t}{2\lambda_n}\right),$$
$$\Psi_1(\xi', n, t) = \frac{2e^{-\nu\left(|\xi'|^2 + \lambda^2{}_n\right)t}}{\lambda_n\Lambda}\sin\left(\frac{\Lambda t}{\lambda_n}\right),$$
$$\Psi_2(\xi', n, t) = \frac{2e^{-\nu\left(|\xi'|^2 + \lambda^2{}_n\right)t}}{\lambda_n^2}\cos\left(\frac{\Lambda t}{\lambda_n}\right),$$
$$\Lambda = \sqrt{\omega^2\lambda_n^2 + \gamma|\xi'|^2}.$$

For the following, we assume $v_i^0 \in W_1^4(\Omega)$, $i = 1,2,4,5,$

$$\int_0^h \left[\frac{\partial v_1^0}{\partial x_1} + \frac{\partial v_2^0}{\partial x_2}\right] dx_3 = 0,$$

We also suppose that the condition of consistency of the initial data and boundary values is fulfilled.

After solving (6) and applying the inverse Fourier and Laplace transforms $F_{\xi'\to x'}^{-1}L_{\lambda\to t}^{-1}$, we can represent the solution of the problem (4)-(5) as

$$\hat{v}_k(x', n, t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left\{\tilde{v}_k^0 e^{Ht} - \left(\gamma\xi_k^2 + \omega^2\lambda_n^2\right)\tilde{v}_k^0\Psi_0 - \right.$$
$$-(-1)^k\left[\lambda_n^2\omega\Psi_1 + (-1)^k\xi_1\xi_2\gamma\Psi_0\right]\tilde{v}_{3-k}^0 +$$
$$\left. +\lambda_n\left[i\xi_k\Psi_1 - (-1)^k i\xi_{3-k}\omega\Psi_0\right]\tilde{U}_3^0\right\} d\xi'\ k = 1,2,$$

$$\hat{v}_3(x',n,t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\begin{bmatrix}\lambda_n\left(\omega\tilde{U}_2^0\Psi_1 - \tilde{U}_1^0\Psi_2\right) + \\ +|\xi'|^2\tilde{U}_3^0\Psi_1\end{bmatrix} d\xi',$$

$$\hat{p}(x',n,t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left[\gamma\left(\omega\tilde{U}_2^0\Psi_0 - \tilde{U}_1^0\Psi_1\right) - \right.$$
$$\left. -\lambda_n\left(\omega^2\Psi_0 + \Psi_2\right)\tilde{U}_3^0\right]d\xi',$$

$$\hat{v}_4(x',n,t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left[\tilde{v}_4{}^0 e^{Ht} + |\xi'|^2\Psi_0\left(\alpha_4\tilde{U}_4^0 - \gamma\tilde{v}_4{}^0\right) + \right.$$
$$\left. +\alpha_3\lambda_n\left(\tilde{U}_1^0\Psi_1 - \omega\tilde{U}_2^0\Psi_0\right)\ \right]d\xi',$$

$$\hat{v}_5(x',n,t) = \frac{1}{(2\pi)^2}\iint_{R^2} e^{i(x',\xi')}\left[\tilde{v}_5{}^0 e^{Ht} - |\xi'|^2\Psi_0\left(\alpha_3\tilde{U}_4^0 + \gamma\tilde{v}_5{}^0\right) + \right.$$
$$\left. +\alpha_4\lambda_n\left(\tilde{U}_1^0\Psi_1 - \omega\tilde{U}_2^0\Psi_0\right)\ \right]d\xi',$$

where

$$\tilde{U}_1^0(\xi',n) = i\xi_1\tilde{v}_1^0 + i\xi_2\tilde{v}_2^0\ ,\ \tilde{U}_2^0(\xi',n) = i\xi_1\tilde{v}_2^0 - i\xi_2\tilde{v}_1^0,.$$
$$\tilde{U}_3^0(\xi',n) = \alpha_3\tilde{v}_4{}^0 + \alpha_4\tilde{v}_5{}^0\ ,\ \tilde{U}_4^0(\xi',n) = \alpha_4\tilde{v}_4{}^0 - \alpha_3\tilde{v}_5{}^0,$$
$$H = -\nu\left(|\xi'|^2 + \lambda_n^2\right).$$

In this way, the solution of the problem (1)-(3) can be represented as follows ([13]):

$$(v_i,p)(x,t) = \frac{1}{h}(\hat{v}_i,\hat{p})(x',0,t) + \frac{2}{h}\sum_{n=1}^{\infty}(\hat{v}_i,\hat{p})(x',n,t)\cos(\lambda_n x_3),$$
$$i = 1,2,$$
$$(v_3,v_4,v_5)(x,t) = \frac{2}{h}\sum_{n=1}^{\infty}(\hat{v}_3,\hat{v}_4,\hat{v}_5)(x',n,t)\sin(\lambda_n x_3). \tag{8}$$

We denote $Q_\tau = \Omega\times\{0<t<\tau\}$,

$$\vec{U}^0(x',x_3) = \left(v_1^0,v_2^0,v_4^0,v_5^0\right)(x',x_3),\ \|f\|_k = \|f\|_{W_1^k(\Omega)},$$
$$\hat{V}(Q_\tau) = \{v_i \in C([0,\tau],L_2(\Omega)) \cap L_2\left((0,\tau),W_2^1(\Omega)\right),\ i=1,2,$$
$$v_3 \in L_2\left((0,\tau),\overset{0}{W}{}_2^1(\Omega)\right),\ \mathrm{div}\vec{v} = 0\ ,$$
$$v_i \in C([0,\tau],L_2(\Omega)) \cap L_2\left((0,\tau),\overset{0}{W}{}_2^1(\Omega)\right),\ i=4,5\ \},$$
$$V(Q_\tau) = \{(\vec{v},v_4,v_5) \in \hat{V}(Q_\tau) : D_t v_i \in L_2(Q_\tau)\ ,\ i=1,2,4,5\},$$
$$\mathrm{A}_1 = \frac{\alpha_1}{\alpha_3}\ ,\ \mathrm{A}_2 = \frac{\alpha_2}{\alpha_4}.$$

We define a *strong solution* of the problem (1)-(3) as a system of the functions $\{\vec{v},p,v_4,v_5\}$ such that

$$v_i \in C_{x,t}^{2,1}(Q) \cap C_{x,t}^{1,0}(\bar{Q})\ ,\ i=1,2,\ p \in C_{x,t}^{1,0}(Q),$$
$$v_3 \in C_{x,t}^{1,0}(Q) \cap C(\bar{Q})\ ,\ v_i \in C_{x,t}^{2,1}(Q) \cap C(\bar{Q})\ ,\ i=4,5$$

satisfy (1) and the conditions (2), (3).

We define a *weak solution* of the problem (1)-(3) as a system of the functions $\{\vec{v},v_4,v_5\} \in V(Q_\tau)$ which satisfy the condition (2) and the integral identity

$$\int_{Q_\tau}\{\sum_{i=1}^{2}\frac{\partial v_i}{\partial t}\Phi_i + \mathrm{A}_1\frac{\partial v_4}{\partial t}\Phi_4 + \mathrm{A}_2\frac{\partial v_5}{\partial t}\Phi_5 + \nu\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial \Phi_i}{\partial x_j} +$$
$$+\nu\sum_{i=1}^{2}(\mathrm{A}_1\frac{\partial v_4}{\partial x_i}\frac{\partial \Phi_4}{\partial x_i} + \mathrm{A}_2\frac{\partial v_5}{\partial x_i}\frac{\partial \Phi_5}{\partial x_i}) + \nu(\mathrm{A}_1\frac{\partial v_4}{\partial x_3}\frac{\partial \Phi_4}{\partial x_3} + \mathrm{A}_2\frac{\partial v_5}{\partial x_3}\frac{\partial \Phi_5}{\partial x_3}) +$$
$$+\omega(v_1\Phi_2 - v_2\Phi_1) + \alpha_1(v_3\Phi_4 - v_4\Phi_3) + \alpha_2(v_3\Phi_5 - v_5\Phi_3)\}dxdt = 0$$

for all $t\in[0,\tau]$ and for every vector function

$$\vec{\Phi}(x,t) = (\Phi_i)_{i=1}^5 \in \hat{V}(Q_\tau).$$

Our aim now is to study the properties of existence and uniqueness of the strong and weak solutions for (1)-(3).

III. Problem Solution

Theorem 1 The system of functions (8) defines a strong solution of the problem (1)-(3).

Proof. Evidently, it is sufficient to show that the series (8) converge uniformly with respect to *x* and *t*, together with their term-by-term derivatives in *x* and *t* , and that the initial conditions (2) are satisfied. Let us investigate the first component of the solution, since the rest of the components are analogous. For $|\alpha|\le 2$, $t \ge t_0 > 0$, the derivatives of the series which define $v_1(x,t)$, are estimated in the following way:

$$\left|D^\alpha\hat{v}_1(x',n,t)\cos\lambda_n x_3\right| \le C_0 n^{\alpha_3}\int_{R^2} e^{-\nu t\left(|\xi'|^2+\lambda_n^2\right)}|\xi'|^{|\alpha'|}\{|\tilde{v}_1^0| + |\tilde{v}_2^0| +$$
$$+|\tilde{v}_4^0| + |\tilde{v}_5^0|\}d\xi' \le Cn^{\alpha_3}t_0^{-\left(1+\frac{|\alpha'|}{2}\right)}e^{-\nu\lambda_n^2 t_0}\left\|\vec{U}^0\right\| = C_1 n^{\alpha_3}e^{-\nu\lambda_n^2 t_0}. \tag{9}$$

Similarly,

$$\left|D_t\hat{v}_1(x',n,t)\right| \le C\int_{R^2} e^{-\nu t\left(|\xi'|^2+\lambda_n^2\right)}\left(|\xi'|^2 + \lambda_n^2 + \frac{|\xi'|}{\lambda_n}\right)\{|\tilde{v}_1^0| + |\tilde{v}_2^0| +$$
$$+|\tilde{v}_4^0| + |\tilde{v}_5^0|\}d\xi' \le C_1\left(1+n^2\right)e^{-\nu\lambda_n^2 t_0}\left\|\vec{U}^0\right\|. \tag{10}$$

We observe that the constants C_1 in (9) and (10), in general, depend on t_0 . Due to the arbitrary choice of $t_0 > 0$, it follows from (9), (10), that the series (8) converge uniformly in *x* and *t*, together with the series obtained as a result of term-by-term differentiation with respect to *x* and *t*.

Let us prove that $v_1(x,t)$ satisfies the initial condition (2). For that, we represent the general term of the series as follows.

$$\frac{2-\delta_{n,0}}{h}\hat{v}_1\left(x',n,t\right)=\sum_{k=1}^{2}\hat{v}_{1,k}\left(x',n,t\right),$$
$$\hat{v}_{1,1}\left(x',n,t\right)=\frac{2-\delta_{n,0}}{h}\frac{1}{(2\pi)^2}\int_{R^2}e^{i(x',\xi')}e^{-\nu t\left(|\xi'|^2+\lambda_n^2\right)}\tilde{v}_1^0\left(\xi',n\right)d\xi'=$$
$$=\frac{2-\delta_{n,0}}{h}\left[\left(\int_0^h v_1^0\left(x',x_3\right)\cos\lambda_n x_3 dx_3\right)\underset{R^2}{*}G\left(x',t\right)\right]e^{-\nu\lambda_n^2 t},$$

where $\delta_{i,j}$ is the Kronecker symbol and $G(x',t)$ is the singular solution of the heat transfer equation.

Since

$$\lim_{t\to 0}\hat{v}_{1,1}\left(x',n,t\right)=\frac{2-\delta_{n,0}}{h}\hat{v}_1^0\left(x',n\right)$$

uniformly in $x'\in R^2$, then

$$\lim_{t\to 0}\sum_{n=0}^{\infty}\hat{v}_{1,1}\left(x',n,t\right)\cos\lambda_n x_3=\sum_{n=0}^{\infty}\frac{2-\delta_{n,0}}{h}\hat{v}_1^0\left(x',n\right)\cos\lambda_n x_3=v_1^0\left(x\right).\quad(11)$$

To estimate the term $\hat{v}_{1,2}$ for $t\le t_0$, we use the explicit form of Ψ_i, the inequalities $|\sin\alpha|\le\alpha$, $x^\beta e^{-\varepsilon x}\le C, x\ge 0,\beta\ge 0,\varepsilon>0$, and the estimates

$$\left|\tilde{W}\left(\xi',n\right)\right|\le\frac{C}{n\left(1+|\xi'|^\alpha\right)}\|W\|_3,$$

where $\tilde{W}\left(\xi',n\right)$ is any of the functions $\tilde{v}_i^0\left(\xi',n\right)$, $i=1,2,4,5$, and $W(x)$ is any of the functions $v_i^0(x)$, $i=1,2,4,5$.
In this way, we obtain

$$\left|\hat{v}_{1,2}\left(x',n,t\right)\right|\le C\int_{R^2}e^{-\nu t\left(|\xi'|^2+\lambda_n^2\right)}\{\frac{t^2|\xi'|^2}{n^2}\left|\tilde{v}_1^0\right|+\frac{t|\xi'|^2}{n^2}\left|\tilde{v}_2^0\right|+$$
$$+\frac{t|\xi'|}{n}\left(\left|\tilde{v}_4^0\right|+\left|\tilde{v}_5^0\right|\right)\}d\xi'\le C_1\left\|\vec{U}^0\right\|_3\frac{t^{1/4}}{n^2}\int_{R^2}\frac{d\xi'}{|\xi'|^{1/2}\left(1+|\xi'|^2\right)}=\frac{C_2 t^{1/4}}{n^2}.$$

From the last inequality, the relation (11), and from the representation $v_1(x,t)=\sum_{n=0}^{\infty}\left[\hat{v}_{1,1}^0(x',n,t)+\hat{v}_{1,2}^0(x',n,t)\right]\cos\lambda_n x_3$, it follows that, for the function $v_1(x,t)$, the initial conditions (2) are satisfied, which completes the proof.

Theorem 2 The weak solution of the problem (1)-(3), is unique.

Proof. Let $\left(\vec{v},v_4,v_5\right)$ be a weak solution of the problem (1)-(3) for

$$v_i^0\left(x\right)=0,\ i=1,2,4,5.$$

Our aim is to verify that $v_i\left(x,t\right)=0$, $i=1,2,3,4,5.$

We take $\left(\vec{v},v_4,v_5\right)$ as test functions Φ_i. In this way, we obtain

$$\frac{1}{2}\int_\Omega\left(\sum_{i=1}^{2}v_i^2+\mathrm{A}_1v_4^2+\mathrm{A}_2v_5^2\right)dx+\int_{Q_\tau}\{\nu\sum_{i=1}^{2}\sum_{j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2+$$
$$+\nu\sum_{i=1}^{2}(\mathrm{A}_1\left(\frac{\partial v_4}{\partial x_i}\right)^2+\mathrm{A}_2\left(\frac{\partial v_5}{\partial x_i}\right)^2)+\quad(12)$$
$$+\nu(\mathrm{A}_1\left(\frac{\partial v_4}{\partial x_3}\right)^2+\mathrm{A}_2\left(\frac{\partial v_5}{\partial x_3}\right)^2)\}dxdt=0.$$

It follows from (12) that $\frac{\partial v_4}{\partial x_i}=\frac{\partial v_5}{\partial x_i}=0$, $1\le i\le 3$, which implies $v_4\left(x,t\right)=v_5\left(x,t\right)=0$, due to the boundary conditions.

Additionally, it follows from (2) that

$$\frac{\partial v_i}{\partial x_j}=0,\ \int_\Omega\sum_{i=1}^{2}v_i^2dx=0\ \text{ for all } t\in[0,\tau]\ ;\ i=1,2,1\le j\le 3,$$

which implies $v_i\left(x,t\right)=0$, $i=1,2$. From the equation of continuity, we have that $\frac{\partial v_3}{\partial x_3}=0$. Therefore, $v_3\left(x,t\right)=\varphi(x',t)$, and from the boundary conditions, we finally obtain that $v_3(x,t)=0$, and thus, the theorem is proved.

Theorem 3 The strong solution of the problem (1)-(3), is unique and belongs to the class $V\left(Q_\tau\right)$.

Proof. Let us consider the component $v_1(x,t)$ of the solution. Using the Parseval formula and the explicit representation (8), we have

$$\left\|v_1\left(x,t\right)\right\|_{L_2(\Omega)}^2=\frac{1}{h}\left(\left\|\hat{v}_1\left(x',0,t\right)\right\|_{L_2(\Omega)}^2+2\sum_{n=1}^{\infty}\left\|\hat{v}_1\left(x',n,t\right)\right\|_{L_2(\Omega)}^2\right)=$$
$$=\frac{(2\pi)^2}{h}\left(\left\|\overline{v}_1\left(\xi',0,t\right)\right\|_{L_2(\Omega)}^2+2\sum_{n=1}^{\infty}\left\|\overline{v}_1\left(\xi',n,t\right)\right\|_{L_2(\Omega)}^2\right).$$

Let us estimate the general term of the last series. With the help of the obvious inequality $(a+b)^2\le 2\left(a^2+b^2\right)$ and the explicit form of the functions Ψ_i, we obtain

$$\left\|\overline{v}_1\left(\xi',n,t\right)\right\|_{L_2}^2\le C\int_{R^2}e^{-2\nu t\left(|\xi'|^2+\lambda_n^2\right)}\{\left|\tilde{v}_1^0\right|^2+\left|\tilde{v}_2^0\right|^2+\left|\tilde{v}_4^0\right|^2+\left|\tilde{v}_5^0\right|^2\}d\xi'.$$

From the last relation and the proof of Theorem 1, we have

$$\left\|\overline{v}_1\left(\xi',n,t\right)\right\|_{L_2}^2\le\left\|\vec{U}^0\right\|_3\frac{C_1}{n^2}\int_{R^2}\frac{d\xi'}{\left(1+|\xi'|^2\right)^2}=\frac{C_2}{n^2},$$

which implies that $v_1(x,t) \in C([0,\tau], L_2(\Omega))$.

Let $\Pi = R_2 \times \{0 < t < \tau\}$. Analogously, for $|\alpha| \le 1$, we obtain

$$\left\|D_x^{\alpha} v_1(x,t)\right\|_{L_2(Q_\tau)}^2 = \frac{1}{h}\left(\left(1-\delta_{1,\alpha_3}\right)\left\|D_{x'}^{\alpha'} \hat{v}_1(x',0,t)\right\|_{L_2(\Pi)}^2 + 2\sum_{n=1}^{\infty}\left\|\lambda_n^{\alpha_3} D_{x'}^{\alpha'} \hat{v}_1(x',n,t)\right\|_{L_2(\Pi)}^2\right) =$$

$$= \frac{(2\pi)^2}{h}\left(\left(1-\delta_{1,\alpha_3}\right)\left\|(i\xi')^{\alpha'} \bar{v}_1(\xi',0,t)\right\|_{L_2(\Pi)}^2 + 2\sum_{n=1}^{\infty}\left\|(i\xi')^{\alpha'} \lambda_n^{\alpha_3} \bar{v}_1(\xi',n,t)\right\|_{L_2(\Pi)}^2\right).$$

Due to the inclusion property $W_1^4(\Omega) \to W_2^2(\Omega)$, the general term of the series may be estimated as follows:

$$\left\|(i\xi')^{\alpha'} \lambda_n^{\alpha_3} \bar{v}_1(\xi',n,t)\right\|_{L_2(\Pi)}^2 \le C\int_0^\tau \int_{R^2} |\xi'|^{2|\alpha'|} e^{-2\nu t\left(|\xi'|^2+\lambda_n^2\right)} \{\sum_{i=1,2,4,5} |\tilde{v}_i^0|^2\} d\xi' dt =$$

$$C\int_{R^2} \frac{|\xi'|^{2|\alpha'|}\lambda_n^{2\alpha_3}}{2\nu\left(|\xi'|^2+\lambda_n^2\right)}\left(-e^{-2\nu t\left(|\xi'|^2+\lambda_n^2\right)}\Big|_{t=0}^{t=\tau}\right)\left\{\sum_{i=1,2,4,5}|\tilde{v}_i^0|^2\right\} d\xi' \le$$

$$\le C_1\int_{R^2} \sum_{i=1,2,4,5} |\tilde{v}_i^0|^2 d\xi' \le \frac{C_2}{n^2}\left\|U^0\right\|_{W_2^1(\Omega)}^2 = \frac{C_3}{n^2}.$$

Therefore, we have obtained that $v_1(x,t) \in L_2\left((0,\tau), W_2^1(\Omega)\right)$.

Repeating the same reasoning, we verify that the derivatives $D_t v_1(x,t)$ belong to the functional space $L_2(Q_\tau)$. Thus, we obtain that $v_1(x,t) \in V(Q_\tau)$. The rest of the components for the solutions are estimated analogously. The uniqueness of the solution follows from Theorem 2. In this way, the theorem is proven.

Now, let us consider the initial system of fluid dynamics for compressible fluid

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 - \nu\Delta u_1 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 - \nu\Delta u_2 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} - \nu\Delta u_3 + \dfrac{\partial p}{\partial x_3} - \alpha_1\rho + \alpha_2 W = 0 \\ \alpha^2 \dfrac{\partial p}{\partial t} + \operatorname{div}\vec{u} = 0 \\ \dfrac{\partial \rho}{\partial t} + \alpha_3 u_3 = 0 \\ \dfrac{\partial W}{\partial t} - \nu\Delta W + \alpha_4 u_3 = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases} \tag{13}$$

in a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class C^1. We associate system (13) to the boundary conditions

$$\vec{u}\cdot\vec{n}\big|_{\partial\Omega} = 0 \tag{14}$$

where $\vec{n}$ is the exterior normal to the surface $\partial\Omega$. Let us consider the following problem of normal vibrations

$$\begin{aligned} \vec{u}(x,t) &= \vec{v}(x)e^{-\lambda t} \\ p(x,t) &= \frac{1}{\alpha} v_4(x)e^{-\lambda t} \\ \rho(x,t) &= v_5(x)e^{-\lambda t} \\ W(x,t) &= v_6(x)e^{-\lambda t} \quad , \quad \lambda \in C. \end{aligned} \tag{15}$$

We denote $\tilde{v} = (\vec{v}, v_4, v_5, v_6)$ and write the system (13) in the matrix form

$$L\tilde{v} = 0 \tag{16}$$

where

$$L = M - \lambda I$$

and

$$M = \begin{pmatrix} -\nu\Delta & -\omega & 0 & \dfrac{1}{\alpha}\dfrac{\partial}{\partial x_1} & 0 & 0 \\ \omega & -\nu\Delta & 0 & \dfrac{1}{\alpha}\dfrac{\partial}{\partial x_2} & 0 & 0 \\ 0 & 0 & -\nu\Delta & \dfrac{1}{\alpha}\dfrac{\partial}{\partial x_3} & -\alpha_1 & \alpha_2 \\ \dfrac{1}{\alpha}\dfrac{\partial}{\partial x_1} & \dfrac{1}{\alpha}\dfrac{\partial}{\partial x_2} & \dfrac{1}{\alpha}\dfrac{\partial}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & -\nu\Delta \end{pmatrix}. \tag{17}$$

We define the domain of the differential operator *M* with the boundary condition (14) as follows.

$$D(M) = \left\{ \begin{array}{l} \vec{v} \in \left(\overset{0}{W}{}_2^1(\Omega)\right)^3, v_5 \in \overset{0}{W}{}_2^1(\Omega), v_6 \in \overset{0}{W}{}_2^1(\Omega) \\ v_4 \in L_2(\Omega) \,:\, M\tilde{v} \in \left(L_2(\Omega)\right)^6 \end{array} \right\}.$$

The consideration of the separated variables of the form (15) permits to interpret every non-stationary process as a linear sum of the normal oscillations. The spectrum of inner vibrations may be used for investigating the properties of the stability of the solutions. As well, the spectral properties of M may be used in the studying of weakly non-linear flows, since the points of bifurcation are the points of the spectrum of the operator M.

We observe that the above defined operator M is a closed operator, and its domain is dense in $\left(L_2(\Omega)\right)^6$.

Let us denote by $\sigma_{ess}(N)$ the essential spectrum of a closed linear operator *N*. We recall that, according to the definition of the essential spectrum [15], [16],

$$\sigma_{ess}(N) = \left\{\lambda \in C : (N - \lambda I) \text{ is not of Fredholm type}\right\},$$

it consists of the eigenvalues of infinite multiplicity, limit points of the point spectrum, and the points of the continuous spectrum.

Therefore, the spectral points outside of the essential spectrum, are eigenvalues of finite multiplicity. For calculating the essential spectrum of *M*, we would like to refer to the property [17]:

$$\sigma_{ess}(M) = Q \cup S ,$$

where

$$Q = \begin{Bmatrix} \lambda \in C : (M - \lambda I) \text{is not elliptic} \\ \text{in sense of Douglis-Nirenberg} \end{Bmatrix}$$

and

$$S = \begin{Bmatrix} \lambda \in C \setminus Q : \text{the boundary conditions of } (M - \lambda I) \\ \text{do not satisfy Lopatinski conditions} \end{Bmatrix}.$$

Theorem 4 The essential spectrum of the operator M is composed of one real point $\sigma_{ess}(M) = \left\{ \frac{1}{\nu\alpha^2} \right\}$.

Proof. We observe that, according to the definition of the ellipticity in sense of Douglis-Nirenberg [18], the main symbol of the operator $L = M - \lambda I$ will be expressed as:

$$\tilde{L}(\xi) = \begin{pmatrix} -\nu|\xi|^2 & 0 & 0 & \frac{1}{\alpha}\xi_1 & 0 & 0 \\ 0 & -\nu|\xi|^2 & 0 & \frac{1}{\alpha}\xi_2 & 0 & 0 \\ 0 & 0 & -\nu|\xi|^2 & \frac{1}{\alpha}\xi_3 & 0 & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -\nu|\xi|^2 \end{pmatrix}.$$

We calculate the determinant of the last matrix:

$$\det\left(\widetilde{M - \lambda I}\right)(\xi) = \frac{\nu^3}{\alpha^2}|\xi|^8\left(1 - \nu\lambda\alpha^2\right),$$

and thus, we can see that for only one point $\lambda = \frac{1}{\nu\alpha^2}$ the operator $L = M - \lambda I$ is not elliptic in sense of Douglis-Nirenberg. Now, we will show, additionally, that the conditions of Lopatinski [17] are satisfied.

The boundary condition (14) can be written in a matrix form

$$G\tilde{v}\big|_{\partial\Omega} = 0,\ G = \begin{pmatrix} n_1 & n_2 & n_3 & 0 & 0 & 0 \end{pmatrix}.$$

If we denote $\tilde{\xi} = (\xi_1, \xi_2)$, $\xi_3 = \tau$; then

$$\det\left(\widetilde{M - \lambda I}\right)\left(\tilde{\xi}, \tau\right) = \frac{\nu^3}{\alpha^2}\left(\left|\tilde{\xi}\right|^2 + \tau^2\right)^4\left(1 - \nu\lambda\alpha^2\right),$$

and thus, the equation $\det\left(\widetilde{M - \lambda I}\right)\left(\tilde{\xi}, \tau\right) = 0$ for $\lambda \neq \frac{1}{\nu\alpha^2}$ has one root $\tau = i\left|\tilde{\xi}\right|$ of multiplicity four in the upper half of the complex plane.

In this way, $M^+\left(\tilde{\xi}, \tau\right) = \left(\tau - i\left|\tilde{\xi}\right|\right)^4$. Since the elements of the matrices $\widetilde{M - \lambda I}$ and *G* are homogeneous functions with respect to $\tilde{\xi}, \tau$, then it is sufficient to verify the Lopatinski conditions for unitary vectors $\tilde{\xi}$. Let us choose a local system of coordinates so that $\xi_1 = 1$, $\xi_2 = 0$.

For the matrix $\left(\widetilde{M - \lambda I}\right)$, we construct first the adjoint matrix $(M - \lambda I)^*$, then we multiply $(M - \lambda I)^*$ by the boundary conditions matrix *G* and thus obtain the following.

$$G(M - \lambda I)^* = \left(n_1 B^3\left[B\lambda + \frac{\tau^2}{\alpha^2}\right], 0, -n_3 B^3\frac{\tau}{\alpha^2}, 0, 0, 0\right),$$

where $B = -\nu\left(1 + \tau^2\right)$.

Since $G(M - \lambda I)^*$ is a vector row, then evidently, the Lopatinski conditions are satisfied, and thus, the theorem is proved.

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