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Forced Vibration of a Planar Curved Beam on Pasternak Foundation

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***Abstract*—**The objective of this study is to investigate the forced vibration analysis of a planar curved beam lying on elastic foundation by using the mixed finite element method. The finite element formulation is based on the Timoshenko beam theory. In order to solve the problems in frequency domain, the element matrices of two nodded curvilinear elements are transformed into Laplace space. The results are transformed back to the time domain by the well-known numerical Modified Durbin's transformation algorithm. First, the presented finite element formulation is verified through the forced vibration analysis of a planar curved Timoshenko beam resting on Winkler foundation and the finite element results are compared with the results available in the literature. Then, the forced vibration analysis of a planar curved beam resting on Winkler-Pasternak foundation is conducted.

***Keywords*—**Curved beam, dynamic analysis, elastic foundation, finite element method.

I. Introduction

CURVED beams are preferred in many engineering applications due to architectural or structural reasons. Many studies exist in the literature examining the static analysis of curved beams resting on elastic foundation [1]-[9] and the free vibration analysis of curved beams resting on elastic foundation [8], [10]-[17], while the number of researches concerning the forced vibration analysis curved beam on elastic foundation is limited. Celep [18] considered the problem of a thin circular ring resting on a tensionless Winkler foundation. This ring is subjected to time dependent in-plane loads. Çalım [19] investigated the forced vibration problem of Timoshenko curved beam on Winkler foundation subjected to triangle impulsive loading. The rocking influence is also considered. Ordinary differential equations are obtained in Laplace space and solved by using the complementary functions method. The results are transformed to the time space by using Durbin's numerical Laplace inverse transformation algorithm.

The aim of this study is to investigate the dynamic behavior of a planar curved Timoshenko beam resting on elastic foundation (Winkler-Pasternak). The rocking influence of the foundation is also considered. The solution under the triangular impulsive type loading is carried out in Laplace space by using the mixed finite element method. The planar curved beam is discretized by a two-noded curvilinear mixed finite element. Each node of the element contains 12 degrees of freedom. In detail, six displacement type variables involving three translations and three rotations, six stress resultant type variables composed of two shear forces, one axial force, two bending moments, and one torsional moment. The results are transformed back to the time domain numerically by the modified Durbin's transformation algorithm [20]-[22]. First, the verification of the mixed finite element formulation by means of a comparison with results from the literature is carried out for the forced vibration analysis of a planar curved beam on Winkler foundation. Then, the effects of some parameters (e.g. the opening angle of curved beam, the radius of the curved beam to the height of the rectangular cross-section ratio and foundation parameter) on the dynamic behavior of the planar curved beam resting on Winkler-Pasternak foundation are investigated.

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II. Formulation in Laplace Space

A. The Field Equations

The field equations based on the Frenet coordinate system for the isotropic homogenous spatial Timoshenko beam exist in [23] and [24]. As an extension of those formulations, additional terms due to the foundation interaction are involved in the field equations of spatial beam by transforming them to the Laplace space. The field equations become Laplace space

$$
\begin{aligned}
-\bar{\mathbf{T}}_{,s} - \bar{\mathbf{q}} + (\mathbf{k}_W)^T \bar{\mathbf{u}} - (\mathbf{k}_P)^T \bar{\mathbf{u}}_{,ss} + \rho A z^2 \bar{\mathbf{u}} = \mathbf{0} \\
-\bar{\mathbf{M}}_{,s} - \mathbf{t} \times \bar{\mathbf{T}} - \bar{\mathbf{m}} + (\mathbf{k}_R)^T \bar{\mathbf{\Omega}} + \rho \mathbf{I} z^2 \bar{\mathbf{\Omega}} = \mathbf{0} \\
\bar{\mathbf{u}}_{,s} + \mathbf{t} \times \bar{\mathbf{\Omega}} - \bar{\mathbf{C}}_\gamma \bar{\mathbf{T}} = \mathbf{0} \\
\bar{\mathbf{\Omega}}_{,s} - \bar{\mathbf{C}}_\kappa \bar{\mathbf{M}} = \mathbf{0}
\end{aligned}
\qquad (1)
$$

The Laplace transformed variables are denoted by the over bars. z is the Laplace transformation parameter, s denotes the axis parameter along the arc length of the spatial beam. $\bar{\mathbf{u}} = \bar{u}_t\,\mathbf{t} + \bar{u}_n\,\mathbf{n} + \bar{u}_b\,\mathbf{b}$, $\bar{\mathbf{\Omega}} = \bar{\Omega}_t\,\mathbf{t} + \bar{\Omega}_n\,\mathbf{n} + \bar{\Omega}_b\,\mathbf{b}$, and $\bar{\mathbf{T}} = \bar{T}_t\,\mathbf{t} + \bar{T}_n\,\mathbf{n} + \bar{T}_b\,\mathbf{b}$ are the displacement, rotation, and force vectors, respectively. $\bar{\mathbf{M}} = \bar{M}_t\,\mathbf{t} + \bar{M}_n\,\mathbf{n} + \bar{M}_b\,\mathbf{b}$ is the moment vector in the Laplace space, ρ is the density of homogeneous material, A is the area of the cross section, $\mathbf{I} = I_t\,\mathbf{t} + I_n\,\mathbf{n} + I_b\,\mathbf{b}$ is the moment of inertia of the cross section, $\bar{\mathbf{q}}$ and $\bar{\mathbf{m}}$ are the distributed external force and moment vectors in the Laplace space, $\bar{\mathbf{C}}_\gamma$ and $\bar{\mathbf{C}}_\kappa$ are the compliance matrices. $\mathbf{k}_W(k_{Wt}, k_{Wn}, k_{Wb})$ and $\mathbf{k}_P(k_{Pt}, k_{Pn}, k_{Pb})$ are foundation parameter vectors of Winkler and Pasternak, respectively. $\mathbf{k}_R(k_{Rt}, k_{Rn}, k_{Rb})$ is foundation rocking stiffness vector.

B. Functional

The field equations in (1) are written down in operator form as $\mathbf{Q} = \mathbf{Ly} - \mathbf{f}$. After showing the potentiality of the operator, the functional of the structural problem can be obtained in the Laplace space as:

$$\begin{aligned} \mathbf{I}(\bar{\mathbf{y}}) = & -[\bar{\mathbf{u}}, \bar{\mathbf{T}}_{,s}] + [\mathbf{t} \times \bar{\mathbf{\Omega}}, \bar{\mathbf{T}}] - [\bar{\mathbf{M}}_{,s}, \bar{\mathbf{\Omega}}] - \tfrac{1}{2}[\mathbf{C}_\kappa \bar{\mathbf{M}}, \bar{\mathbf{M}}] - \tfrac{1}{2}[\mathbf{C}_\gamma \bar{\mathbf{T}}, \bar{\mathbf{T}}] \\ & + \tfrac{1}{2}\rho A z^2 [\bar{\mathbf{u}}, \bar{\mathbf{u}}] + \tfrac{1}{2}\rho z^2 [\mathbf{I}\bar{\mathbf{\Omega}}, \bar{\mathbf{\Omega}}] - [\bar{\mathbf{q}}, \bar{\mathbf{u}}] - [\bar{\mathbf{m}}, \bar{\mathbf{\Omega}}] \\ & + \tfrac{1}{2}\left[(\mathbf{k}_W)^T \bar{\mathbf{u}}, \bar{\mathbf{u}}\right] + \tfrac{1}{2}\left[(\mathbf{k}_P)^T \bar{\mathbf{u}}_{,s}, \bar{\mathbf{u}}_{,s}\right] + \tfrac{1}{2}\left[(\mathbf{k}_R)^T \bar{\mathbf{\Omega}}, \bar{\mathbf{\Omega}}\right] \\ & + [(\bar{\mathbf{T}} - \hat{\bar{\mathbf{T}}}), \bar{\mathbf{u}}]_\sigma + [(\bar{\mathbf{M}} - \hat{\bar{\mathbf{M}}}), \bar{\mathbf{\Omega}}]_\sigma - \left[\mathbf{k}_P \hat{\bar{\mathbf{u}}}_{,s}, \bar{\mathbf{u}}\right]_\sigma \\ & + [\hat{\bar{\mathbf{u}}}, \bar{\mathbf{T}}]_\varepsilon + [\hat{\bar{\mathbf{\Omega}}}, \bar{\mathbf{M}}]_\varepsilon - \left[\mathbf{k}_P \bar{\mathbf{u}}_{,s}, (\bar{\mathbf{u}} - \hat{\bar{\mathbf{u}}})\right]_\varepsilon \end{aligned} \quad (2)$$

The terms with hats in (2) define the known values on the boundary. The subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively. The details of the variational formulation and functional can be found in [25].

C. Mixed Finite Element Formulation

A two-nodded curved element is employed to discretize the beam domain. The curved element has 2×12 degrees of freedom. Linear shape functions are employed for the interpolation. The curvatures are satisfied exactly at the nodal points and linearly interpolated through the element [24], [25].

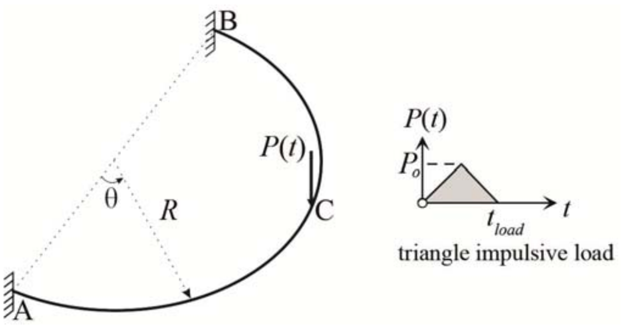


Fig. 1 A planar curved beam and impulsive load

III. Numerical Examples

The forced vibration analysis of a planar curved beam resting on Pasternak foundation is performed. The curved beam with various opening angles is subjected to a triangular impulsive type of external dynamic load $P = P(t)$ acting at the midpoint of beam (Fig. 1). The fixed-fixed end condition is employed. The analyses are carried out in the Laplace space, and the results are transformed back to the time space numerically using modified Durbin's algorithms. Firstly, the dynamic analysis results of the planar curved beam on two-parameter elastic foundation are verified with the literature [19]. Next, the influence of the opening angle of curved beam, the radius of the curved beam to the height of the rectangular cross-section ratio (R/h) and foundation parameter are investigated on the dynamic response of the planar curved beam on Winkler-Pasternak foundation. The rocking influence of the foundation is also considered in the solutions.

The common parameters for the examples are: The modulus of elasticity of the beam $E = 47.24\,\text{GPa}$, its Poisson's ratio $\upsilon = 0.2$, the density of material $\rho = 5000\,\text{kg/m}^3$, the radius of curved beam having rectangular cross-section $R = 7.63\,\text{m}$. The opening angles (θ) are $45°, 90°, 135°, 180°$. The component of Winkler foundation parameter in the direction of b $k_{Wb} = 23.623\,\text{MPa}$, the foundation rocking stiffness constant in the direction of t $k_{Rt} = 1143\,\text{kNm/m}$. The intensity and the duration of the loading $P_o = 100\,\text{kN}$ and $t_{load} = 0.064\,\text{s}$, respectively. The dynamic response of the beam is determined within $0 \le t \le 0.25\,\text{s}$. The parameters used in the analysis for inverse Laplace transformation algorithm are chosen $N = 2^{11}$ and $aT = 6$. These parameters are verified by the authors in [25].

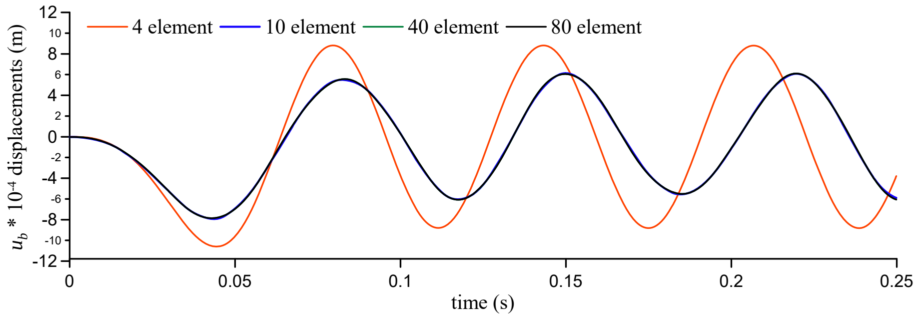
A. Convergence Test and Comparison with the Literature

The dimensions of rectangular cross-section are $b = h = 0.762\,\text{m}$. The dynamic analysis of the planar curved beam with $\theta = 180°$ is carried out using 4, 10, 40 and 80 finite elements. The time history curves of the displacement (u_b) at the midpoint of the beam (at point C) and the shear force (T_b) and the moments (M_t, M_n) at the fixed end of the beam (at point A) are presented in Fig. 2. From the time variation curves, the first maximum values for u_b, T_b, M_t, M_n are tabulated in Table I. In the following examples, 80 elements are employed.

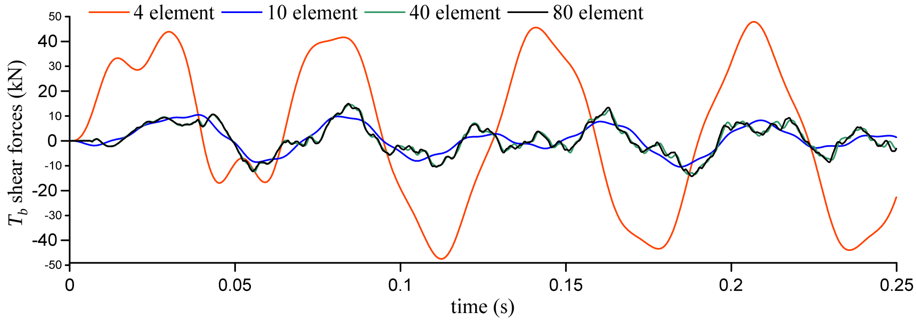
TABLE I
Convergence Analysis

Number of element	$u_b \times 10^{-4}$ (M)	T_b (kN)	M_t (kNm)	M_n (kNm)
4	-10.5958	43.8939	-12.4514	0.76248
10	-7.93354	10.4702	-9.39795	28.3368
20	-7.86019	9.50552	-9.89036	37.1206
40	-7.85772	9.58955	-9.84413	36.8867
60	-7.85641	9.45788	-9.84480	37.1374
80	-7.85560	9.37392	-9.84470	37.2756
100	-7.85571	9.36965	-9.84396	37.3439

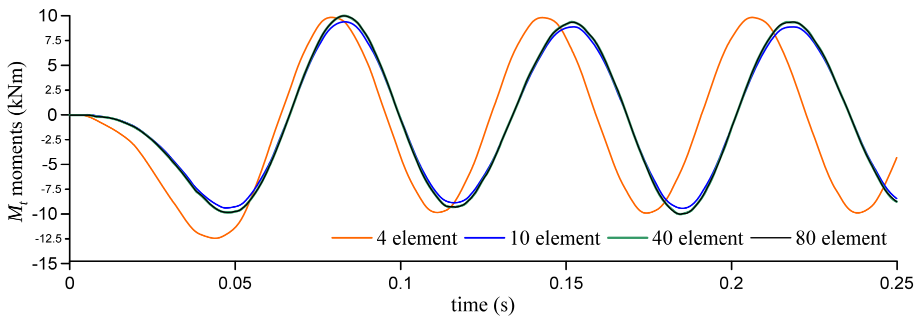
The dynamic analysis of curved beam is investigated for four different opening angle values ($\theta = 45°, 90°, 135°, 180°$). The influence of the opening angle on the dynamic behavior is shown in Fig. 3 by plotting the time histories of u_b and M_t. In [19], this problem is solved, but the results are given graphically. In order to compare our mixed finite element results with [19], we got in touch with the author. The author of [19] has shared his results with us. The comparison for the absolute values of displacement u_b and moment M_t of $t = 0.05\text{s}$ is tabulated for different opening angle values in Table II. The absolute percent differences with respect to the [19] results are also provided in Table II. It is observed that the results of both studies are in agreement with each other.



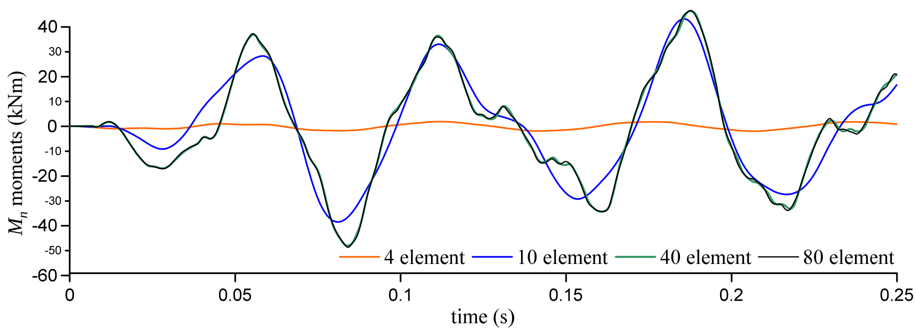
(a) u_b displacement at the midpoint of the beam



(b) T_b shear force at the fixed end of the beam

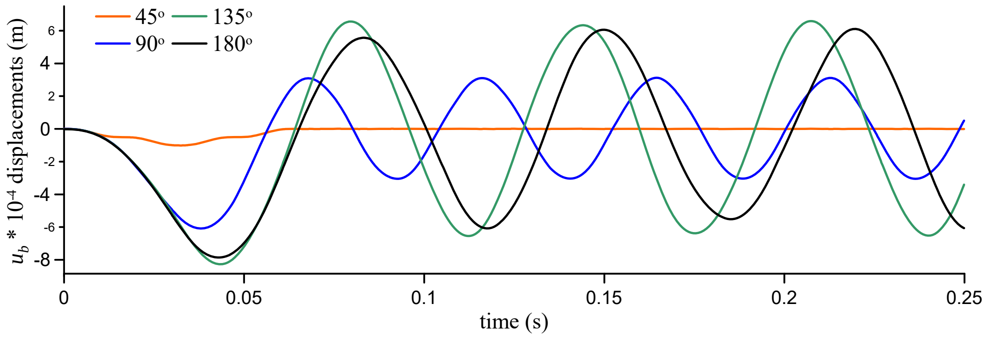


(c) M_t moment at the fixed end of the beam

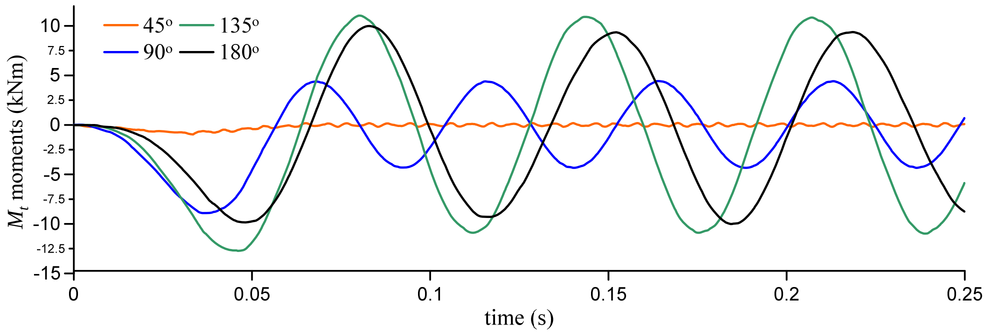


(d) M_n moment at the fixed end of the beam

Fig. 2 Convergence analysis of transverse triangle type impulsive point load applied at the midpoint of the curved beam, for 4, 10, 40, 80 elements



(a) u_b displacement at the midpoint of the beam



(b) M_t moment at the fixed end of the beam

Fig. 3 Time histories of the curved beam for different values of opening angle (θ)

TABLE II
COMPARISON WITH THE LITERATURE

θ		[19]	ANSYS[19]	This study	% diff.
45°	$u_b \times 10^{-4}$ (m)	0.492	0.490	-0.493	0.20
	M_t (kNm)	-0.337	-0.119	-0.409	17.6
90°	$u_b \times 10^{-4}$ (m)	3.150	3.160	-3.156	0.19
	M_t (kNm)	-4.110	-4.053	-4.538	9.43
135°	$u_b \times 10^{-4}$ (m)	7.160	7.130	-7.164	0.06
	M_t (kNm)	-11.30	-11.22	-11.84	4.56
180°	$u_b \times 10^{-4}$ (m)	6.950	6.910	-6.950	0.00
	M_t (kNm)	-9.790	-9.843	-9.736	-0.55

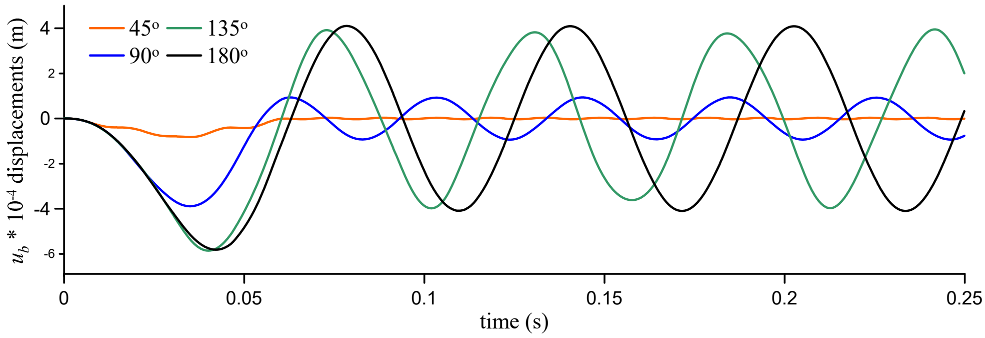
B. Curved Beam on Pasternak Foundation

The dynamic behavior of the curved beam for different opening angles ($\theta = 45°, 90°, 135°, 180°$), the radius of the curved beam to the height of the rectangular cross-section ratios ($R/h = 5, 10, 15$) keeping $R = 7.63\,\text{m}$ and $b = 0.762\,\text{m}$, and Pasternak foundation parameters ($k_{Pb} = 2362.3\,\text{kN}$, $23623\,\text{kN}$, $236230\,\text{kN}$) are investigated.

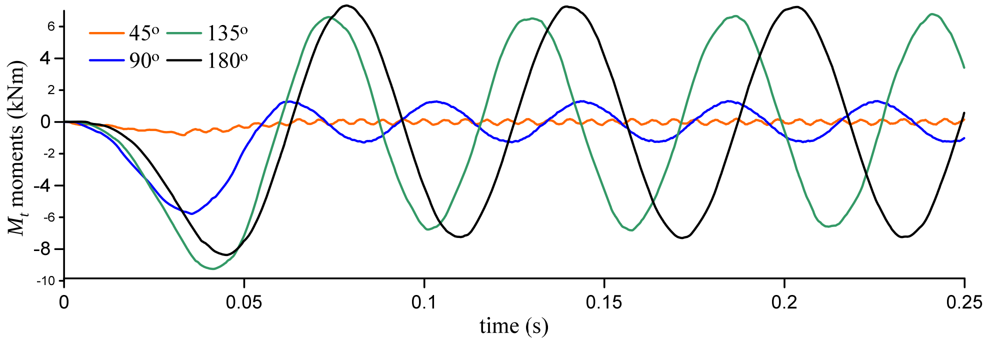
For four different opening angle values, the time histories of u_b and M_t are given in Fig. 4. It is observed that increasing the opening angle of the curved beam enlarges the vibration periods of u_b and M_t. The values of first extrema of the forced vibration zone corresponding to four different opening angle values are determined from Figs. 4 (a), (b) and the MFEM results for $\theta = 180°$ are compared with the results associated with $\theta = 45°, 90°, 135°$. As the opening angle values (except $\theta = 135°$) increases, an increasing trend is observed for the displacement u_b and the moment M_t. If the displacement u_b in each opening angles θ are compared with respect to the results of $\theta = 180°$, the absolutely percent reduction for the cases $\theta = 45°$ and $90°$ are 85% and 33%, respectively. The percent increase for $\theta = 135°$ is -0.8%. A similar situation is seen in the moment M_t. If the moments M_t in each opening angles θ are compared with respect to the results of $\theta = 180°$, the absolute percent reduction for the cases $\theta = 45°$ and $90°$ are 90.1% and 30.9%, respectively. The absolute percent increase for $\theta = 135°$ is -10.5%.

For the ratios of radius to the height of the rectangular cross-section ($R/h = 5, 10, 15$), the time histories of u_b and M_t are given in Fig. 5. As R/h ratios increase (or the thicknesses of the beam decrease), it is observed that the displacements u_b increase and the vibration periods of u_b decrease. The values of first extrema of the forced vibration zone corresponding to three R/h ratios are determined from Figs. 5 (a), (b) and the MFEM results for $R/h = 5$ are compared with the results that correspond to the $R/h = 10, 15$.

If the displacements u_b in each R/h ratios are compared with respect to the results of $R/h=5$, the absolute percent increase for the cases $R/h=10$ and 15 are 31% and 41%, respectively. As R/h ratios increase, it is observed that the moments and the vibration periods of M_t decrease. If the moments M_t in each R/h ratios are compared with respect to the results of $R/h=5$, the absolutely percent reduction for the cases $R/h=10$ and 15 are 83% and 95%, respectively.

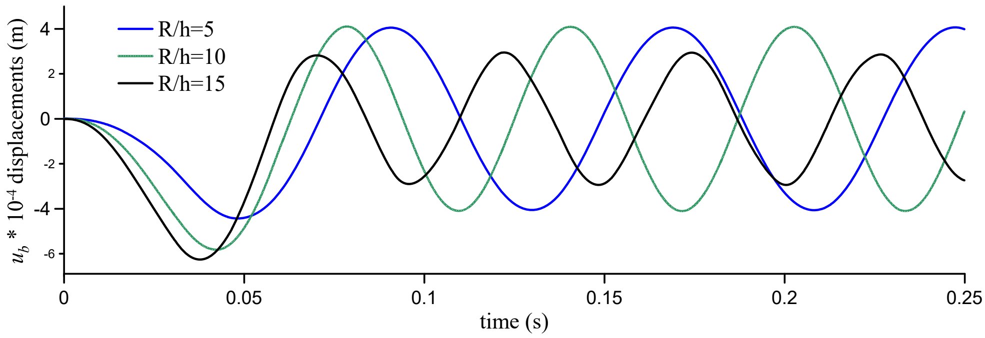


(a) u_b displacements at the midpoint of the beam

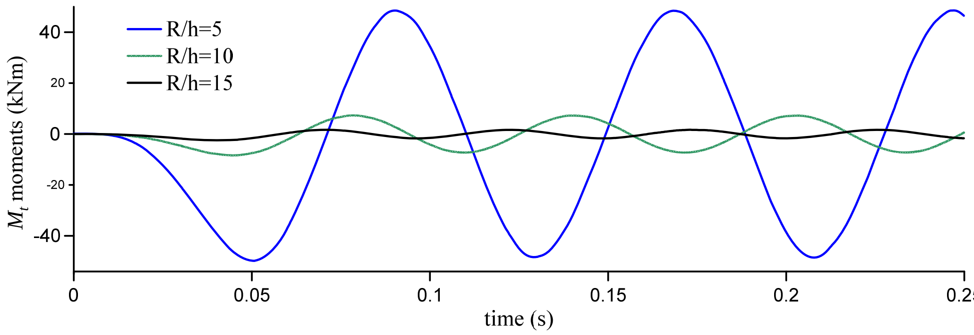


(b) M_t moments at the fixed end of the beam

Fig. 4 Time histories of the curved beam for different values of opening angle



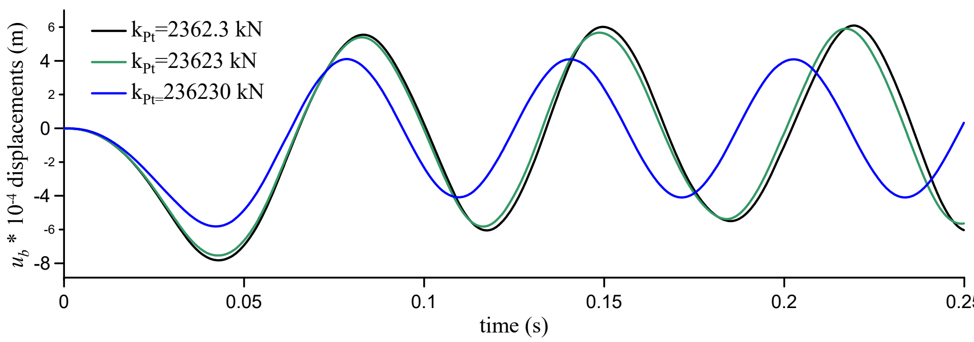
(a) u_b displacements at the midpoint of the beam



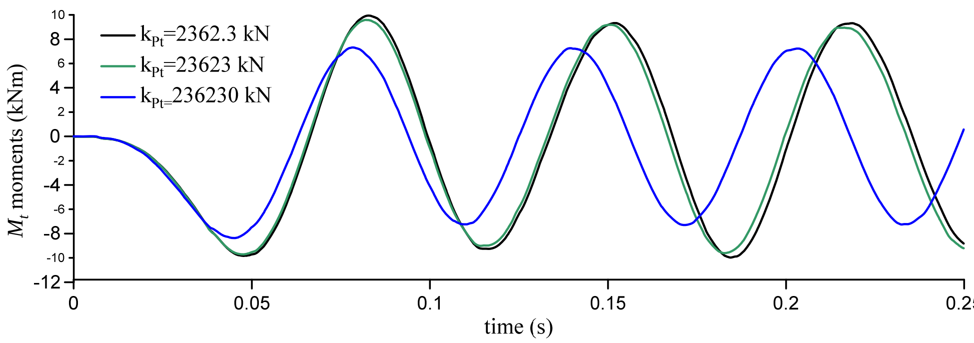
(b) M_t moments at the fixed end of the beam

Fig. 5 Time histories of the curved beam for different values of R/h ratios

For each Pasternak foundation constants, the displacements u_b of $k_{Pb} = 23623\,\text{kN}, 236230\,\text{kN}$ are compared with the results that correspond to $k_{Pb} = 2362.3\,\text{kN}$; in the cases of $k_{Pb} = 23623\,\text{kN}, 236230\,\text{kN}$, the percent reductions for the values of first extrema of the forced vibration zone are 3.6% and 26%, respectively. Similar comparison can be made for the moments M_t and the percent reductions are 1.2% and 15%, respectively. It is also observed that due to an increase of Pasternak foundation constants of the curved beam, the vibration periods of u_b and M_t decrease (see Fig. 6).



(a) u_b displacements at the midpoint of the beam



(b) M_t moments at the fixed end of the beam

Fig. 6 Time histories of the curved beam for different values of Pasternak foundation parameters

IV. Conclusion

Dynamic behavior of a planar curved Timoshenko beam on elastic foundation with rectangular cross-section is investigated using the mixed finite element method. The solutions are obtained in Laplace space and the results are transformed back to time space by using modified Durbin's algorithm. Regarding as a converge test, a semicircular beam resting on Winkler foundation is handled, the mixed finite element results for different opening angle values of curved beam are compared with the literature and a good agreement is observed. The rocking influence is also considered in the formulation. Next, some examples are solved to investigate the influence of the radius of the curved beam to the height of the rectangular cross-section ratio (R/h), the opening angle of the curved beam (θ) and Pasternak foundation parameter (k_p) on the dynamic analysis of a planar curved beam having rectangular cross-section. Following remarks can be cited:

- As the opening angle values increase, an increasing trend is observed for the magnitude of displacements u_b and the moments M_t.
- As the ratio R/h increase, an increase of the displacements u_b and a reduction of the moments M_t are observed.
- An increase of Pasternak foundation constant caused a reduction of the displacements u_b and the moments M_t.
- The change of opening angle, R/h and Pasternak foundation constant affect the vibration period of displacements u_b and the moments M_t.

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