International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:11, No:8, 2017

Zero Divisor Graph of a Poset with Respect to Primal Ideals

Hossein Pourali

***Abstract*—In this paper, we extend the concepts of primal and weakly primal ideals for posets. Further, the diameter of the zero divisor graph of a poset with respect to a non-primal ideal is determined. The relation between primary and primal ideals in posets is also studied.**

***Keywords*—Zero divisors graph, ideal, prime ideal, semiprime ideal, primal ideal, weakly primal ideal, associated prime ideal, primary ideal.**

I. Introduction

THE study of zero-divisor graphs was initiated by Istvan Beck [4] in 1988, when he proposed a method for coloring a commutative ring by associating the ring to a simple graph, the vertices of which were defined to be the elements of the ring, with vertices x and y joined by an edge when $x \wedge y = 0$. In 1999 Anderson and Livingston [1] changed this definition, restricting the set of vertices to the non-zero zero divisors of the ring, and from their paper work is proceeded in two directions. Specifically, Redmond [17] investigated zero-divisor graphs of non-commutative rings, while DeMeyer, McKenzie, and Schneider [5] looked at the zero-divisor graphs of commutative semigroups with 0. The other direction is the work for posets with 0.

Nimbhorkar, Wasadikar and DeMeyer [16] introduced the zero divisor graphs for meet-semilattice L with 0 and proved a form of Beck's Conjecture. They associated a zero divisor graph to a meet-semilattice L with 0, whose vertices are the elements of L and two distinct elements $x, y \in L$ are adjacent if and only if $x \wedge y = 0$

This work was further extended by Halaš and Jukl [8] to posets with 0(see also, [7]). Halaš and Jukl [8] introduced the concept of zero divisor graph to posets with 0 where vertex set of the zero divisor graph $G(P)$ is the poset P and two vertices x and y are adjacent if and only if $(x, y)^\ell = \{0\}$.

The zero divisor graph with respect to an ideal was first defined in the context of commutative rings by Redmond [17]. In [9], Joshi introduced a similar graph in the context of posets, which consider both definition of zero divisor graphs given by Lu and Wu [15].

Initially, the concept of primal and weakly primal ideals over commutative semirings introduced by Attani [2], [3]. The concept of primal ideals on lattices already studied by Pourali, Joshi and Waphare in [11]. To see more, [12]-[14].
In this paper, we extend the concept of primal ideal in terms of posets and we show that the result holds for the class of posets. At the end of this paper, we find the relation between primal and primary ideals of posets.

Hossein Pourali is with the Department of Mathematics, Azad University of Tehran, North branch, Iran (e-mail: hosseinypourali@gmail.com).

II. Properties of a Zero Divisor Graph of a Poset

We begin with necessary concepts and terminology in a poset P. Let $A \subseteq P$. The set $A^u = \{x \in P \mid x \geq a$ for every $a \in A\}$ is called *upper cone* of A. Dually, we have the concept of *lower cone* A^ℓ of A. The upper cone $\{a\}^u$ is simply denoted by a^u and $\{a, b\}^u$ is denoted by $(a, b)^u$. Similar notations are used for lower cones.

A non-empty subset I of a poset P is called a *semi-ideal*, if for $x \in I$, $y \in P$, $y \leq x$ implies $y \in I$. A proper semi-ideal I of a poset P is called a *prime semi-ideal*, if $(a, b)^\ell \subseteq I$ implies $a \in I$ or $b \in I$; Venkatnarasimhan [18].

A non-empty subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a, b)^{u\ell} \subseteq I$. An ideal $I \neq P$ is called *prime* if $(a, b)^\ell \subseteq I$ implies either $a \in I$ or $b \in I$; Halaš [6]. Dually, we have concepts of filters and prime filters.

Now, the concept of a zero divisor graph of a poset P with respect to an ideal I is due to Joshi [9].

Definition 1: Let I be an ideal of a poset P with 0. We associate an undirected graph, called the *zero divisor graph of P with respect to the ideal I*, denoted by $G_I(P)$ in which the set of vertices is $\{x \in P \setminus I \mid (x, y)^\ell \subseteq I$ for some $y \in P \setminus I\}$ and two distinct vertices x, y are adjacent if and only if $(x, y)^\ell \subseteq I$. When $I = \{0\}$ then the zero divisor graph is denoted by $G(P)$.

Definition 2: For an ideal I and a non-empty subset A of a poset P, define a subset $I : A$ of P as follows :

$$I : A = \{z \in P | (a, z)^\ell \subseteq I; \forall a \in A\}.$$

If $A = \{x\}$ then we write $I : x$ instead of $I : \{x\}$. Note that, if $x \leq y$ for $x, y \in P$, then $I : y \subseteq I : x$. Observe that $I \subseteq I : A$ and $I : A = \cap_{x \in A} I : x$ however, $I : A$ need not be an ideal but it is a semi-ideal. Moreover, $I = (0]$ then $I : x$ is nothing but $Ann(x) = \{y | (x, y)^\ell = 0\}$.

Definition 3: Let I be an ideal of a lattice L with 0. We associate an undirected graph, called the *zero divisor graph of L with respect to the ideal I*, denoted by $G_I(L)$ in which the set of vertices is $V(G_I(L)) = \{x \notin I \mid x \wedge y \in I$ for some $y \notin I\} = Z_I(L)^*$ and two distinct vertices x, y are adjacent if and only if $x \wedge y \in I$. When $I = \{0\}$ then the corresponding zero divisor graph is denoted by $G_{\{0\}}(L)$.

We recall the following concepts from graph theory, see D. B. West [19].

Definition 4: Let G be a graph. Let x, y be distinct vertices in G. We denote by $d(x, y)$ the length of a shortest path from x to y (if it exists) and put $d(x, y) = \infty$ otherwise we write $d(x, x) = 0$ for $x \in V(G)$. The *diameter* of G is denoted by $diam(G)$, $diam(G) = sup\{d(x, y) \mid \ x, y \in V(G)$. A *cycle* in a graph G is a path that begins and ends at the same vertex.

The *girth* of G, denoted $gr(G)$, is the length of a shortest cycle in G (and $gr(G) = \infty$ if G has no cycle).

III. Zero Divisor Graph of a Poset with Respect to Primal Ideals

Definition 5: Let I be an ideal of a poset P. A set J of P is called *prime* to I if $I : J = I$.

In particular if $J = \{a\}$, then we say that $a \in P$ is prime to I if $I = I : a$. We will define $F_I = \bigcup\{J \subseteq P \mid I : J = I\}$. Further, the set of elements of P that are *not prime to I* will denote by $S(I)$, where $S(I) = \bigcup\{J \subseteq P \mid I \subsetneqq I : J\}$.

Lemma 1: $S(I)$ is a semi-ideal .

Proof: Let $x \in S(I)$ and $y \leq x$ we want to show that $y \in S(I)$. By applying definition of $S(I)$ there exists J such that $I \subsetneqq I : J$ and $x \in J$. Therefore there exists $t \in I : J$ such that $t \notin I$. Hence $(t, x)^\ell \subseteq I$. Therefore $(t, y)^\ell \subseteq I$. Therefore $y \in I : t$ for some $t \notin I$. Hence $y \in S(I)$. ■

Lemma 2: Let I be a proper ideal of a poset P with 1. If I is prime ideal then $S(I)$ is an ideal. Further, $S(I)$ as an ideal, is prime.

Proof: Let $a, b \in S(I)$. We have to show that $(a, b)^{u\ell} \subseteq S(I)$. Assume on the contrary that $(a, b)^{u\ell} \nsubseteq S(I)$, then there exists $t \in (a, b)^{u\ell}$ such that $t \notin S(I)$. Since $a, b \in S(I)$, there exist some sets say, J_1 and J_2 such that $I \subsetneqq I : J_1$ and $I \subsetneqq I : J_2$. Therefore there exist $x \in I : J_1$ such that $x \notin J_1$ and $y \in I : J_2$ such that $y \notin J_2$. Therefore $(x, j_1)^\ell \subseteq I$ for every $j_1 \in J_1$ and $(y, j_2)^\ell \subseteq I$ for every $j_2 \in J_2$.
In particular since $a \in J_1$ and $b \in J_2$, we have $(x, a)^\ell \subseteq I$ and $(y, b)^\ell \subseteq I$ $(*)$.
Since $x, y \notin I$ and I is prime, therefore $(x, y)^\ell \nsubseteq I$. There exists $z \in (x, y)^\ell$ such that $z \notin I$. From $(*)$ we get, $(z, a)^\ell \subseteq I$ and $(z, b)^\ell \subseteq I$. Since every prime ideal is semi prime, we have $\{z, (a, b)^u\}^\ell \subseteq I$. Since $t \notin S(I)$, then $\{t\}$ is prime to I, which means $I = I : t$. Further, $t \in (a, b)^{u\ell}$ and $\{z, (a, b)^u\}^\ell \subseteq I$, we have $(z, t)^\ell \subseteq I$. Therefore, $z \in I : t = I$. Thus $z \in I$, a contradiction to $z \notin I$.

Next, to show that $S(I)$ is a prime ideal. Let $(a] \cap (b] \subseteq S(I)$ and $a, b \notin S(I)$, therefore we have $I = I : a$ and $I = I : b$. Put $J = (a, b)^\ell$. Let $x \in I : J$, therefore $x \in I : (a, b)^\ell$. Then we get $(x, a, b)^\ell \subseteq I$ and hence $(x, a)^\ell \subseteq I : b = I$. Therefore $(x, a)^\ell \subseteq I$, and we get $x \in I : a = I$. Thus $I : J \subseteq I$. Converse is always true, thus $I : J = I$. Hence $J = (a, b)^\ell$ is prime to I. ■

Lemma 3: Let I be a proper ideal of poset P with 1, then the following statements are true:

(i) $I \subseteq S(I)$

(ii) $V(G_I(P)) = S(I)\backslash I$. In particular, $V(G_I(P)) \cup I = S(I)$.

Proof: (i) Let $x \in I$. To show that $x \in S(I)$. Since I is proper ideal of P, therefore there exists $y \in P$ such that $y \notin I$. Put $J = \{x\}$. Clearly we have $I \subset I : J = I : x = P$. Now to show that $I \subsetneqq I : J$, then there exists $t \in I : J$ such that $t \notin I$. But $I : J = I : \{x\} = \{z \in P | (z, x)^\ell \subseteq I\}$. In particular $z = y$ then $y \in I : J$ and since $y \notin I$ then $I \subsetneqq I : J$.

(ii) Let $r \in V(G_I(L))$, then there exist $r \notin I$ such that $(r, x)^\ell \subseteq I$ for some $x \notin I$. Put $J = \{x\}$, then $r \in I : J = I : x$ and $r \notin I$. Therefore $I \subsetneqq I : J$.

Conversely, let $a \in S(I)\backslash I$ that means there exists $J \subseteq P$ such that $I \subsetneqq I : J$ and $a \in J$. Therefore, there exists $t \in I : J$ such that $t \notin I$. That means $(t, j)^\ell \subseteq I$ for every $j \in J$. In particular, put $j = a$, then we have $(t, a)^\ell \subseteq I$. Hence $a \in V(G_I(P))$. ■

Definition 6: A proper ideal I of a poset P is said to be *primal* if $S(I)$ forms an ideal. In this case we also say that I is a Q-*primal* ideal of P, where Q is a prime ideal of a poset P.

Lemma 4: Let I and Q be ideals of a poset P with 1 and $I \subseteq Q$. Then I is a Q-primal ideal of P if and only if $V(G_I(L)) = Q\backslash I$.

Proof: If I is a Q-primal ideal of P, then by Lemma 3, $V(G_I(P)) = S(I)\backslash I = Q\backslash I$

Conversely, assume that $V(G_I(P)) = Q\backslash I$. It suffices to show that Q is exactly the set of elements of P that are not prime to I. Since , every element of I is not prime to I, hence we can assume that $c \in Q\backslash I = V(G_I(P))$. Then there exists $z \notin I$ such that $(c, z)^\ell \subseteq I$. Put $J = \{c\}$. Then there exists $z \in I : J = I : c$ and $z \notin I$. Therefore $I \subsetneqq I : J$. So we could find $J \subseteq P$ such that $c \in J$ and $I \subsetneqq I : J$. Hence $c \in S(I)\backslash I$, which yields that $Q\backslash I \subseteq S(I)$. Therefore $Q \subseteq S(I)\backslash I$.
Next, suppose that $a \in S(I)$. If $a \in I$ then $a \in Q$ as $I \subseteq Q$. When $S(I) \subseteq Q$ as we are through. If $a \notin I$ and $a \in S(I)$, hence there exists $J \subseteq P$ such that $a \in J$ and $I \subsetneqq I : J$. There exists $t \in I : J$ such that $t \in I$. Then we have $(t, j)^\ell \subseteq I$, $\forall j \in J$. In particular, put $J = \{a\}$, then we have $(t, a)^\ell \subseteq I$. Hence $a \in V(G_I(L)) = Q\backslash I$. Therefore $S(I)\backslash I \subseteq Q\backslash I$. Hence $S(I) \subseteq Q$. ■

Corollary 1: Let I be an ideal of a poset P. Then I is Q-primal of P if and only if $V(G_I(P)) \cup I$ is an (prime) ideal.

Theorem 1: (Joshi and Mundlik [10])Let Q be a prime ideal of a poset P. Then the following statements are equivalent.

(a) Q is minimal prime ideal belonging to I.

(b) For each $x \in Q$, there exists $y \notin Q$ such that $(x, y)^\ell \subseteq I$.

(c) Exactly one of them $(x]$ or $I : x$ are contained in Q.

Theorem 2: Let I and J be ideals of poset P. Then $V(G_I(P)) = V(G_J(P))$ if and only if $I = J$.

Proof: Let $V(G_I(P)) = V(G_J(P))$ and assume on the contrary that $I \neq J$. Then there exists $x \in I$ such that $x \notin J$. Let $a \in V(G_I(P)) = V(G_J(P))$. Hence, for every $a \in V(G_I(P))$, there exists $b \in V(G_I(P))$ such that $(a, b)^\ell \subseteq I$ and since $x \in I$. Therefore we have $(x, a)^\ell, (x, b)^\ell \subseteq I$. We claim that $(x, a)^\ell, (x, b)^\ell \nsubseteq J$.

If possible, $(x, a)^\ell, (x, b)^\ell \subseteq J$. Since $x \notin J$, therefore we get $a \in V(G_J(P))$. Without loss of generality assume that $(x, a)^\ell \subseteq J$, which yields $x \in V(G_J(P)) = V(G_I(P))$, a contradiction to $x \in J$. Thus $(x, a)^\ell, (x, b)^\ell \nsubseteq J$.

Now, note that as $(a, b)^\ell \subseteq J$, we have that $(x, a, b)^\ell \subseteq J$, and $(x, a)^\ell, (x, b)^\ell \nsubseteq J$. Therefore $(x, a)^\ell, (x, b)^\ell \subseteq V(G_I(P)) = V(G_J(P))$. Hence $(x, a)^\ell, (x, b)^\ell \subseteq J$, a contradiction to the fact that $(x, a)^\ell, (x, b)^\ell \subseteq J$. Thus $I = J$.

Conversely, if $I = J$ then it is easy to see that $V(G_I(P)) = V(G_J(P))$. ■

Corollary 2: Let I be Q-primal ideal and J be an ideal of a poset P such that $V(G_I(P)) = V(G_J(P))$. Then J is a Q-primal ideal of P.

Corollary 3: Let I and J be ideals of a poset P with 1. Then $V(G_I(P)) = V(G_J(P))$ if and only if $S(I) = S(J)$.

Proposition 1: Let I be a semi prime ideal of a poset P. If there are non-adjacent elements $a, b \in V(G_I(P))$ such that the ideal $(a,b)^{u\ell}$ is prime to I. Then diam $(G_I(P)) = 3$.

Proof: Since a and b are non-adjacent, we must have $d(a,b) \neq 1$. Let $d(a,b) = 2$. Then there is an element $c \in P \backslash I$, such that $(a,c)^\ell \subseteq I$ and $(c,b)^\ell \subseteq I$. Since I is semi prime ideal, then we have $(c,(a,b)^u)^\ell \subseteq I$. Therefore, $c^\ell \cap (a,b)^{u\ell} \subseteq I$, then we get $c^\ell \subseteq I : (a,b)^{u\ell}$. Since $(a,b)^{u\ell}$ is prime to I, by applying definition, there exists $J \subseteq P$ such that $I = I : J$. Put $J = (a,b)^{u\ell}$. Then $c^\ell \subseteq I : (a,b)^{u\ell} = I$. Hence $c \in I$, a contradiction. Thus $d(a,b) \neq 2$. Since $diam(G_I(P)) \leq 3$, therefore $diam(G_I(P)) = 3$, as required. ■

Lemma 5: Let I be an ideal of a poset P. If I is not Q-primal, there exist elements a and b of $V(G_I(P))$ such that the ideal $(a,b)^{u\ell}$ is prime to I.

Proof: Since I is not Q-primal, therefore $S(I)$ is not an ideal. Therefore there exist $a, b \in S(I)$ such that $(a,b)^{u\ell} \nsubseteq S(I)$. Hence there exists $t \in (a,b)^{u\ell}$ such that $t \notin S(I)$. Therefore $t \notin I$, for otherwise $t \in I \subseteq S(I)$, therefore $t \in S(I)$, which is a contradiction. Thus $t \notin J$, $\forall J$ with $I \subsetneq I : J$ $(*)$

To show that $(a,b)^{u\ell}$ is prime to I, we have to show that $I = I : (a,b)^{u\ell}$. Suppose on the contrary that, $I \subsetneq I : (a,b)^{u\ell}$. Hence there exists $z \in I : (a,b)^{u\ell}$ such that $z \notin I$. Now $\{z, (a,b)^u\}^\ell \subseteq I$. Since $t \in (a,b)^{u\ell}$, we have $(z,t)^\ell \subseteq I$. Hence $t \in I : z$ and also $t \notin I$. Put $J = \{z\}$. Then we have an element $t \in I : J$ and $t \notin I$. Therefore, $t \in J$ with $I \subsetneq I : J$ a contradiction to $(*)$. ■

Definition 7: A proper ideal Q of a poset P is said to be *weakly prime* if $0 \neq (a,b)^\ell \subseteq Q$ implies that $a \in Q$ or $b \in Q$. We assume that $(0]$ is always weakly prime.

Definition 8: An element $a \in P$ is called *weakly prime* to an ideal I if $0 \neq (r,a)^\ell \subseteq I$ implies that $r \in I$. Denote $W(I)$, the set of all elements of P that are not weakly prime to I.

Definition 9: A proper ideal I of P is called *Q-weakly primal* if the set $Q = W(I) \cup \{0\}$, forms an ideal. This ideal is always a weakly prime ideal.

Definition 10: Let I be an ideal of a poset P. We define set $Z_I(P) = \{r \in P \backslash I | (r,a)^\ell = 0 \text{ for some } a \in P \backslash I\}$.

Lemma 6: Let I be a Q-weakly primal ideal of a poset P. Then $V(G_I(P)) = (W(I) \backslash I) \cup Z_I(P)$.

Proof: Assume that I is a Q-weakly primal ideal of P. Let $r \in V(G_I(P))$, then there is an element $a \in P \backslash I$ with $(r,a)^\ell \subseteq I$. If $(r,a)^\ell \neq 0$, then $r \in W(I)$. If $(r,a)^\ell = 0$, then $r \in Z_I(P)$. Hence $V(G_I(P)) \subseteq (W(I) \backslash I) \cup Z_I(P)$.

Let $s \in (W(I) \backslash I) \cup Z_I(P)$. If $s \in (W(I) \backslash I)$ then $0 \neq (s,b)^\ell \subseteq I$ for some $b \in P \backslash I$, hence $s \in V(G_I(P))$. If $s \in Z_I(P)$, then there is an element $c \in P \backslash I$ such that $(s,c)^\ell = 0 \in I$ hence $s \in V(G_I(P))$. ■

Proposition 2: Let I be an ideal of a poset P and Q be an ideal of P with $W(I) \subseteq Q$ and $(Q \backslash I) \cap Z_I(P) = \phi$. Then $V(G_I(P)) = (Q \backslash I) \cup Z_I(P)$ if and only if I is a Q-weakly primal ideal of L.

Proof: By Lemma 6, it suffices to show that if $V(G_I(P)) = (Q \backslash I) \cup Z_I(P)$ then I is a Q-weakly primal ideal of P. We show that $Q \backslash \{0\}$ consists exactly of elements of P that are not weakly prime to I . Let $s \in Q \backslash \{0\}$. Since every non-zero element of I is not weakly prime to I, we can assume that $s \notin I$. Therefore, $s \in Q \backslash I \subseteq V(G_I(P))$ implies that $(s,b)^\ell \subseteq I$ for some $b \in Q \backslash I$. Since, $(Q \backslash I) \cap Z_I(P) = \phi$, we must have $(s,b)^\ell \neq 0$. Hence s is not weakly prime to I. Hence $Q \backslash \{0\} \subseteq W(I)$. Since $W(I) \subseteq Q \backslash \{0\}$, we have I is Q-weakly primal ideal of P. ■

The following definition is due to Joshi and Mundlik [10]

Definition 11: An ideal I in a poset P is *primary* if

(i) $I \neq P$

(ii) $(x,y)^\ell \subseteq I$ then either $x \in I$ or $y \in r(I)$ where $r(I) = \cap_{I \subseteq Q} Q$.

In particular, if $r(I) = Q$ then I is called a Q-primary ideal.

The following result is due to Joshi and Mundlik [10]:

Theorem 3: Q is minimal prime ideal of a finite poset P if and only if for any $x \in Q$ there exists $y \notin Q$ such that $(x,y)^\ell = \{0\}$.

Proposition 3: Let I be a Q-primary ideal of a poset P then I is a Q-primal ideal of a poset P.

Proof: It suffices to show that the set of elements of P that are not prime to I is just Q. Assume that a is an element of P that is not prime to I, then $a \in S(I)$. Therefore there exists $J \subseteq P$ such that $I \subsetneq I : J$ and $a \in J$. Put $J = \{a\}$, then we have $I \subsetneq I : a$. Hence there is an element $b \in I : a$ with $b \notin I$ and $(a,b)^\ell \subseteq I$. Thus I be Q-primary gives $a \in Q$. We have proved that $S(I)$, set of elements of P that are not prime to I is subset of Q. Now we will show that $S(I)$ can not be properly subset of Q.

Suppose $S(I) \subsetneq Q$, then there exist an element $x \in Q$ such that $x \notin S(I)$. Therefore there exists $J \subseteq P$ such that $I = I : J$ and $x \in J$. Put $J = \{x\}$. Therefore $I = I : x$ and $x \in Q = r(I) = r(I : x) \supseteq I$. Since $r(I)$ is the smallest prime ideal containing I. So by using Theorem 3, for any $x \in Q$ there exists $y \notin Q$ such that $(x,y)^\ell \subseteq I$. Since I is Q-primary then either $x \in I$ or $y \in r(I)$ but $x \notin I$ since if $x \in I$ then we must have $I = P$ which is a contradiction.
Hence $y \in r(I) = Q$ a contradiction to $y \notin Q$ ■

Proposition 4: Let I be an ideal of a poset P. Then I is Q-primary if and only if $G_I(P) = r(I) \backslash I$ where $r(I) = \cap_{I \subseteq Q} Q$.

Proof: If I is Q-primary then by using Proposition 3, I is Q-primal ideal of P. Since $I \subseteq r(I) = Q$ then by lemma 3, I is Q-primal of P if and only if $G_I(P) = Q \backslash I = r(I) \backslash I$.

Conversely, suppose that $a, b \in L$ are such that $(a,b)^\ell \subseteq I$ but $a \notin I$ and $b \notin r(I)$. So $b \notin I$. Then $b \in G_I(P) = r(I) \backslash I$, which is a contradiction. Thus, I is primary. ■

Definition 12: The set of associated primes of a poset P is denoted by $Ass(P)$ and it is the set of prime ideals q of P such that there exists $x \in P$ with $q = Ann(x)$.

Definition 13: A Graph G is said to be planar if it can be drown in such a way that no two edges meet except at vertex with which they are both incident .

Theorem 4: Let P be a poset . Then the following hold:

(a) If $|Ass(P)| \geq 2$ and $p = Ann(x)$ and $q = Ann(y)$ are two distinct elements of $Ass(P)$ then $(x, y)^\ell = 0$

(b) If $|Ass(P)| \geq 3$ then $girth(G(P)) = 3$

(c) If $|Ass(P)| \geq 5$ then $G(P)$ is not planar.

Proof: (a) Assume on the contrary that $(x, y)^\ell \neq 0$. Therefore, $x \notin Ann(y)$ and $y \notin Ann(x)$. Since $Ann(x)$ and $Ann(y)$ both are prime, we conclude that $Ann(x) \subseteq Ann(y)$ and $Ann(y) \subseteq Ann(x)$. Hence $Ann(y) = Ann(y)$, which is a contradiction. Therefore $(x, y)^\ell = 0$.

(b) Let $p_1 = Ann(x_1)$ and $p_2 = Ann(x_2)$ and $p_3 = Ann(x_3)$ belong to $Ass(P)$. Then $x_1 - x_2 - x_3 - x_1$ is a cycle of length 3.

(c) Since $|Ass(P)| \geq 5$, K_5 is a subgraph of $G(P)$ and hence by Kuratowski's Theorem(A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$), $G(P)$ is not planar. ■

The following result is due to Joshi [9]:

Theorem 5: Let I be an ideal of a poset P with 0. Then the following hold:

(α) If P_1 and P_2 are prime semi ideals and $I = P_1 \cap P_2$. Then $G_I(P)$ is a complete bipatite graph.

(β) If I is a semiprime ideal then $G_I(P)$ is a complete bipartite graph if and only if there exist prime ideals P_1 and P_2 such that $I = P_1 \cap P_2$.

Theorem 6: Let P be a poset and $Ass(P) = \{p_1, p_2\}$, $|p_i| \geq 3$ for $i = 1, 2$ and $p_1 \cap p_2 = \{0\}$ then $gr(G(P)) = 4$.

Proof: Let $p_i = Ann(x_i)$, $i = 1, 2$. Since p_1, p_2 are two distinct elements of $Ass(P)$ then by using Theorem 4, we have $(x_1, x_2)^\ell = 0$. Let $a \in p_1 \backslash \{0, x_2\}$ and $b \in p_2 \backslash \{0, x_1\}$. Since $(a, b)^\ell \subseteq p_1 \cap p_2 = \{0\}$ we have $a - x_1 - x_2 - b - a$. Thus $G(P)$ has a cycle, moreover by Theorem 5, $G(P)$ is a complete bipartite graph and hence $gr(G(P)) = 4$. ■

Theorem 7: Let P be a poset and $Ass(P) = \{p_1, p_2\}$ with $p_1 \cap p_2 \neq 0$. If $|p_1 \cap p_2| > 3$, then $gr(G(P)) = 3$.

Proof: Let $p_1 = Ann(x_1)$ and $p_1 = Ann(x_1)$ and $|p_1 \cap p_2| > 3$. Since p_1 and p_2 are distinct elements of $Ass(P)$ then by Theorem 4, we have $(x_1, x_2)^\ell = 0$. Now take $a \neq 0 \in p_1 \cap p_2$, then we have $a \in p_1 = Ann(x_1)$ and $a \in p_2 = Ann(x_2)$. Therefore, $(a, x_1)^\ell = 0$ and $(a, x_2)^\ell = 0$. Hence we get $a - x_1 - x_2 - a$. Therefore $gr(G(P)) = 3$. ■

Definition 14: A poset P with 0 is called 0-*distributive poset* if, for $x, y, z \in P$, $(x, y)^\ell = 0$ and $(x, z)^\ell = 0$ together imply that $(x, (y, z)^u)^\ell = 0$.

Definition 15: Let P be a poset then for all $a \in P$, we denote by $a^\perp = \{x | (a, x)^\ell = 0\}$

.

The following two results is due to Joshi and Mundlik [10]:

Lemma 7: Let L be a 0-distributive finite poset then for every elements $a, b \in P$ we have $((a] \vee (b]]^\perp = a^\perp \cap b^\perp$.

Theorem 8: Let P be a 0-distributive finite poset. Then a prime ideal is minimal prime if and only if it contains exactly one of a or $a^\perp$.

Theorem 9: Let P be a 0-distributive finite poset. If P has more than two minimal primes and there are non zero elements $a, b \in G(P)$ such that $((a] \vee (b]]$ has no non zero annihilator, then $diam(G(P)) = 3$.

Proof: **Case**(I) Let P contains a pair of zero divisors a and b such that $(a, b)^\ell \neq 0$ and $((a] \vee (b]]^\perp = \{0\}$. Then we get, $a^\perp \cap b^\perp = 0$.

Now $a, b \in V(G(P))$, therefore there exists $x \neq 0$, $y \neq 0$ such that $(x, a)^\ell = 0$ and $(y, b)^\ell = 0$. Therefore, $x \in a^\perp$ and $y \in b^\perp$. Clearly $x \neq y$. For otherwise, $x \in a^\perp \cap b^\perp = \{0\}$, a contradiction to the fact that $x \neq 0$.

subcase(1) If $(x, y)^\ell = 0$, then we have a path $a - x - y - b$. Therefore, $d(a, b) = 3$ and hence $diam(G(P)) = 3$.

subcase(2) If $(x, y)^\ell \neq 0$ then $(x, y)^\ell \in V(G(P))$ as $(a, x, y)^\ell = 0$ also $(b, x, y)^\ell = 0$ then we have $a - t - b$, where $t \in (x, y)^\ell$. Hence $(x, y)^\ell \subseteq a^\perp \cap b^\perp = \{0\}$ a contradiction.

Case(II) Consider when $(a, b)^\ell = 0$ and P has more than two minimal prime ideals. We may assume three minimal primes as p, q and r. Since $(a, b)^\ell = 0 \in p$ therefore either $a \in p$ or $b \in p$. We claim that each minimal prime ideal contains only one of a and b but not both of them. If possible $(a, b)^\ell \nsubseteq p$ and p is minimal prime ideal and P is 0-distributive poset therefore we have $a^\perp \notin p$ and $b^\perp \notin p$. Hence $a^\perp \cap b^\perp \notin p$ as p is prime. But $a^\perp \cap b^\perp = \{0\}$ a contradiction.

Without loss of generality, let $a \in (p \cap q) \backslash r$ and $b \in r \backslash (p \cup q)$ and $c \in (q \cap r) \backslash p$. Consider $x \in (a, (b, c)^\ell)^u$. Clearly $(b, c)^\ell \neq 0$. For otherwise, $(b, c)^\ell = 0 \in p$. Since p is prime, therefore either $b \in p$ or $c \in p$. Therefore we get a contradiction. Hence $(b, c)^\ell \neq 0$. Further $(a, b)^\ell = 0$. Clearly $x \in (a, (b, c)^\ell)^u \neq 0$. Moreover $(x] \vee (b] = (a] \vee (b]$ then $((x] \vee (b]] = ((a] \vee (b]]$. Since $((a] \vee (b]]^\perp = \{0\}$ therefore $((x] \vee (b]]^\perp = \{0\}$. Then, $x^\perp \cap b^\perp = 0$. We claim that $(x, b)^\ell \neq 0$. If $(x, b)^\ell = 0$ then $x \in b^\perp$. Therefore, $x \in (a, (b, c)^\ell)^u \in b^\perp$, and we get $(b, c)^\ell \subseteq b^\perp$ which yields $(b, c)^\ell = 0$ a contradiction. Hence $(x, b)^\ell \neq 0$. Thus we have $x \neq 0$, $b \neq 0$ such that $(x, b)^\ell \neq 0$, further $((x] \vee (b]]^\perp = \{0\}$ by case(I), $d(x, b) = 3$. Thus $diam(G(P)) = 3$. ■

Lemma 8: Let P be a 0-distributive poset and $Z(P)$ be the set of zero divisors of P. Then $Z(P) = \bigcup q$; where q is minimal prime.

Proof: Let $a \in Z(P)$ to show that $a \in \bigcup q$. Since $a \in Z(P)$ there exists a non zero element b such that $(a, b)^\ell = 0$. Suppose $a \notin \bigcup q$; for every poset P. Then $a \notin q$. By using Theorem 8, we get $a^\perp \in q$. Then $b \in a^\perp$ and hence, $b \in q$. Therefore $b \in \bigcup P$; for every poset P. Since P is 0-distributive poset and $\bigcup q = 0$ therefore $b = 0$ a contradiction. Conversely, suppose $a \in \bigcup q$ so there exists a minimal prime q such that $a \in q$. Therefore $a^\perp \nsubseteq q$. Then there exists a non zero element b such that $b \in a^\perp$ but $b \notin q$. Then $a \in Z(P)$. Hence $Z(P) = \bigcup q$; where q is minimal prime. ■

Theorem 10: Let P be a 0-distributive poset and $Z(P)$ be the set of zero divisors of P. If $Z(P)$ is not an ideal, then diameter of $G(P)$ is less than or equal to 2 if and only if P has exactly two minimal primes.

Proof: Suppose $Z(P)$ is not an ideal. So there exists a non zero elements $a, b \in Z(P)$ such that $(a, b)^{u\ell}$ is not a zero divisor. Hence $(a \vee b]^\perp = \{0\}$.
By applying Theorem 7, $a^\perp \cap b^\perp = \{0\}$. Since $a, b \in Z(P)$, we have non zero elements $c, d \in P$ such that $(a, c)^\ell = 0$ and $(b, d)^\ell = 0$. Clearly $c \neq d$. For otherwise, since P is 0-distributive poset, we get that $(c, (a, b)^u)^\ell = \{0\}$. Therefore

we have, $c \in (a \vee b]^{\perp}$, a contradiction. Hence $c \neq d$. By using Theorem 9, there exists two minimal ideals p and q such that $c \notin p$ and $d \notin q$. By applying Theorem 8, we get $c^{\perp} \subseteq p$ and $d^{\perp} \subseteq q$. Since $a \in c^{\perp}$ and $b \in d^{\perp}$, we have $a \in p$ and $b \in q$. Therefore $a^{\perp} \nsubseteq p$ and $b^{\perp} \nsubseteq q$. This implies that the minimal prime ideals p and q are distinct. If possible $p = q$ then $a^{\perp}$,$b^{\perp} \nsubseteq p$. Therefore $a^{\perp} \cap b^{\perp} \nsubseteq p$. Then we have, $(a \vee b]^{\perp} = 0 \nsubseteq p$, a contradiction. Therefore p and q are distinct. Since $diam(G(P)) \leq 2$. By using Theorem 9, $diam(G(P))$ is exactly 2.

Conversely, assume that poset P has exactly two distinct minimal prime ideals, say p and q. Therefore there exists elements a and b such that $a \in p \backslash q$ and $b \in q \backslash p$. But then $(a, b)^{\ell} \subseteq p \cap q = \{0\}$. Let $x, y \in V(G(P))$ and x, y be distinct elements. Then clearly x or y can not be in both p and q. Without loss of generality, we may assume that $x \in p \backslash q$. If $(x, y)^{\ell} = 0$, then they are adjacent and so $d(x, y) = 1$. Now if $(x, y)^{\ell} \neq 0$ since we have $x \in p \backslash q$ and $b \in q \backslash p$, therefore we have $(x, b)^{\ell} \subseteq p \cap q = \{0\}$. We claim that $y \notin q$. For otherwise, if $y \in q$ then $(x, y)^{\ell} \subseteq q$. Also since $x \in p$, we have $(x, y)^{\ell} \subseteq p$. Therefore $(x, y)^{\ell} \subseteq p \cap q = \{0\}$, a contradiction. Moreover $Z(P) = p \cup q$ and since $y \notin q$, therefore $y \in p$ which means $(x, y)^{u\ell} \subseteq p$. Therefore $((x, y)^{u}, b)^{\ell} \subseteq p$. Moreover since $b \in q \backslash p$, we have $((x, y)^{u}, b)^{\ell} \subseteq q$. Therefore $((x, y)^{u}, b)^{\ell} \subseteq p \cap q = \{0\}$. Therefore $(x, b)^{\ell} = 0$ and $(y, b)^{\ell} = 0$. Therefore $d(x, y) = 2$. Hence $diam(G(P)) \leq 2$. ■

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