Study of Rayleigh-Bénard-Brinkman Convection Using LTNE Model and Coupled, Real Ginzburg-Landau Equations

P. G. Siddheshwar, R. K. Vanishree, C. Kanchana

Abstract—A local nonlinear stability analysis using a eight-mode expansion is performed in arriving at the coupled amplitude equations for Rayleigh-Bénard-Brinkman convection (RBBC) in the presence of LTNE effects. Streamlines and isotherms are obtained in the two-dimensional unsteady finite-amplitude convection regime. The parameters' influence on heat transport is found to be more pronounced at small time than at long times. Results of the Rayleigh-Bénard convection is obtained as a particular case of the present study. Additional modes are shown not to significantly influence the heat transport thus leading us to infer that five minimal modes are sufficient to make a study of RBBC. The present problem that uses rolls as a pattern of manifestation of instability is a needed first step in the direction of making a very general non-local study of two-dimensional unsteady convection. The results may be useful in determining the preferred range of parameters' values while making rheometric measurements in fluids to ascertain fluid properties such as viscosity. The results of LTE are obtained as a limiting case of the results of LTNE obtained in the paper.

Keywords—Rayleigh-Bénard convection, heat transport, porous media, generalized Lorenz model, coupled Ginzburg-Landau model.

I. INTRODUCTION

THE onset of thermoconvective instability in a horizontal fluid saturated porous layer heated from below has been extensively studied. This is due to the fact that it finds wide variety of applications because of its interdisciplinary nature ranging from geophysical research to biophysical applications as well as petroleum and heat transfer engineering. The study of natural convection in a porous medium has been understood and well documented in the works of Vadasz [1], Crolet [2], Kaviani [3], Straughan [4], Ingham and Pop [5], Vafai [6] and Nield and Bejan [7]. In most of the situations it is observed that temperature fields of solid and fluid phase of the porous medium are assumed to be identical such a situation is generally known as local thermal equilibrium (LTE). However in many practical situations involving porous material and also media in which there is a large temperature difference between the fluid and the solid phases, it has been realized that the assumption of LTE model is inadequate for proper understanding of the heat transfer problems. In

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such circumstances the local thermal non-equilibrium (LTNE) effects are to be taken into consideration in which case the single energy equation has to be replaced by two, one for each phase.

The LTNE effects on convective flows in a porous medium have been studied by many authors. Banu and Rees [8] have studied the criterion for onset of convection in a Darcy porous medium using LTNE model. Free convection in a square porous cavity using LTNE was considered by Baytas and Pop [9]. Nield [10] has made a note on the modelling of LTNE in a structured porous medium. The onset of DarcyBrinkman convection in a porous layer using a thermal nonequilibrium model for stress-free boundaries was analysed by Postelnicu and Rees [11]. Malashetty et. al.[12]-[13] studied the onset of convection in an anisotropic porous layer and LapwoodBrinkman convection using a thermal non-equilibrium model. Explicit conditions for LTNE in porous media heat conduction were obtained by Vadasz [14]. Rees and Pop [15] studied the LTNE in porous medium convection. The effect of mechanical and thermal anisotropy on the stability of gravity driven cinvection in a rotating porous media in the presence of LTNE was analysed by Govender and Vadasz [16]. Rees et. al. [17] have made an analysis on the LTNE effects arising from the injection of a hot fluid into a porous medium. The effect of a horizontal pressure gradient on the onset of a Darcy-Bénard convection in thermal non-equilibrium conditions was investigated by Postelnicu [18]. Kuzentsov and Nield [19] studied the effect of LTNE on the onset of convection in a porous medium layer saturated by a nanofluid. Malashetty and Mahantesh Swamy [20] used LTNE model to study the effect of rotation on the onset of thermal convection in a sparsely packed porous layer. Boundary and thermal non-equilibrium effects on convective instability in an anisotropic porous layer was investigated by Shivakumara et al. [21]. Barletta and Celli [22] studied the local thermal non-equilibrium flow with viscous dissipation in a plane horizontal porous layer. Effects of thermal non-equilibrium and non-uniform temperature gradients on the onset of convection in a heterogeneous porous medium was investigated by Shivakumara et al. [23]. Lee et al. [24] have considered a LTNE to study the effect of nonuniform temperature gradient on thermogravitational convection in a porous layer. Convective transport in a nanofluid saturated porous layer with thermal non-equilibrium was analysed by Bhadauria and Agarwal [25]. Saravanan and Sivakumar [26] studied the onset of thermovibrational

filtration convection: departure from thermal equilibrium. Local thermal non-equilibrium effects in the Darcy - Bénard instability with isoflux boundary conditions was investigated by Barlette and Rees [27]. Nield [28] made a note on LTNE in porous media near boundaries and interfaces. The effect of LTNE was considered by Patil and Rees [29] to study the linear stability analysis of a horizontal boundary layer formed by vertical throughflow in a porous medium. Celli et al. [30] studied the LTNE effects in the Darcy-Bénard instability in a porous layer heated from below by a uniform flux. Thermorheological effect on thermal nonequilibrium porous convection with heat generation was analysed by Saravanan and Senthilnayaki [31]. Thermoconvective instability and local thermal non-equilibrium in a porous layer with isoflux-isothermal boundary conditions was studied by Celli et al. [32]. Dehgan et al. [33] studied the perturbation analysis of the LTNE condition in a fluid-saturated porous medium bounded by an iso-thermal channel. LTNE effects in the Horton-Rogers-Lapwood problem with a free surface was analysed by Celli et al. [34].

In the present paper we study linear and weakly nonlinear stability analyses of Rayleigh-Bénard-Brinkman convection in the presence of LTNE effect using 5-mode and 8-mode Lorenz models. We show 8-mode Lorenz model will produce qualitatively similar result like 5-mode Lorenz model. Hence its sufficient to use minimal modes for perform weakly nonlinear analysis. Later the analytically intractable 5-mode Lorenz model is transformed to analytically tractable first order Ginzburg-Landau equation using multiscal method using which we quantify heat transport. Thus the main objectives of the present paper are the following:

1) Studying the effect of additional modes on local nonlinear stability analysis,
2) Deriving Ginzburg-Landau model from Lorenz model,
3) Showing the influence of parameters on onset of convection and heat transport and
4) Obtaining LTE as a limiting case of LTNE.

II. NOMENCLATURE

Latin symbols

A, B, C	amplitudes
$\mathbf{g}$	acceleration due to gravity $(0, 0, -g)$
J	jacobian
K	permeability of the porous media
Nu	Nusselt number
p	dynamic pressure
Pr	Prandtl number
$\mathbf{q}$	velocity vector
Ra	Rayleigh number
t	time
T	temperature
T_0	reference temperature
u	horizontal velocity
w	vertical velocity
x	horizontal coordinate
z	vertical coordinate
X	non-dimensional horizontal coordinate
Z	non-dimensional vertical coordinate

Greek symbols

α	thermal diffusivity
β	coefficient of thermal expansion
k	thermometric conductivity
Λ	Brinkman number
μ_f	dynamic viscosity of fluid
μ_p	dynamic viscosity of fluid in a porous medium
γ	porosity-modified conductivity ratio
ϕ	porosity of the porous medium
ψ	dimensional stream function
σ^2	Porous parameter
Ψ	non-dimensional stream function
ρ	fluid density
ρ_0	reference fluid density
Θ	non-dimensional temperature

Subscripts

b	basic state
$'$	perturbed state
l	liquid
s	solid

III. MATHEMATICAL FORMULATION

The governing equations for studying two-dimensional, unsteady Rayleigh-Brinkman-Bénard convection (RBBC) in the case when there is local thermal non-equilibrium (LTNE) between liquid and solid phases are:

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

$$\rho_{l_0} \left[\frac{1}{\phi} \frac{\partial q}{\partial t} + \frac{1}{\phi^2} (\vec{q} \cdot \nabla) \vec{q} \right] = \mu_l' \nabla^2 \vec{q} - \frac{\mu_l}{K} \vec{q} + \rho_l \vec{g} - \nabla P, \tag{2}$$

$$(\rho C_p)_l \frac{\partial T_l}{\partial t} = \phi \kappa_l \nabla^2 T_l + h(T_s - T_l) - (\rho c_p)_l (\vec{q} \cdot \nabla) T_l, \tag{3}$$

$$(\rho C_p)_s \frac{\partial T_s}{\partial t} = (1 - \phi) \kappa_s \nabla^2 T_s - h(T_s - T_l), \tag{4}$$

$$\rho(T) = \rho(T_0)(1 - \beta(T_l - T_0)). \tag{5}$$

Considering velocity, temperature, density and pressure fields in the quiescent basic state to be:

$$\left. \begin{array}{l} \vec{q} = \vec{q}_b = (0, 0), \\ T_l(z) = T_{l_b}(z), \quad T_s(z) = T_{s_b}(z), \\ \rho(z) = \rho_b(z), \quad P(z) = P_b(z), \end{array} \right\}, \tag{6}$$

we obtain the quiescent state solution for the temperature distributions in the form:

$$T_{l_b}(z) = T_0 + \Delta T \left(\frac{1}{2} - \frac{z}{d} \right), \tag{7}$$

$$T_{s_b}(z) = T_0 + \Delta T \left(\frac{1}{2} - \frac{z}{d} \right). \tag{8}$$

We now superimpose perturbation on the quiescent basic state quantities and so we write:

$$\left. \begin{array}{l} \vec{q} = \vec{q}_b + \vec{q}', T_l(z) = T_{l_b} + T'(z), T_s(z) = T_{s_b} + T'(z), \\ \rho = \rho_b + \rho', P = P_b + P', \end{array} \right\}, \tag{9}$$

where the primes indicate a perturbed quantity. Eliminating the pressure term in (2) and introducing the stream function,

ψ, as follows

$$u = \frac{\partial \psi}{\partial z}, \ w = -\frac{\partial \psi}{\partial x}. \tag{10}$$

and making (1)-(5) dimensionless using

$$(X, Z) = \left(\frac{x}{d}, \frac{z}{d}\right), \ \Psi = \frac{\phi\psi}{\alpha_l}, \ \Theta_l = \frac{T_l}{\Delta T}, \ \Theta_s = \frac{T_s}{\Delta T}, \tag{11}$$

the dimensionless form of the vorticity and heat transport equations can be obtained in the form

$$\frac{1}{Pr}\frac{\partial}{\partial \tau}(\nabla^2 \Psi) = \Lambda \nabla^4 \Psi - \sigma^2 \nabla^2 \Psi - Ra\frac{\partial \Theta_l}{\partial X}, \tag{12}$$

$$\frac{\partial \Theta_l}{\partial \tau} = -\frac{\partial \Psi}{\partial X} + \nabla^2 \Theta_l + H(\Theta_s - \Theta_l) + \frac{\partial \Psi}{\partial X}\frac{\partial \Theta_l}{\partial Z} - \frac{\partial \Psi}{\partial Z}\frac{\partial \Theta_l}{\partial X}, \tag{13}$$

$$\frac{a_1}{\gamma}\frac{\phi}{(1-\phi)}\frac{\partial \Theta_s}{\partial \tau} = \nabla^2 \Theta_s + \gamma H(\Theta_l - \Theta_s) = 0, \tag{14}$$

where

$$\Lambda = \frac{\mu'}{\mu}, \ \sigma^2 = \frac{d2}{K}, Ra = \frac{\rho_l(\rho c_p)_l \beta g d^3 \Delta T}{\phi\kappa\mu},$$

$$H = \frac{hd^2}{\phi\kappa_l}, \ \gamma = \frac{\phi\kappa_l}{(1-\phi)\kappa_s}.$$

The stationary convection is the preferred mode at onset. In the following section we perform linear stability analysis to study the condition for onset of convection.

IV. LINEAR STABILITY ANALYSIS

We make a linear stability analysis by considering minimal double Fourier series expansion as follows:

$$\Psi = A \sin(\kappa_c X) \sin\left(\pi Z + \frac{\pi}{2}\right), \tag{15}$$

$$\Theta_l = B \cos(\kappa_c X) \sin\left(\pi Z + \frac{\pi}{2}\right), \tag{16}$$

$$\Theta_s = L \cos(\kappa_c X) \sin\left(\pi Z + \frac{\pi}{2}\right). \tag{17}$$

Substituting (15)-(17) in linearized version of (12)-(14) and taking the orthogonality condition with the eigenfunctions associated with the considered minimal modes, we get

$$\begin{bmatrix} \delta^2(\delta^2\Lambda + \sigma^2) & -\kappa_c Ra & 0 \\ \kappa_c & -(H + \delta^2) & H \\ 0 & \gamma H & -\delta_1^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

where $\delta^2 = (\kappa_c^2 + \pi^2)$ and $\delta_1^2 = \kappa_c^2 + \pi^2 + \gamma H$.

The critical Rayleigh number, Ra_c, is given by

$$Ra_c = \frac{\delta^6}{(\Lambda + \sigma')\kappa_c^2}\left(1 + \frac{H}{\delta_1^2}\right), \tag{19}$$

where $\sigma' = \frac{\sigma^2}{\delta^2}$. In the next section we discuss the non-linear stability analysis in order to find the amplitude equation of Ginzburg-Landau and thereby estimate the heat transport.

V. LOCAL NONLINEAR STABILITY ANALYSIS USING FIVE MODES (MINIMAL REPRESENTATION)

We make a local nonlinear stability analysis by considering minimal double Fourier series expansion as follows:

$$\Psi = \frac{-\sqrt{2}\delta^2}{\pi\kappa}A(\tau)\sin(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right), \tag{20}$$

$$\Theta_l = \frac{\sqrt{2}}{\pi r_l}B(\tau)\cos(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right) - \frac{1}{\pi r_l}C(\tau)\sin(2\pi Z + \pi), \tag{21}$$

$$\Theta_s = \frac{\sqrt{2}}{\pi}L(\tau)\cos(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right) + \frac{1}{\pi}M(\tau)\sin(2\pi Z + \pi), \tag{22}$$

where $r_l = \frac{Ra\kappa^2}{\delta^6}$ and the amplitudes $A(\tau), B(\tau), C(\tau), L(\tau)$ and $M(\tau)$ are to be determined.

Substituting (20)-(22) in (12)-(14) and taking the orthogonality condition with the eigenfunctions associated with the considered minimal modes, we get

$$\frac{1}{Pr}\frac{dA}{d\tau_1} = [-(\Lambda + \sigma')A + B], \tag{23}$$

$$\frac{dB}{d\tau_1} = [r_l A - (H' + 1)B + r_l H'L - AC], \tag{24}$$

$$\frac{dC}{d\tau_1} = [-(H' + b)C - r_l H'M + AB], \tag{25}$$

$$r_l a_1 \frac{dL}{d\tau_1} = [\gamma H'B - r_l(1 + \gamma H')L], \tag{26}$$

$$r_l a_1 \frac{dM}{d\tau_1} = [\gamma H'C - (b + H')M], \tag{27}$$

where $\tau_1 = \delta^2\tau$, $a_1 = \frac{\alpha_l}{\alpha_s}$, $H' = \frac{H}{\delta^2}$ and $b = \frac{4\pi^2}{\delta^2}$.

We now use the following regular perturbation expansion in (23)-(27):

$$\begin{bmatrix} A \\ B \\ C \\ L \\ M \\ r_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \epsilon \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ L_1 \\ M_1 \\ 0 \end{bmatrix} + \epsilon^2 \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ L_2 \\ M_2 \\ r_2 \end{bmatrix} + \epsilon^3 \begin{bmatrix} A_3 \\ B_3 \\ C_3 \\ L_3 \\ M_3 \\ 0 \end{bmatrix} + \cdots \tag{28}$$

and we assume the time variations only at the small time scale $\tau_1^* = \epsilon^2\tau_1$.

Let us now use the following notation:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & 1 & 0 & 0 & 0 \\ r_0 & \mathcal{L}_2 & 0 & r_0 H' & 0 \\ 0 & 0 & \mathcal{L}_3 & 0 & -r_0 H' \\ 0 & \gamma H' & 0 & \mathcal{L}_4 & 0 \\ 0 & 0 & -\gamma H' & 0 & \mathcal{L}_5 \end{bmatrix}, \tag{29}$$

$$V_i = [A_i \ B_i \ C_i \ L_i \ M_i]^{Tr}, (i = 1, 2, 3) \tag{30}$$

where $\mathcal{L}_1 = -(\Lambda + \sigma')$, $\mathcal{L}_2 = -(H' + 1)$, $\mathcal{L}_3 = -(H' + b)$, $\mathcal{L}_4 = -r_0(1 + \gamma H')$ and $\mathcal{L}_5 = -r_o(H' + b)$.

Substituting (28) in (23)-(27) and on comparing the like powers of ϵ on both the sides of the resulting equations, we get the following equations at various orders:

First-order system:

$$\mathcal{L}V_1 = 0, \tag{31}$$

Second-order system:

$$\mathcal{L}V_2 = [R_{21}, R_{22}, R_{23}, R_{24}, R_{25}]^{Tr}, \tag{32}$$

Third-order system:

$$\mathcal{L}V_3 = [R_{31}, R_{32}, R_{33}, R_{34}, R_{35}]^{Tr}, \tag{33}$$

where

$$R_{21} = 0, R_{22} = A_1 C_1, R_{23} = -A_1 B_1, R_{24} = 0, R_{25} = 0, \tag{34}$$

$$\left. \begin{aligned} R_{31} &= \frac{1}{Pr}\frac{dA_1}{d\tau_1^*}, \\ R_{32} &= \frac{dB_1}{d\tau_1^*} - r_2 A_1 - r_2 H' L_1 + A_1 C_2 + A_2 C_1, \\ R_{33} &= \frac{dC_1}{d\tau_1^*} - (A_1 B_2 + A_2 B_1), \\ R_{34} &= a_1 r_0 \frac{dL_1}{d\tau_1^*} - r_2(1 + \gamma H')L_1, \\ R_{35} &= a_1 r_0 \frac{dM_1}{d\tau_1^*} + r_2(H' + b)M_1. \end{aligned} \right\} \tag{35}$$

The solution of the first- and second-order systems subject to appropriate initial condition are obtained as follows:
First-order solution:

$$V_1 = [A_1, \ (\Lambda + \sigma')A_1, \ 0, \ \frac{\gamma H'(\Lambda + \sigma')}{r_0(1 + \gamma H')}A_1, \ 0]^{Tr}, \tag{36}$$

Second-order solution:

$$V_2 = \left[A_2, \ (\Lambda + \sigma')A_2, \ \frac{(\Lambda + \sigma')(b + H')}{[(b + H')^2 - \gamma H'^2]}A_1^2, \right. \tag{37}$$

$$\left. \frac{\gamma H'(\Lambda + \sigma')}{r_0(1 + \gamma H')}A_1^2, \ \frac{-\gamma H'(\Lambda + \sigma')}{r_0[(b + H')^2 - \gamma H'^2]}A_1^2 \right]^{Tr}, \tag{38}$$

where A_1 and A_2 are arbitrary functions of τ_1^*. We are not interested in finding the solution of the third order system. However, for the purpose of determining the amplitude, A_1, it is sufficient to consider the Fredholm solvability condition and this yields the Ginzburg-Landau equation in the form:

$$\frac{dA_1}{d\tau_1^*} = Q_1 A_1 - Q_2 A_1^3, \tag{39}$$

where

$$Q_1 = \frac{Pr(\Lambda + \sigma')(1 + \gamma H')^2}{P_1}, \tag{40}$$

$$Q_2 = \frac{Pr(\Lambda + \sigma')^2(H' + b)(1 + \gamma H')^2}{P_1[(H' + b)^2 - \gamma H'^2]}, \tag{41}$$

$$P_1 = Pr(\Lambda + \sigma'^2)^2 \left[a_1 \gamma H'^2 + (1 + \gamma H')^2 \right] + r_0(1 + \gamma H')^2,$$

$$r_2 = \frac{Ra_2 \kappa_c^2}{\eta^6},$$

We introduce an additional mode in each of the representations of the stream function and the temperature in succeeding section to verify whether the results from such a study are qualitatively different from the results of the model involving the most minimal mode.

VI. LOCAL NONLINEAR STABILITY ANALYSIS WITH EIGHT-MODES

An eight-mode truncated Fourier series expansion is given by:

$$\Psi = \frac{-\sqrt{2}\delta^2}{\pi\kappa}A(\tau)\sin(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right)$$
$$-\frac{\sqrt{2}\delta^2}{\pi\kappa}A'(\tau)\cos(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right), \tag{42}$$

$$\Theta_l = \frac{\sqrt{2}}{\pi r_l}B(\tau)\cos(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right)$$
$$-\frac{\sqrt{2}}{\pi r_l}B'(\tau)\sin(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right)$$
$$-\frac{1}{\pi r_l}C(\tau)\sin(2\pi Z + \pi), \tag{43}$$

$$\Theta_s = \frac{\sqrt{2}}{\pi}L(\tau)\cos(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right)$$
$$-\frac{\sqrt{2}}{\pi}L'(\tau)\sin(\kappa X)\sin\left(\pi Z + \frac{\pi}{2}\right)$$
$$+\frac{1}{\pi}M(\tau)\sin(2\pi Z + \pi), \tag{44}$$

Substituting (42)-(44) in (12)-(14) and taking the orthogonality condition with the eigenfunctions associated with the considered modes, we get

$$\frac{1}{Pr}\frac{dA}{d\tau_1} = [-(\Lambda + \sigma')A + B], \tag{45}$$

$$\frac{1}{Pr}\frac{dA'}{d\tau_1} = [-(\Lambda + \sigma')A' + B'], \tag{46}$$

$$\frac{dB}{d\tau_1} = [r_l A - (H' + 1)B + r_l H'L - AC], \tag{47}$$

$$\frac{dB'}{d\tau_1} = [r_l A' - (H' + 1)B' + r_l H'L' - A'C], \tag{48}$$

$$\frac{dC}{d\tau_1} = [-(H' + b)C - r_l H'M + AB + A'B'], \tag{49}$$

$$r_l a_1 \frac{dL}{d\tau_1} = [\gamma H'B - r_l(1 + \gamma H')L], \tag{50}$$

$$r_l a_1 \frac{dL'}{d\tau_1} = [\gamma H'B' - r_l(1 + \gamma H')L'], \tag{51}$$

$$r_l a_1 \frac{dM}{d\tau_1} = [\gamma H'C - (b + H')M], \tag{52}$$

Following the method adopted in Section (V) for getting

the real, Ginzburg-Landau equation using the five mode Lorenz model, we get the following coupled system of Ginzburg-Landau equations in the form:

$$\frac{dA_1}{d\tau_1^*} = Q_1 A_1 - Q_2(A_1^3 - A_1^2 A_1'), \qquad (53)$$

$$\frac{dA_1'}{d\tau_1^*} = Q_1 A_1' - Q_2(A_1'^3 - A_1'^2 A_1), \qquad (54)$$

where Q_1 and Q_2 are defined in (40) and (41). Equations (53) and (54) form the coupled Ginzburg-Landau model for nonlinear convection. Equations (53) and (54) can be combined into a single equation given by:

$$\frac{d\mathcal{A}}{d\tau_1^*} = Q_1 \mathcal{A} - Q_2 \mathcal{A}|\mathcal{A}|^2, \qquad (55)$$

where $\mathcal{A} = A_1 + iA_1'$.

In phase-amplitude form, $\mathcal{A}$, can be written as:

$$\mathcal{A} = |\mathcal{A}|e^{i\varphi}. \qquad (56)$$

Substituting (56) in (55), we get the amplitude equation:

$$\frac{d|\mathcal{A}|}{d\tau_1^*} = Q_1|\mathcal{A}| - Q_2|\mathcal{A}|^3. \qquad (57)$$

In the next section we quantify the heat transport in terms of the Nusselt number at the lower boundary for the stationary mode of convection.

A. Estimation of Enhanced Heat Transport in Nanoliquids at Lower Boundary

$$Nu_{nl} = \frac{\text{Heat transport by (conduction+convection)}}{\text{Heat transport by conduction}}. \qquad (58)$$

Using Fourier law for the conductive and convective fluxes, we may write the expressions for the liquid an solid phases in the form:

$$Nu_l = 1 + \frac{\left[-\kappa_l \int_0^{\frac{2\pi}{k}} \frac{\partial \Theta_l}{\partial Z} dX\right]_{Z=-\frac{1}{2}}}{\left[-\kappa_l \int_0^{\frac{2\pi}{k}} \frac{d\Theta_{l_b}}{dZ} dX\right]_{Z=-\frac{1}{2}}}, \qquad (59)$$

$$Nu_s = 1 + \frac{\left[-\kappa_s \int_0^{\frac{2\pi}{k}} \frac{\partial \Theta_s}{\partial Z} dX\right]_{Z=-\frac{1}{2}}}{\left[-\kappa_s \int_0^{\frac{2\pi}{k}} \frac{d\Theta_{s_b}}{bZ} dX\right]_{Z=-\frac{1}{2}}}, \qquad (60)$$

where Nu_l is Nusselt number of the liquid phase and Nu_s is that of the solid phase.

The weighted-average Nusselt number, Nu_w, for stationary mode of convection evaluated at lower boundary $Z = -\frac{1}{2}$ for a single wavelength is given by

$$Nu_w = \phi Nu_l + (1-\phi)Nu_s. \qquad (61)$$

Substituting (6), (21) and (22) in (59) and (60) and

completing the integration, we get

$$Nu_l = 1 + \frac{2}{r_l}\epsilon^2 C_2, \qquad (62)$$

$$Nu_s = 1 - \epsilon^2 M_2. \qquad (63)$$

$$Nu = 1 - \frac{\epsilon^2}{\pi}\left[M_2 - \left(\frac{2C_2\pi}{r_l} + M_2\right)\phi\right] \qquad (64)$$

With the necessary background for analysing the results prepared in the previous sections, in what follows we discuss the results obtained and draw a few conclusions.

VII. RESULTS AND DISCUSSION

Rayleigh-Bénard-Brinkman convection (RBBC) in the presence of LTNE effects is studied analytically in the paper. The stationary convection is the preferred mode at onset. The expression for critical Rayleigh number is derived using minimal Fourier series expansion. The influence of parameters on onset of convection is explained through Rayleigh number.

Fig. 2 is a plot of Rayleigh number versus wave number for different values of porous parameter, σ^2. The figure shows that as we increase σ^2 the critical Rayleigh number increases and this means porous medium delays the onset of convection. Brinkman number, Λ, has an effect analogous to that of σ^2.

The effect of porosity-modified thermal conductivity ratio, γ, is explained in Fig. 4. The figure shows that as we increase γ, the critical Rayleigh number decreases. After a certain range of value of γ there is no great change in the value of the critical Rayleigh number.

The result of LTE can be obtained as a limiting case of that of LTNE by taking $H \to 0$. Fig. 5 very clearly shows that LTE underpredicts the onset of convection compared to LTNE.

The streamlines and the isotherms in the unsteady finite-amplitude convective regime are shown in Figs. 6-9. Streamline plots 6 and 7 are for time $\tau = 0.5$ and $\tau = 1$ respectively. From these plots we observe that as time progresses the convective activity is deep into the center of cell. Similar observation can also be made for isotherms (see plots 8 and 9).

The influence of various parameters on heat transport is explained through plots of Nusselt number. Fig. 10 is a plot of Nusselt number versus time for different values of σ^2. As we increase σ^2 we observe that Nusselt number decreases. A similar effect is observed in the case of Brinkman number. The effect of porosity modified conductivity ratio is to enhance the heat transport. By taking small value of H the result of LTE is obtained and comparison of the result of LTE and LTNE is shown in Fig. 13. The figure reveals that LTE model underpredicts heat transfer compare to that predicted by LTNE.

VIII. CONCLUSION

1) The analytically intractable Lorenz model can be reduced to the tractable Ginzburg-Landau equation using the multiscale method, thus circumventing the need to do a numerical study of the problem.
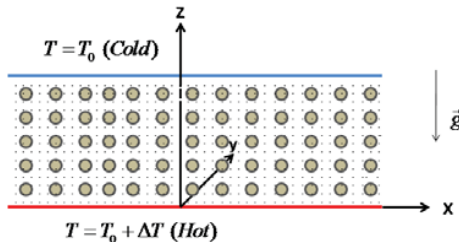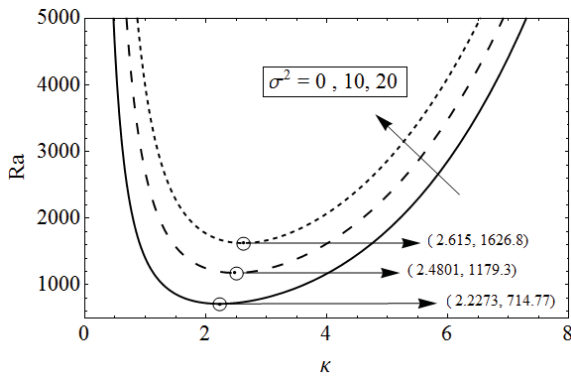2) The 5-mode and 8-mode Lorenz models estimate heat transport identically. Hence the 5-mode Lorenz model is

Fig. 1 Physical configuration

Fig. 2 Plot of Ra_l versus κ for different value of σ and for $H = 10$, $\Lambda = 1$, $\gamma = 20$

a good enough truncated model for a weakly nonlinear study of convection.

3) The effect of porous parameter and Brinkman number is to delay the onset of convection.
4) The effect of porous modified conductivity ratio is to advance the onset of convection.
5) The effect of porous parameter and Brinkman number is to diminish heat transport.
6) The effect of porosity modified conductivity ratio is to enhance the heat transport.
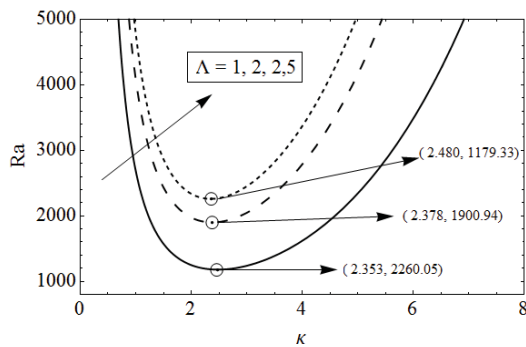
Fig. 3 Plot of Ra_l versus κ for different value of Λ and for $H = 10$, $\sigma^2 = 10$, $\gamma = 20$
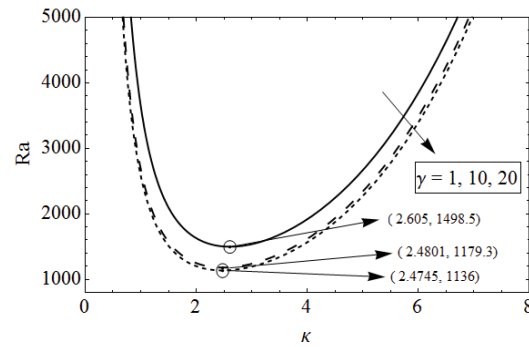
Fig. 4 Plot of Ra_l versus κ for different value of γ and for $H = 10$, $\sigma^2 = 10$, $\Lambda = 1$
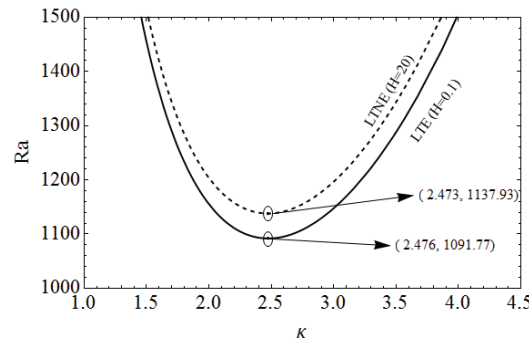
Fig. 5 Plot of thermal Rayleigh number, Ra_l, versus wave number, κ, for different value of H and for $\sigma^2 = 10$, $\Lambda = 1$, $\gamma = 20$
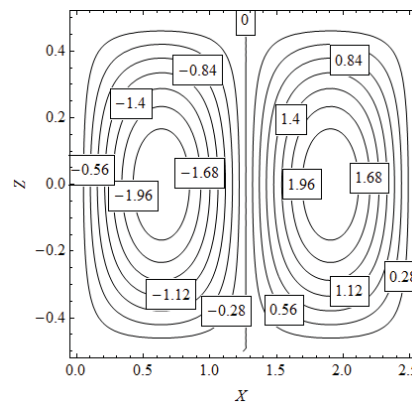
Fig. 6 Contour plot of Stream function for $H = 10$, $\sigma^2 = 10$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 0.5$
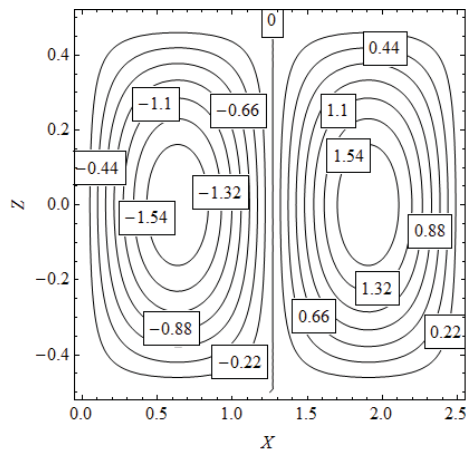
Fig. 7 Contour plot of Stream function for $H = 10$, $\sigma^2 = 10$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 1$
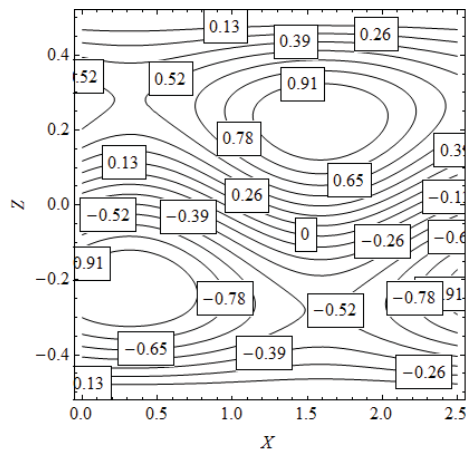

Fig. 8 Isotherms of unsteady convection for $H = 10$, $\sigma^2 = 10$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 0.5$
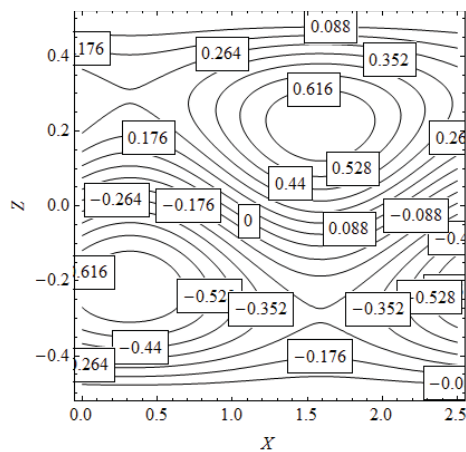

Fig. 9 Isotherms of unsteady convection for $H = 10$, $\sigma^2 = 10$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 1$
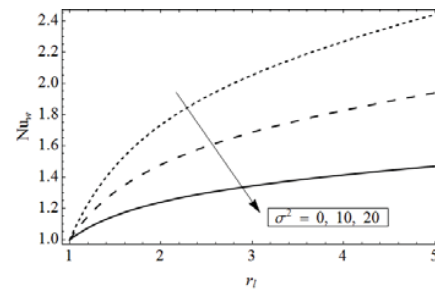

Fig. 10 Plot of Nu versus r_l for different value of σ and for $H = 10$, $\gamma = 20$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 1$

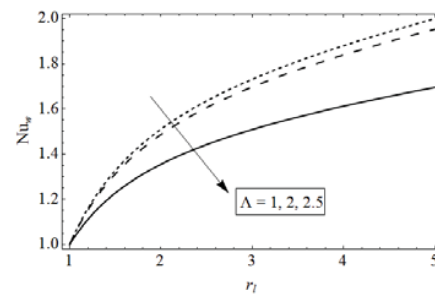

Fig. 11 Plot of Nu versus r_l for different value of Λ and for $H = 10$, $\gamma = 20$, $\sigma^2 = 10$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 1$
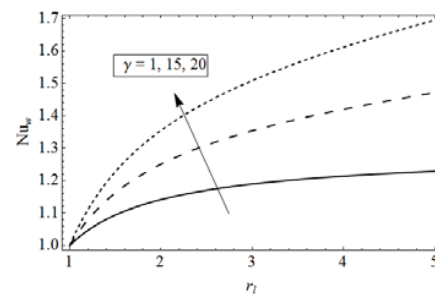

Fig. 12 Plot of Nu versus r_l for different value of γ and for $H = 10$, $\sigma^2 = 10$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 1$
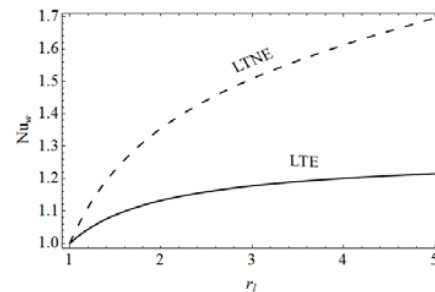

Fig. 13 Plot of Nusselt number, Nu, versus scaled Rayleigh number, r_l, for different value of H and for $\gamma = 20$, $\sigma^2 = 10$, $\Lambda = 1$, $Pr = 4$, $\phi = 0.88$, $a_1 = 1.4$, $A_{1_0} = 1$ and $\tau = 1$

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