Application of Method of Symmetries at a Calculation and Planning of Circular Plate with Variable Thickness

Kirill Trapezon, Alexandr Trapezon

***Abstract*—**A problem is formulated for the natural oscillations of a circular plate of linearly variable thickness on the basis of the symmetry method. The equations of natural frequencies and forms for a plate are obtained, providing that it is rigidly fixed along the inner contour. The first three eigenfrequencies are calculated, and the eigenmodes of the oscillations of the acoustic element are constructed. An algorithm for applying the symmetry method and the factorization method for solving problems in the theory of oscillations for plates of variable thickness is shown. The effectiveness of the approach is demonstrated on the basis of comparison of known results and those obtained in the article. It is shown that the results are more accurate and reliable.

***Keywords*—**Vibrations, plate, thickness, symmetry, factorization, approximation.

I. Introduction

PLATES of variable thickness as components of the structural elements of devices for applied purposes (vibration isolators [1], plate vibration absorbers [2], rotor turbines [3], hydraulic machines [3], tank bottoms [4], bellows, pressure sensors [5]) have wide practical applications in various fields of industry [6]. For example, in the aircraft industry, some thin-walled structural elements are made in the form of plate-like parts of variable thickness. In this example, the plates operate under vibration conditions under resonance conditions, from which the need arises to evaluate the stress-strain state of the elements. The analysis of the state consists of finding the solution of the problem of own flexural vibrations.

The main problem in the cases of plate vibrations is inextricably linked with the search for the solution of fourth-order differential equations.

The purpose of the article is to formulate an algorithm for calculating circular plates of a special configuration. In addition, the problem of axisymmetric vibrations of plates must be solved with the help of comparatively simple analytical dependences, which allow one to find the frequencies, deflections, and stresses of a number of forms of natural oscillations.

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II. Formulation of the Problem

Differential equation of the forms of the proper axisymmetric oscillations of a circular plate of linearly variable thickness, varying according to the law $h = H_0(1-\rho)$, where the constant coefficient H_0; $\rho = r/R$ - the relative variable radius (r – variable radius, R – constant radius), can be written in the form [3]:

$$\left[(1-\rho)\frac{d^2}{d\rho^2}+\left(\frac{1}{\rho}-3\right)\frac{d}{d\rho}\right]\cdot\left[(1-\rho)\frac{d^2W}{d\rho^2}+\left(\frac{1}{\rho}-3\right)\frac{dW}{D\rho}\right]-\lambda^4W=0 \quad (1)$$

where $W = W(\rho)$ is displacement.

Fig. 1 Graphic image experiment of plate

It is obvious that (1) can be replaced by two equations of the second order according to the method of factorization

$$(1-\rho)W''+\left(\frac{1}{\rho}-3\right)W'\pm\lambda^2W=0 \;. \quad (2)$$

Then, the general solution of (1) can be defined as the sum of the general solutions $W = W_1 + W_2$ of two equations, where the solution W_1 of (2) with the plus sign near λ^2, and W_2 with the minus sign.

Finding exact solutions to these equations is difficult from a technical point of view, but the symmetry method obtained by the authors allows solving the problem with an accuracy sufficient for technical applications.

III. Algorithm for Applying the Method

Equation (2) can be rewritten in the form [4]:

$$W_{xx} + \frac{F_x}{F} W_x + k^2 W = 0, \tag{3}$$

where

$$W_x = \frac{dW}{dx}; \; k^2 = \pm 4\lambda^2; \tag{4}$$

and

$$D = \sqrt{F} = D_0 \sqrt{(1-x)^3 - (1-x)^5}. \tag{5}$$

Formally, (3) is analogous to the equation in the forms of longitudinal oscillations for a bar of variable cross-section with an area $F(x)$ whose solution can be found through the symmetry method [5]. To construct a general solution, it is necessary to provide for the functions $W = W_1 + W_2$ the corresponding boundary conditions for x_1 and x_2.

Obviously, (4) is not solvable in elementary or known tabulated functions, but one can find the solution in an approximate way. To do this, we must approximate $D(x)$ by a function $D_1(x)$ in which the solution of these equations will be found in a closed form. As such a function, on the basis of the symmetry method [5],

$$D^* = D_0^* \frac{2n \sin n(x-1)}{n(x-1) - \sin n(x-1) \cos n(x-1) + C^*}, \tag{6}$$

where D_0^*, C^*, n are free constants.

It is important to note that expression (6) on the accepted interval $\rho = 0.1 \div 0.5$ $(x = 0.0513 \div 0.2929)$ satisfactorily corresponds to (5) at $D_0 = 1$, if we assume that $D_0^* =$ -0.164877; $C^* = 4.4375$; $n = 2.849$. It is obtained that the solution of the problem of natural vibrations of a plate rigidly fixed at $\rho_1 = 0.1$ and free at $\rho_2 = 0.5$, obtained on the basis of the approximating function (6), is more accurate than the solution obtained directly on the basis of the rows method.

An approximating function $D_1(x)$ for which (3) has an exact solution is obtained on the basis of the symmetry method

$$D_1 = D_{01} \sqrt{x} \left[J_0(mx) - \chi Y_0(mx) \right] = D_{01} \sqrt{x} Z_0(mx). \tag{7}$$

Fig. 2 shows the variations of $D(x)$ and $D_1(x)$ at $D_0 = 1$; $D_{01} = 1.0173$; $m = 3.35$; $\chi = 0.2322$. As can be seen, on the interval $x = 0.0513 \div 0.2929$, the coincidence of D and D_1 is quite satisfactory.

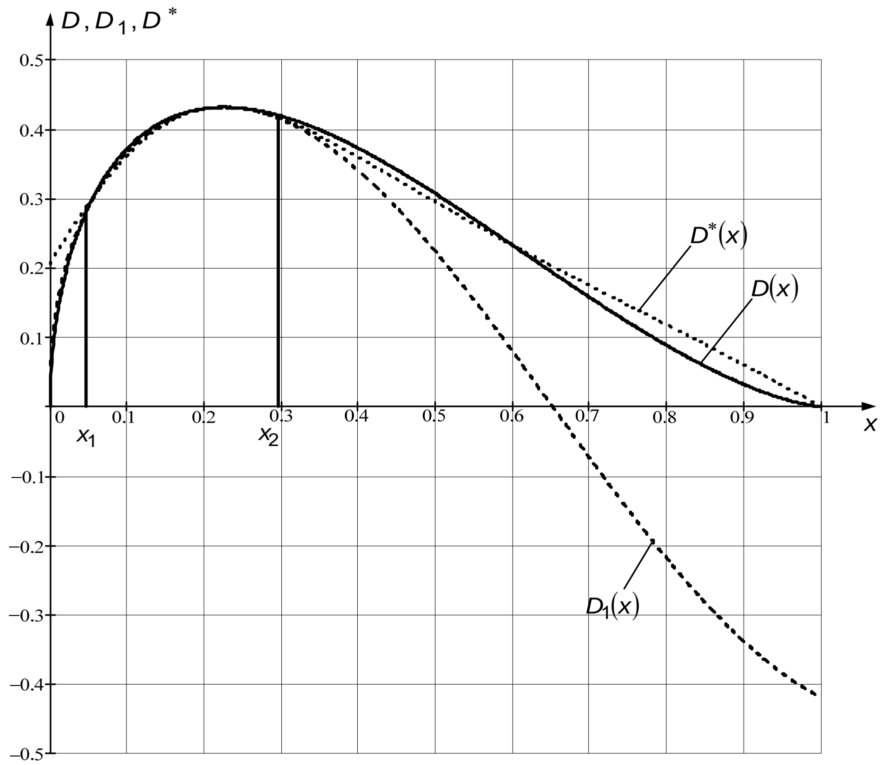

Fig. 2 Graphic image of building functions

Tables I and II show the values of functions D, D_1, from which it can be concluded that the quantitative discrepancy D and D_1 on the average does not exceed the absolute values of $\delta_{average} = 0.41\%$.

Equation (3) with the choice of the function $D_1(x)$ has the following form

$$W_{1,2}'' + 2\frac{\left[\sqrt{x}Z_0(mx)\right]'}{\sqrt{x}Z_0(mx)}W_{1,2}' + k_{1,2}^2 W_{1,2} = 0\,, \quad (8)$$

where

$$Z_0(mx) = J_0(mx) - \chi Y_0(mx)\,; \quad (9)$$

and

$$m = 3.35;\ \chi = 0.2322;\ k_1^2 = 4\lambda^2;\ k_2^2 = -4\lambda^2\,. \quad (10)$$

On the basis of the symmetry method, an exact solution of (8)

$$W_1 = \frac{AJ_0(\alpha x) + BY_0(\alpha x)}{Z_0(mx)}\,; \quad (11)$$

$$W_2 = \frac{A_1 I_0(\beta x) + B_1 K_0(\beta x)}{Z_0(mx)}\,, \quad (12)$$

where

$$\alpha^2 = 4\lambda^2 + m^2;\ \beta^2 = 4\lambda^2 - m^2. \quad (13)$$

TABLE I
NUMERICAL VALUES OF FUNCTIONS

x	0.0513	0.1	0.15	0.2	0.2254
D	0.29216	0.37217	0.41282	0.42933	0.43116
D_1	0.29193	0.6733	0.40985	0.42901	0.43134
$\delta = \left(\frac{D}{D_1} - 1\right)\times 100\%$	0.081	1.316	0.7249	0.07254	-0.04

TABLE II
NUMERICAL VALUES OF FUNCTIONS

x	0.25	0.26	0.27	0.2929
D	0.42962	0.42816	0.42627	0.42045
D_1	0.4293	0.42733	0.42472	0.41642
δ	0.0727	0.19449	0.365	0.966

The boundary conditions of the problem are

$$(W)_{\rho=\rho_1} = 0\,;\ (W_\rho)_{\rho=p_1} = 0\,, \quad (14)$$

and on the free edge

$$\left(W_{\rho\rho} + \frac{\nu}{\rho}W_\rho\right)_{\rho=\rho_2} = 0\,; \quad (15)$$

$$\left(W_{\rho\rho\rho} + \frac{1}{\rho}W_{\rho\rho} - \frac{1}{\rho^2}W_\rho\right)_{\rho=\rho_2} = 0\,. \quad (16)$$

When passing to the variable $x(\rho)$, the conditions (14)-(16) take the form, starting from the expressions $x = 1 - \sqrt{1-\rho}$ and $\rho = 1 - (x-1)^2 = 1 - (1-x)^2$.

For convenience, it is assumed that

$$W_\rho = x_\rho W_x;\quad W_{\rho\rho} = x_\rho^2 W_{xx} + x_{\rho\rho}W_x; \quad (17)$$

$$W_{\rho\rho\rho} = x_\rho^3 W_{xxx} + 3x_\rho x_{\rho\rho}W_{xx} + x_{\rho\rho\rho}W_x; \quad (18)$$

where

$$x = 1 - \sqrt{1-\rho}; \quad (19)$$

$$x_\rho = \frac{1}{2(1-x)};\ x_\rho^2 = \frac{1}{4(1-x)^2};\ x_\rho^3 = \frac{1}{8(1-x)^3}; \quad (20)$$

$$x_{\rho\rho} = \frac{1}{4(1-x)^3};\ x_{\rho\rho\rho} = \frac{3}{8(1-x)^5}; \quad (21)$$

$$3x_\rho x_{\rho\rho} = \frac{3}{8(1-x)^4}. \quad (22)$$

IV. ANALYSIS OF RESULTS

It is found that the general solution of (1) has the form

$$W = \frac{1}{Z_0(mx)}\left[AJ_0(\alpha x) + BY_0(\alpha x) + A_1 I_0(\beta x) + B_1 K_0(\beta x)\right] \quad (23)$$

where A, B, A_1, B_1 are the constants whose values depend on the boundary conditions, and they can be found from the solution of a system of homogeneous equations:

$$\left.\begin{aligned} AJ_0(\alpha x_1) + BY_0(\alpha x_1) + A_1 I_0(\beta x_1) + B_1 K_0(\beta x_1) = 0 \\ AJ_0'(\alpha x_1) + BY_0'(\alpha x_1) + A_1 I_0'(\beta x_1) + B_1 K_0'(\beta x_1) = 0. \end{aligned}\right\} \quad (24)$$

For the first form of oscillation ($\lambda_1 = 4.317126$), we obtained

$$\frac{A_1}{B_1} = 0.095498;$$

$$\frac{B}{B_1} = 1.072439;\ \frac{A}{B_1} = \text{-}0.719443. \quad (25)$$

To construct specific forms of oscillations, we use the deflection function (23)

$$W_i = \frac{B_1}{Z_0(mx)}\left[\frac{A}{B_1}J_0(\alpha_i x) + \frac{B}{B_1}Y_0(\alpha_i x) + \frac{A_1}{B_1}I_0(\beta_i x) + K_0(\beta_i x)\right] \quad (26)$$

where the parameter B_1 is a freely selectable parameter. We select the parameter value from the normalization condition of the function W_i in such a way that

$$W_i(\rho = 0.5) = 1 \quad (27)$$

Fig. 3 shows the first three forms of natural vibrations of a plate. The values of the normalizing coefficient B_1 for the forms W_1, W_2, W_3 are: 0.796202, -2.275333, and 5.365396, respectively.

The value of the found frequency parameter of the plate on the first form of oscillations, as compared with $\lambda_1 = 4{,}3859$, calculated when solving the problem of the rows methods [7], is lower by 0.96%.

The practical significance of such a discrepancy is unimportant [8]; however, in some cases, when a reliable estimate of the stress-strain state of the plate is required, it is necessary to use the vibration parameters obtained in the article on the basis of the function $D_1(x)$.

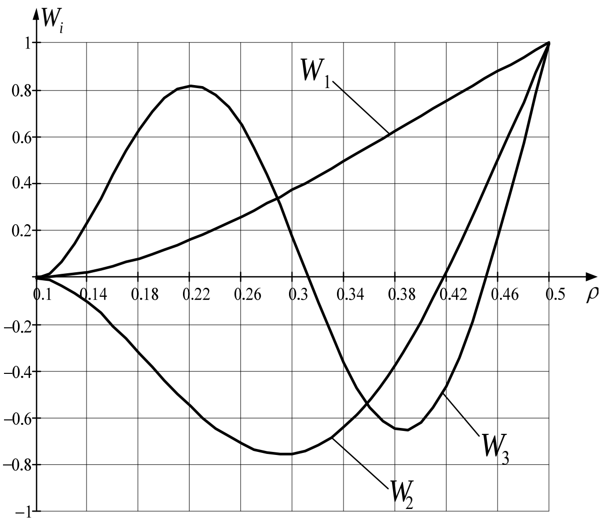

Fig. 3 Graphic image of own forms

V. Conclusions

The scientific novelty and practical value of the results consists of obtaining a new version of the application of the symmetry method for solving the problem of oscillations of an axisymmetric plate of linearly variable thickness.

The practical value of theoretical results includes the possibility of direct use of computational models, in particular, for the rational design of resonance sound and ultrasonic systems based on plates. Thus, based on the results obtained and the study conducted, the following main conclusions can be formulated:

- A simple solution of the problem is found for the self-axisymmetric oscillations of a circular plate on the basis of the symmetry method;
- Equations of frequencies and forms of natural oscillations are obtained for an annular plate with rigid fixation along the inner contour;
- The first three frequencies are calculated and the corresponding eigenmodes of oscillations are constructed;
- The effectiveness of the solution of the problem using the symmetry method is confirmed;
- A calculation model for the rational design of plates as acoustically active elements for sound and ultrasound systems is constructed.

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References

[1] B. Radg, V. Radgendran, P. Palanichami, *Applications of ultrasound.* Moscow, Russia: Technosfera, 2006.
[2] L. Bergman, *Ultrasound and his application are in science and technique.* Moscow, USSR: Nauka, 1957.
[3] A. Trapezon, *Calculation of resilient elements at resonant fatigue tests.* Kyiv, Ukraine: Naukova dumka, 1983.
[4] S. Timoshenko, D. Young, *Vibration problems in engineering.* New York, USA: Van Nostrand, 1955.
[5] K. Trapezon, “Method of symmetries at a calculation and planning of acoustic thickeners,” *Acoustic Bulletin*, vol. 9, no. 4, pp. 50–55, Dec. 2006.
[6] K. Trapezon, A. Trapezon, “Ultrasonic oscillating system,” in 2016 *Proc. RTPSAS-2016* Conf., pp. 69–71.
[7] A. Kovalenko, *Round plates with variable thickness.* Moscow, USSR: Phizmatgiz, 1959.
[8] K. Trapezon, “Method of factorization at the decision of tasks on own values,” *Electronics and Communications*, vol. 68, no. 3, pp. 19–23, Sept. 2012.