On CR-Structure and F-Structure Satisfying Polynomial Equation

Manisha Kankarej

Abstract—The purpose of this paper is to show a relation between CR structure and F-structure satisfying polynomial equation. In this paper, we have checked the significance of CR structure and F-structure on Integrability conditions and Nijenhuis tensor. It was proved that all the properties of Integrability conditions and Nijenhuis tensor are satisfied by CR structures and F-structure satisfying polynomial equation.

Keywords—CR-submainfolds, CR-structure, Integrability condition & Nijenhuis tensor.

2000 AMS Mathematics Subject Classification—53C40, 53D10.

I. Introduction

THE study of F structure and CR structure is done by many mathematicians. In this paper the study of these structures are considered with polynomial equations, the study of Integrability and Nijenhuis Tensor is also extended to polynomial equation. Yano [1] initiated the study of F structure. Nikie [8] and Das [9] further studied the properties of F structure.

Let *F* be a non zero tensor field of type (1,1) and of class C^∞ dimensional manifold *M* such that

$$a_nF^n + a_{n-1}F^{n-1} \dots . a_2F^2 + a_1F^1 = 0 \tag{1}$$

where n is a fixed positive integer greater than or equal to 1. Such a structure on M is called an F-structure. If the rank of *F* is constant and *r=F(r),* then *M* is called an *F* structure manifold of degree *n.*

Let us define the operator on *M* as:

$$l = -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) \tag{2}$$

$$m = I + \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) \tag{3}$$

where I denotes the identify operator on M.

Theorem 1. Let *M* be an *F(*$a_n, a_{n-1}, \dots a_1$*)* structure manifold satisfying (1) then

a) $l + m = I$
b) $l^2 = l$
c) $m^2 = m$
d) $l.m = 0$

Manisha M. Kankarej is with the Department of Mathematics and Statistics, University College, Dubai, UAE (e-mail: Manisha.Kankarej@zu.ac.ae).

Proof.

a) $l + m = I$

$$l + m = -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) + I + \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) = I$$

$$\Rightarrow l + m = I \tag{4}$$

b) $l^2 = l$

$$l^2 = -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) * -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) =$$

$$-\left(\frac{-a_1}{a_1}\right) * -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) = -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) = l$$

$$\Rightarrow\ l^2 = l \tag{5}$$

c) $m^2 = m$

$$m^2 = \left[I + \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right)\right] * \left[I + \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right)\right] =$$

$$I^2 + 2I\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) + \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right)^2 =$$

$$I^2 + 2I\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) + l^2 = I^2 + 2I\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) + l =$$

$$I^2 + 2I\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) + -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) =$$

$$I + \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots \dots a_3F^2+a_2F^1}{a_1}\right) = m$$

$$\text{So } m^2 = m \tag{6}$$

d) $l.m = l.\,(I-l)$

$$= l - l^2 = l - l \Rightarrow l.m = 0 \tag{7}$$

For $F \neq 0$ satisfying (1) there exist complimentary distributions D_l & D_m corresponding to the projection operator *l* & *m* respectively. If Rank *F* = constant and $r = r(F)$. Then, dim $D_l = r$ and $D_m = n - r$.

Theorem 2. We have-

a) (I) $lF = Fl = F,$
(II) $mF = Fm = 0$

b) (I) $\left(\frac{a_nF^{n}+a_{n-1}F^{n-1}+\cdots \dots a_3F^3+a_2F^2}{a_1}\right) * m = 0$

(II) $\left(\frac{a_nF^n+a_{n-1}F^{n-1}+\cdots\ldots a_3F^3+a_2F^2}{a_1}\right)*l$ = - l

Proof.

a) (I) $lF = Fl = F$

$$lF = \left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots\ldots a_3F^2+a_2F^1}{a_1}\right)*F= -\left(\frac{a_nF^n+a_{n-1}F^{n-1}+\cdots\ldots a_3F^3+a_2F^2}{a_1}\right)= F$$

So $lF = Fl = F$ (8)

(II) $mF = Fm = 0$

$$mF = \left[I+\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots\ldots a_3F^2+a_2F^1}{a_1}\right)\right]*F= \left[F+\left(\frac{a_nF^n+a_{n-1}F^{n-1}+\cdots\ldots a_3F^3+a_2F^2}{a_1}\right)\right]= F + (-F)= 0$$

So $mF = Fm = 0$ (9)

b) (I)$\left(\frac{a_nF^n+a_{n-1}F^{n-1}+\cdots\ldots a_3F^3+a_2F^2}{a_1}\right)*m = 0$

$= -F * m = 0$ (10)

(II)$\left(\frac{a_nF^n+a_{n-1}F^{n-1}+\cdots\ldots a_3F^3+a_2F^2}{a_1}\right)*l$

$= -F* l = - F$ (11)

Thus, F acts on D_l as an almost complex structure and on D_m as a null operator.

II. Nijenhius Tensor

The Nijenhius tensor N(X,Y) of F satisfying (1) in M is expressed as follows for every vector field X,Y on M.

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY]+ F^2[X,Y] \quad (12)$$

We state the following theorem without proof

Theorem 3. A necessary & sufficient condition for the f-structure to be integrable is that $N(X,Y)=0$ for any vector field X & Y on M.

III. Lie bracket

If X & Y are two vector fields in M then their lie bracket [X,Y] is defined by

$$[X,Y] = XY - YX \quad (13)$$

IV. CR-Structure

A study of differential geometry of a CR submanifold has been initiated in [4]-[7]. Results on general theory of Cauchy Riemann manifolds have been obtained by [2].

Let M be a differentiable manifold and $T_c(M)$ be its complex field on tangent bundle M. A CR-Structure on M is a complex sub bundle H of $T_c(M)$ such that $H_p \cap H_{\bar{p}} = 0$ & H is involutive i.e. for complex vector field & Y in H, $[X,Y]$ is in H. In this case we say M is a CR-manifold.

Let $F(a_n, a_{n-1},\ldots a_1)$ be an integrable structure satisfying (1) of rank $r = 2m$ on M. We define complex sub bundle H of $T_c(M)$ by

$$H p = \{X - \sqrt{-1}\, FX,\ X \in \chi(Dl)\} \quad (14)$$

where $\chi(D_l)$ is the F(D_m) module for all differentiable sections of D_l. The Re(H) = D_l & $H_p \cap H_{\bar{p}} = 0$, where $H_{\bar{p}}$ denotes the complex conjugate. Intigrability conditions on such submanifolds have been investigated by [4].

Theorem 4. If P & Q are two elements of H then the following relation holds

$$[P, Q] = [X,Y] - [FX, FY] - \sqrt{-1}[X, FY] - \sqrt{-1}\,[FX,Y]$$

Proof. Let us define

$$P = X - \sqrt{-1}FX$$
$$Q = Y - \sqrt{-1}Y$$

then by direct calculation & on simplifying, we obtain-

$$[P,Q] = [X-\sqrt{-1}FX,\ Y-\sqrt{-1}FY]$$
$$= [X,Y] - \sqrt{-1}\,[X,FY] - \sqrt{-1}\,[FX,Y]\ Y - [FX,FY]$$
$$= [X,Y] - [FX,FY] - \sqrt{-1}\,[X,FY] - \sqrt{-1}\,[FX,Y] \quad (15)$$

Theorem 5. If $F(a_n, a_{n-1}\ldots. a_2, a_1)$ structure satisfying (1) is integrable then we have

$$-\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots\ldots a_3F^1+a_2}{a_1}\right)\{[FX*FY]+F^2[X*Y]\} = l\ \{[FX,Y] + [X,FY]\}$$

Proof. From (12) we have,

$$N(X,Y) = [FX,FY] + F^2[X,Y] - F[FX,Y] - F[X,FY]$$

Since $N(X,Y) = 0$ we obtain

$$[FX,FY] + F^2[X,Y] = F[FX,Y] + F[X,FY]$$

Operating $-\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots\ldots a_3F^1+a_2}{a_1}\right)$

$$=-\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots\ldots a_3F^1+a_2}{a_1}\right)\{[FX,FY] + F^2[X,Y]\}$$

$$= -\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots\ldots a_3F^1+a_2}{a_1}\right)\{F[FX,Y] + F[X,FY]\}$$
$$= -\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots\ldots a_3F^1+a_2}{a_1}\right)F\{[FX,Y] + [X,FY]\}$$

$$= -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots\ldots a_3F^2+a_2F}{a_1}\right)\{[FX,Y] + [X,FY]\}$$
$$= -\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots\ldots a_3F^2+a_2F}{a_1}\right)\{[FX,Y] + [X,FY]\}$$
$$= l\ \{[FX,Y] + [X,FY]\}$$

This proves the above theorem.

Theorem 6. The following identities hold

a) $mN(X,Y)= m[FX,FY]$

b) $mN[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots\ldots a_3F^1+a_2}{a_1}\right)X,Y]$

$= m\ [\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots\ldots a_3F^2+a_2F}{a_1}\right),FY]$

Proof.

a) $mN(X,Y) = m\{ [FX,FY] + F^2[X,Y] – F[FX,Y] – F[X,FY]\}$

$$mN (X,Y) = m\{[F, F] + F^2 [X,Y] − F[FX,Y] − F[X, FY]\}$$
$$=m[FX, FY] + m.F.F[X,Y] − mF[FX,Y] − mF[X, FY] = m[FX, FY]$$

$$\Rightarrow mN (X,Y) = m[FX, FY] \tag{16}$$

b) $mN[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X,Y = m[\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots....a_3F^2+a_2F}{a_1}\right),FY]$

$$mN[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X,Y]= mN[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)FX,FY]= F^2[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X,Y] – F[F\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X,Y] – F[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X,FY]\}$$

By the equation $mF = 0 = Fm$

$$mN[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X,Y]= m[\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots....a_3F^2+a_2F}{a_1}\right),FY] \tag{17}$$

Theorem 7. For any two vector field X & Y, the following condition are equivalent –

a) $mN (X,Y)=0$
b) $m[Fx, Fy] = 0$
c) $mN[\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots....a_3F^2+a_2F}{a_1}\right)X, Y] = 0$
d) $m[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X, FY] = 0$
e) $m[\left(\frac{a_nF^{n}+a_{n-1}F^{n-1}+\cdots....a_3F^3+a_2F^2}{a_1}\right) lX, FY] = 0$

Proof. a) => b)

$$mN (X,Y) = 0$$

$$=>m\{[FX, FY] =F^2 [X,Y] −F[FX,Y] −F[X, FY]\} =0$$
$$=>m[FX, FY] = 0 \tag{18}$$

[since $mF = Fm = 0$]

c) ⇒ a)

$$mN[\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots....a_3F^2+a_2F}{a_1}\right)X, Y] = 0$$

By (1)

$$\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots....a_3F^2+a_2F}{a_1} = -I$$

$$==> mN[-X, Y] = 0$$
$$=> mN[X, Y] = 0 \tag{19}$$
$$=>c) =>a)$$

d) => b)

$$m[\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right)X, FY] = 0$$
$$\left(\frac{a_nF^{n-2}+a_{n-1}F^{n-3}+\cdots....a_3F^1+a_2}{a_1}\right) = -F$$

By (1)

$$m[-FX, FY] = 0 \tag{20}$$
$$d) => b)$$

e)=> b)

$$m[\left(\frac{a_nF^{n}+a_{n-1}F^{n-1}+\cdots....a_3F^3+a_2F^2}{a_1}\right) lX, FY] = 0$$
$$m[\left(\frac{a_nF^{n}l+a_{n-1}F^{n-1}l+\cdots....a_3F^3l+a_2F^2l}{a_1}\right)X, FY] = 0$$
$$m[\left(\frac{a_nF^{n-1}Fl+a_{n-1}F^{n-2}Fl+\cdots....a_3F^2Fl+a_2F^1Fl}{a_1}\right)X, FY] = 0$$
$$m[\left(\frac{a_nF^{n-1}Fl+a_{n-1}F^{n-2}Fl+\cdots....a_3F^2Fl+a_2F^1Fl}{a_1}\right)X, FY] = 0$$
$$m[\left(\frac{a_nF^{n-1}(-F)+a_{n-1}F^{n-2}(-F)+\cdots....a_3F^2(-F)+a_2F^1(-F)}{a_1}\right)X, FY] = 0$$
$$\left(\frac{a_nF^{n-1}(-F)+a_{n-1}F^{n-2}(-F)+\cdots....a_3F^2(-F)+a_2F^1\ (-F)}{a_1}\right) = -F$$

$$\Rightarrow m[−FX, FY] = 0 \text{[By (1)]}$$
$$\Rightarrow m[FX, FY] = 0 \tag{21}$$
$$\Rightarrow e) \Rightarrow b)$$

Theorem 8. If F^n acts on D_l as an almost complex structure. Then

$$m[\left(\frac{a_nF^{n}+a_{n-1}F^{n-1}+\cdots....a_3F^3+a_2F^2}{a_1}\right) lX, FY]= m[-X,FY] = 0$$

Proof.

$$m[\left(\frac{a_nF^{n}+a_{n-1}F^{n-1}+\cdots....a_3F^3+a_2F^2}{a_1}\right) lX, FY]$$
$$= m[\left(\frac{a_nF^{n-1}+a_{n-1}F^{n-2}+\cdots....a_3F^2+a_2F^1}{a_1}\right) FlX, FY]$$
$$= m[-FlX, FY]=[−X, Fy] \text{ \{By (8)\}}$$

Theorem 9. For X, Y ∈ x(D_l) we have

$$l ([X, FY] +[FX, FY]) = [X, FY] +[FX,Y]$$

Proof.

$l ([X, FY]) +[FX,Y])= l\{X. FY – FY.X + FX.Y – Y.FX\}$ {By (5)}

$= X. FY – FY.X + FX.Y – Y.FX$ {By (13)}

$=[X, FY] + [FX,Y]$

Theorem 10: The integrable $F(a_n, a_{n−1}.....a_1)$ structure satisfying (1) on M defines a CR-structure H on it. Such that $ReH=D_l$.

Proof. From theorem 4 we have,

$$[P, Q] = [X,Y] −[FX,FY] −\sqrt{−1}[X,FY] −\sqrt{−1} [FX,Y]$$

$l[P, Q] = l[X,Y] - l[FX, FY] - \sqrt{-1}\,([X, FY] + [FX,Y]$
{By theorem (9)}

$=[X,Y] - [FX, FY] - \sqrt{-1}\,([X, FY] + [FX,Y] = [P, Q]$
{By theorem (4)}

Since $l[P, Q] = [P, Q] \Rightarrow [P, Q] \in x(Dl)$. Then, $F(a_n, a_{n-1} \ldots a_1)$ structure satisfying (1) on M defines a CR-structure.

V. Morphism of Vector Bundles

Let $\bar{K}$ be the complementary distribution of $Re(H)$ to TM. We define a morphism of vector bundles $F: TM \to TM$ given by

$$F(X) = 0 \;\forall\; X \in \chi(\bar{K})) \text{ such that-}$$

We have

$$F(X) = \tfrac{1}{2}\sqrt{-1}\,(P - \bar{P})$$

where $P = X + \sqrt{-1}\,Y \,\varepsilon\, x(HP)$ and $\bar{P}$ is the complex of P.

Corollary 1. If $P = X+iY$ and $\bar{P} = X-iY$ belong to H_p and $F(X) = \frac{1}{2}\sqrt{-1}(P-\overline{P}), F(Y) = \frac{1}{2}\sqrt{-1}(P+\overline{P})$ and $F(-Y) = \frac{1}{2}\sqrt{-1}(P+\overline{P})$ then $F(X) = \frac{1}{2}\sqrt{-1}(P-\overline{P}) = -Y$, $F^2(X) = -X$ and $F(-Y) = -X$.

Proof. $P = X + \sqrt{-1}Y$ and $\bar{P} = X - \sqrt{-1}Y \Rightarrow = \frac{(P+\overline{P})}{2}$, $Y = \frac{(P-\overline{P})}{2\sqrt{-1}}$. Since $P + \bar{P} = 2X$ and $P - \bar{P} = 2\sqrt{-1}\,Y$. $F(X) = F[\frac{P+\bar{P}}{2}] = \frac{1}{2}\sqrt{-1}(P-\overline{P}) = -Y$ from the definition of morphism

$$F(-Y) = F\left[-\frac{P-\bar{P}}{2\sqrt{-1}}\right] = -X$$

Theorem 11. If M has a CR-structure H, then we have $a_nF^n + a_{n-1}F^{n-1} \ldots a_2F^2 + a_1F^1 = 0$ and consequently $F(a_n, a_{n-1}, \ldots a_2, a_1)$ structure satisfying (1) is defined on M such that the distribution D_l and D_m coincide with $Re(H)$ and $\bar{K}$ respectively.

Proof. Suppose M has a CR-structure. Then in view of definition of CR manifold & corollary 1 we have-

$$F(X) = -Y;$$

operating above equation by-$\frac{a_nF^{n-1} + a_{n-1}F^{n-2} \ldots a_2F^1}{a_1}$ on both sides we get

$$\left(\frac{a_nF^{n-1} + a_{n-1}F^{n-2} \ldots a_2F^1}{a_1}\right)(F(X) = \left(\frac{a_nF^{n-1} + a_{n-1}F^{n-2} \ldots a_2F^1}{a_1}\right)(-Y)$$

on making use of Corollary 1 the right hand side of the above equation becomes

$$\left(\frac{a_nF^{n-1} + a_{n-1}F^{n-2} \ldots a_2F^1}{a_1}\right)F(X) = \frac{a_nF^{n-2} + a_{n-1}F^{n-3} \ldots a_2}{a_1})F(-Y)$$

which can be written as –

$$\left(\frac{a_nF^{n-1} + a_{n-1}F^{n-2} \ldots a_2F^1}{a_1}\right)F(X) = \frac{a_nF^{n-2} + a_{n-1}F^{n-3} \ldots a_2}{a_1}(-X)$$
$$= -\frac{a_nF^{n-2} + a_{n-1}F^{n-3} \ldots a_2}{a_1}(X) = -\frac{a_nF^{n-3} + a_{n-1}F^{n-4} \ldots a_2F^{-1}}{a_1}$$
$$F(X) = -\frac{a_nF^{n-3} + a_{n-1}F^{n-4} \ldots a_2F^{-1}}{a_1}(-Y) =$$
$$-\frac{a_nF^{n-4} + a_{n-1}F^{n-5} \ldots a_2F^{-2}}{a_1}F(-Y) =$$
$$-\frac{a_nF^{n-4} + a_{n-1}F^{n-5} \ldots a_2F^{-2}}{a_1}(-X)$$
$$= (-)^2\frac{a_nF^{n-4} + a_{n-1}F^{n-5} \ldots a_2F^{-2}}{a_1}(X)$$

We continue simplifying in this manner n times. We get

$$\left(\frac{a_nF^{n-1} + a_{n-1}F^{n-2} \ldots a_2F^1}{a_1}\right)F(X) = -F(X)$$

On simplifying the above equation we get

$$a_nF^n + a_{n-1}F^{n-1} \ldots a_2F^2 + a_1F^1 = 0$$

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