# On CR-Structure and F-Structure Satisfying Polynomial Equation 

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#### Abstract

The purpose of this paper is to show a relation between CR structure and F -structure satisfying polynomial equation. In this paper, we have checked the significance of CR structure and F-structure on Integrability conditions and Nijenhuis tensor. It was proved that all the properties of Integrability conditions and Nijenhuis tensor are satisfied by CR structures and F-structure satisfying polynomial equation.


Keywords-CR-submainfolds, CR-structure, Integrability condition \& Nijenhuis tensor.

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## I. Introduction

THE study of F structure and CR structure is done by many mathematicians. In this paper the study of these structures are considered with polynomial equations, the study of) Integrability and Nijenhuis Tensor is also extended to polynomial equation. Yano [1] initiated the study of F structure. Nikie [8] and Das [9] further studied the properties of $F$ structure.

Let $F$ be a non zero tensor field of type $(1,1)$ and of class $C^{\infty}$ dimensional manifold $M$ such that

$$
\begin{equation*}
a_{n} F^{n}+a_{n-1} F^{n-1} \ldots . a_{2} F^{2}+a_{1} F^{1}=0 \tag{1}
\end{equation*}
$$

where n is a fixed positive integer greater than or equal to 1 . Such a structure on M is called an F-structure. If the rank of $F$ is constant and $r=F(r)$, then $M$ is called an $F$ structure manifold of degree $n$.

Let us define the operator on $M$ as:

$$
\begin{array}{r}
\quad l=-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right) \\
m=I+\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right) \tag{3}
\end{array}
$$

where I denotes the identify operator on M.
Theorem 1. Let $M$ be an $F\left(a_{n}, a_{n-1}, \ldots a_{1}\right)$ structure manifold satisfying (1) then
a) $l+m=I$
b) $l^{2}=l$
c) $m^{2}=m$
d) $\quad l . m=0$

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## Proof.

a) $l+m=I$

$$
\begin{gather*}
l+m=-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)+I \\
+\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)=I \\
\Rightarrow l+m=I \tag{4}
\end{gather*}
$$

b) $l^{2}=l$

$$
\begin{gather*}
l^{2}=-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right) \\
*-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)= \\
-\left(\frac{-a_{1}}{a_{1}}\right) *-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)= \\
-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \cdots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)=l \\
\Rightarrow l^{2}=l \tag{5}
\end{gather*}
$$

$$
\begin{gather*}
m^{2}=m \\
m^{2}=\left[I+\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)\right] *[I+ \\
\left.\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)\right]= \\
I^{2}+2 I\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)+ \\
\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)^{2}= \\
I^{2}+ \\
2 I\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)+l^{2}=I^{2}+ \\
2 I\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)+l= \\
I^{2}+2 I\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)+ \\
 \tag{6}\\
\quad-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)= \\
I+\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)=m \\
\operatorname{So} m^{2}=m
\end{gather*}
$$

d) $\quad l . m=l .(I-l)$

$$
\begin{equation*}
=l-l^{2}=l-l \Rightarrow l \cdot m=0 \tag{7}
\end{equation*}
$$

For $F \neq 0$ satisfying (1) there exist complimentary distributions $D_{l} \& D_{m}$ corresponding to the projection operator $l \& m$ respectively. If Rank $F=$ constant and $r=r(F)$. Then, $\operatorname{dim} D_{l}=r$ and $D_{m}=n-r$.
Theorem 2. We have-
a) (I) $l F=F l=F$,
(II) $m F=F m=0$
b) (I) $\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) * m=0$

$$
\text { (II) }\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) * l=-l
$$

## Proof.

a) (I) $l F=F l=F$

$$
\begin{gather*}
l \mathrm{~F}=\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right) * F= \\
-\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right)=\mathrm{F}  \tag{8}\\
\text { So } l \mathrm{~F}=\mathrm{F} l=\mathrm{F}
\end{gather*}
$$

(II) $m F=F m=0$

$$
\begin{gather*}
m F=\left[I+\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right)\right] * \mathrm{~F}=[F+ \\
\left.\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right)\right]=F+(-F)=0 \\
\text { So } m F=F m=0 \tag{9}
\end{gather*}
$$

b) (I) $\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) * m=0$
(II) $\begin{array}{r}\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) * l \\ =-F^{*} l=-F\end{array}$

Thus, $F$ acts on $D_{l}$ as an almost complex structure and on $D_{m}$ as a null operator.

## II. Nijenhius Tensor

The Nijenhius tensor $\mathrm{N}(\mathrm{X}, \mathrm{Y})$ of F satisfying (1) in M is expressed as follows for every vector field $\mathrm{X}, \mathrm{Y}$ on M .

$$
\begin{equation*}
N(X, Y)=[F X, F Y]-F[F X, Y]-F[X, F Y]+F^{2}[X, Y] \tag{12}
\end{equation*}
$$

We state the following theorem without proof
Theorem 3. A necessary \& sufficient condition for the fstructure to be integrable is that $N(X, Y)=0$ for any vector field $X \& Y$ on $M$.

## III. LIE BRACKET

If $\mathrm{X} \& \mathrm{Y}$ are two vector fields in M then their lie bracket $[\mathrm{X}, \mathrm{Y}]$ is defined by

$$
\begin{equation*}
[X, Y]=X Y-Y X \tag{13}
\end{equation*}
$$

## IV. CR-Structure

A study of differential geometry of a CR submanifold has been initiated in [4]-[7]. Results on general theory of Cauchy Riemann manifolds have been obtained by [2].

Let $M$ be a differentiable manifold and $T_{c}(M)$ be its complex field on tangent bundle M. A CR-Structure on $M$ is a complex sub bundle $H$ of $T_{c}(M)$ such that $H_{p} \cap H_{\bar{p}}=0 \& \mathrm{H}$ is involutive i.e. for complex vector field \& $Y$ in $H,[X, Y]$ is in $H$. In this case we say $M$ is a CR-manifold.

Let $F\left(a_{n}, a_{n-1}, \ldots a_{1}\right)$ be an integrable structure satisfying (1) of rank $r=2 m$ on $M$. We define complex sub bundle $H$ of $T_{c}(M)$ by

$$
\begin{equation*}
H p=\{X-\quad \sqrt{ }-1 F X, X \in \chi(D l)\} \tag{14}
\end{equation*}
$$

where $\chi\left(D_{l}\right)$ is the $\mathrm{F}\left(D_{m}\right)$ module for all differentiable sections of $D_{l}$. The $\operatorname{Re}(H)=D_{l} \& H_{p} \cap H_{\bar{p}}=0$, where $H_{\bar{p}}$ denotes the complex conjugate. Intigrability conditions on such submanifolds have been investigated by [4].
Theorem 4. If $P \& Q$ are two elements of $H$ then the following relation holds

$$
[P, Q]=[X, Y]-[F X, F Y]-\sqrt{ }-1[X, F Y]-\sqrt{ }-1[F X, Y]
$$

Proof. Let us define

$$
\begin{gathered}
P=X-\sqrt{ }-1 F X \\
Q=Y-\sqrt{ }-1 Y
\end{gathered}
$$

then by direct calculation \& on simplifying, we obtain-

$$
\begin{align*}
& {[P, Q]=[X-\sqrt{ }-1 F X, Y-\sqrt{ }-1 F Y] } \\
= & {[X, Y]-\sqrt{ }-1[X, F Y]-\sqrt{ }-1[F X, Y] Y-[F X, F Y] } \\
= & {[X, Y]-[F X, F Y]-\sqrt{ }-1[X, F Y]-\sqrt{ }-1[F X, Y] } \tag{15}
\end{align*}
$$

Theorem 5. If $F\left(a_{n}, a_{n-1} \ldots . . a_{2}, a_{1}\right)$ structure satisfying (1) is integrable then we have

$$
\begin{gathered}
-\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right)\left\{\left[F X^{*} F Y\right]+F^{2}\left[X^{*} Y\right]\right\}=l \\
\{[F X, Y]+[X, F Y]\}
\end{gathered}
$$

Proof. From (12) we have,

$$
N(X, Y)=[F X, F Y]+F^{2}[X, Y]-F[F X, Y]-F[X, F Y]
$$

Since $N(X, Y)=0$ we obtain

$$
[F X, F Y]+F^{2}[X, Y]=F[F X, Y]+F[X, F Y]
$$

$$
\begin{aligned}
& \text { Operating }-\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) \\
& =-\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right)\left\{[F X, F Y]+F^{2}[X, Y]\right\} \\
& =-\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right)\{F[F X, Y]+F[X, F Y]\} \\
& =-\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) F\{[F X, Y]+[X, F Y]\} \\
& =-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} \mathrm{~F}}{a_{1}}\right)\{[F X, Y]+[X, F Y]\} \\
& =-\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} \mathrm{~F}}{a_{1}}\right)\{[F X, Y]+[X, F Y]\} \\
& =l\{[F X, Y]+[X, F Y]\}
\end{aligned}
$$

This proves the above theorem.
Theorem 6. The following identities hold $m N(X, Y)=m[F X, F Y]$
$\left.m N /\left(\frac{a_{F^{F-2}}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, Y\right]$
$=m\left[\left(\frac{a_{F^{n-1}}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F}{a_{1}}\right), F Y\right]$

## Proof.

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a) $m N(X, Y)=m\left\{[F X, F Y]+F^{2}[X, Y]-F[F X, Y]-F[X, F Y]\right\}$
$m N(X, Y)=m\left\{[F, F]+F^{2}[X, Y]-F[F X, Y]-F[X, F Y]\right\}$
$=m[F X, F Y]+m \cdot F \cdot F[X, Y]-m F[F X, Y]-m F[X, F Y]=$ $m[F X, F Y]$

$$
\begin{equation*}
\Rightarrow m N(X, Y)=m[F X, F Y] \tag{16}
\end{equation*}
$$

b) $m N /\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, Y=m$
$\left[\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F}{a_{1}}\right), F Y\right]$

$$
\begin{gathered}
m N\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, Y\right]= \\
m N\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) F X, F Y\right]= \\
F^{2}\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, Y\right]- \\
F\left[F\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, Y\right]- \\
\left.F\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, F Y\right]\right\}
\end{gathered}
$$

By the equation $m F=0=F m$

$$
\begin{align*}
& m N\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, Y\right]=m \\
& {\left[\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots . a_{3} F^{2}+a_{2} F}{a_{1}}\right), F Y\right]} \tag{17}
\end{align*}
$$

Theorem 7. For any two vector field $X \& Y$, the following condition are equivalent -
a) $m N(X, Y)=0$
b) $m[F x, F y]=0$
c) $m N\left[\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F}{a_{1}}\right) X, Y\right]=0$
d) $m\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, F Y\right]=0$
e) $m\left[\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) l X, F Y\right]=0$

Proof. $a)=>b$ )

$$
m N(X, Y)=0
$$

$=>m\left\{[F X, F Y]=F^{2}[X, Y]-F[F X, Y]-F[X, F Y]\right\}=0$
$=>m[F X, F Y]=0$

$$
\begin{equation*}
=>m[F X, F Y]=0 \tag{18}
\end{equation*}
$$

$$
[\text { since } m F=F m=0]
$$

c) $\Rightarrow$ a)

$$
m N\left[\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F}{a_{1}}\right) X, Y\right]=0
$$

By (1)

$$
\begin{gather*}
\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} \mathrm{~F}}{a_{1}}=-\mathrm{I} \\
==>m N[-X, Y]=0 \\
=>m N[X, Y]=0  \tag{1}\\
=>c)=>a)
\end{gather*}
$$

$$
\text { d) }=>b \text { ) }
$$

$$
\begin{gathered}
m\left[\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right) X, F Y\right]=0 \\
\left(\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3}+\cdots \ldots a_{3} F^{1}+a_{2}}{a_{1}}\right)=-F
\end{gathered}
$$

By (1)

$$
\begin{gather*}
m[-F X, F Y]=0  \tag{20}\\
d)=>b)
\end{gather*}
$$

$$
e)=>b)
$$

$$
m\left[\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) l X, F Y\right]=0
$$

$$
m\left[\left(\frac{a_{n} F^{n} l+a_{n-1} F^{n-1} l+\cdots \ldots a_{3} F^{3} l+a_{2} F^{2} l}{a_{1}}\right) X, F Y\right]=0
$$

$$
m\left[\left(\frac{a_{n} F^{n-1} F l+a_{n-1} F^{n-2} F l+\cdots \ldots a_{3} F^{2} F l+a_{2} F^{1} F l}{a_{1}}\right) X, F Y\right]=0
$$

$$
m\left[\left(\frac{a_{n} F^{n-1} F l+a_{n-1} F^{n-2} F l+\cdots \ldots a_{3} F^{2} F l+a_{2} F^{1} F l}{a_{1}}\right) X, F Y\right]=0
$$

$$
m\left[\left(\frac{a_{n} F^{n-1}(-F)+a_{n-1} F^{n-2}(-F)+\cdots \ldots a_{3} F^{2}(-F)+a_{2} F^{1}(-F)}{a_{1}}\right) X, F Y\right]=0
$$

$$
\left(\frac{a_{n} F^{n-1}(-F)+a_{n-1} F^{n-2}(-F)+\cdots \ldots a_{3} F^{2}(-F)+a_{2} F^{1}(-F)}{a_{1}}\right)=-F
$$

$$
\begin{gather*}
\Rightarrow m[-F X, F Y]=0[\mathrm{By}(1)] \\
\Rightarrow m[F X, F Y]=0  \tag{21}\\
\Rightarrow e) \Rightarrow b)
\end{gather*}
$$

Theorem 8. If $F^{n}$ acts on $D_{l}$ as an almost complex structure. Then

$$
m\left[\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) l X, F Y\right]=m[-X, F Y]=0
$$

## Proof.

$$
\begin{gathered}
m\left[\left(\frac{a_{n} F^{n}+a_{n-1} F^{n-1}+\cdots \ldots a_{3} F^{3}+a_{2} F^{2}}{a_{1}}\right) l X, F Y\right] \\
=m\left[\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2}+\cdots \ldots a_{3} F^{2}+a_{2} F^{1}}{a_{1}}\right) F l X, F Y\right] \\
=m[-F l X, F Y]=[-X, F y]\{\mathrm{By}(8)\}
\end{gathered}
$$

Theorem 9. For $X, Y \in x\left(D_{1}\right)$ we have

$$
l([X, F Y]+[F X, F Y])=[X, F Y]+[F X, Y]
$$

## Proof.

$l([X, F Y])+[F X, Y])=l\{X . F Y-F Y . X+F X . Y-Y . F X\}$
$\{\operatorname{By}(5)\}$

$$
=X . F Y-F Y \cdot X+F X . Y-Y . F X
$$

$\{$ By (13) \}

$$
=[X, F Y]+[F X, Y]
$$

Theorem 10: The integrable $F\left(a_{n}, a_{n-l} \ldots . a_{l}\right)$ structure satisfying (1) on $M$ defines a CR-structure $H$ on it. Such that ReH=D .
Proof. From theorem 4 we have,

$$
[P, Q]=[X, Y]-[F X, F Y]-\sqrt{ }-1[X, F Y]-\sqrt{ }-1[F X, Y]
$$

$l[P, Q]=l[X, Y]-l[F X, F Y]-\sqrt{ }-1([X, F Y]+[F X, Y]$
\{By theorem (9) \}

$$
=[X, Y]-[F X, F Y]-\sqrt{ }-1([X, F Y]+[F X, Y]=[P, Q]
$$

\{By theorem (4)\}
Since $l[P, Q]=[P, Q] \Rightarrow[P, Q] \in x(D l)$. Then, $F\left(a_{n}\right.$, $\left.a_{n-1} \ldots . . . a_{l}\right)$ structure satisfying (1) on $M$ defines a CR-structure.

## V. Morphism of Vector Bundles

Let $\bar{K}$ be the complementary distribution of $\operatorname{Re}(H)$ to $T M$. We define a morphism of vector bundles $F: T M \rightarrow T M$ given by

$$
F(X)=0 \forall X \in \chi(\bar{K})) \text { such that- }
$$

We have

$$
F(X)=1 / 2 \sqrt{ }-1(P-\bar{P})
$$

where $P=X+\sqrt{ }-1 Y \varepsilon x(H P)$ and $\bar{P}$ is the complex of $P$.
Corollary 1. If $P=X+i Y$ and $\bar{P}=X-i Y$ belong to $H_{p}$ and $F(X)$ $=\frac{1}{2} \sqrt{-1}\left(P-\overline{P)}, F(Y)=\frac{1}{2} \sqrt{-1}(P+\overline{P)}\right.$ and $F(-Y)=$ $\frac{1}{2} \sqrt{-1}(P+\bar{P})$ then $\mathrm{F}(\mathrm{X})=\frac{1}{2} \sqrt{-1}(P-\bar{P})=-\mathrm{Y}, F^{2}(X)=-X$ and $F(-Y)=-X$.
Proof. $P=X+\sqrt{-1} Y$ and $\bar{P}=X-\sqrt{-1} Y=>=\frac{(P+\bar{P})}{2}, Y=$ $\frac{(P-\overline{P)}}{2 \sqrt{-1}}$. Since $P+\bar{P}=2 X$ and $P-\bar{P}=2 \sqrt{-1} Y . F(X)=F\left[\frac{P+\bar{P}}{2}\right]$ $=\frac{1}{2} \sqrt{-1}(P-\bar{P})=-Y$ from the definition of morphism

$$
F(-Y)=\mathrm{F}\left[-\frac{\mathrm{P}-\bar{P}}{2 \sqrt{-1}}\right]=-X
$$

Theorem 11. If $M$ has a CR-structure $H$, then we have $a_{n} F^{n}+a_{n-1} F^{n-1} \ldots . a_{2} F^{2}+a_{1} F^{1}=0$ and consequently $F\left(a_{n}, a_{n-1}, \ldots . a_{2}, a_{1}\right)$ structure satisfying (1) is defined on $M$ such that the distribution $\mathrm{D}_{1}$ and $\mathrm{D}_{\mathrm{m}}$ coincide with $\operatorname{Re}(H)$ and $\bar{K}$ respectively.
Proof. Suppose $M$ has a CR-structure. Then in view of definition of CR manifold \& corollary 1 we have-

$$
F(X)=-Y ;
$$

operating above equation by- $\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2} \ldots a_{2} F^{1}}{a_{1}}$ on both sides we get

$$
\begin{gathered}
\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2} \ldots a_{2} F^{1}}{a_{1}}\right)(F(X)= \\
\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2} \ldots a_{2} F^{1}}{a_{1}}\right)(-Y)
\end{gathered}
$$

on making use of Corollary 1 the right hand side of the above equation becomes

$$
\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2} \ldots a_{2} F^{1}}{a_{1}}\right) F(X)=
$$

$$
\left.\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3} \ldots . a_{2}}{a_{1}}\right) F(-Y)
$$

which can be written as -

$$
\begin{gathered}
\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2} \ldots a_{2} F^{1}}{a_{1}}\right) F(X)=\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3} \ldots a_{2}}{a_{1}}(-X) \\
=-\frac{a_{n} F^{n-2}+a_{n-1} F^{n-3} \ldots . a_{2}}{a_{1}}(X)=-\frac{a_{n} F^{n-3}+a_{n-1} F^{n-4} \ldots a_{2} F^{-1}}{a_{1}} \\
F(X)=-\frac{a_{n} F^{n-3}+a_{n-1} F^{n-4} \ldots a_{2} F^{-1}}{a_{1}}(-Y)= \\
-\frac{a_{n} F^{n-4}+a_{n-1} F^{n-5} \ldots a_{2} F^{-2}}{a_{1}} F(-Y)= \\
-\frac{a_{n} F^{n-4}+a_{n-1} F^{n-5} \ldots a_{2} F^{-2}}{a_{1}}(-X) \\
=(-)^{2} \frac{a_{n} F^{n-4}+a_{n-1} F^{n-5} \ldots a_{2} F^{-2}}{a_{1}}(X)
\end{gathered}
$$

We continue simplifying in this manner $n$ times. We get

$$
\left(\frac{a_{n} F^{n-1}+a_{n-1} F^{n-2} \ldots a_{2} F^{1}}{a_{1}}\right) F(X)=-F(X)
$$

On simplifying the above equation we get

$$
a_{n} F^{n}+a_{n-1} F^{n-1} \ldots . a_{2} F^{2}+a_{1} F^{1}=0
$$

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