International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:10, No:12, 2016

Numerical Example of Aperiodic Diffraction Grating

Youssef Khmou, Said Safi, Miloud Frikel

Abstract—Diffraction grating is periodic module used in many engineering fields, its geometrical conception gives interesting properties of diffraction and interferences, a uniform and periodic diffraction grating consists of a number of identical apertures that are equally spaced, in this case, the amplitude of intensity distribution in the far field region is generally modulated by diffraction pattern of single aperture. In this paper, we study the case of aperiodic diffraction grating with identical rectangular apertures where theirs coordinates are modeled by square root function, we elaborate a computer simulation comparatively to the periodic array with same length and we discuss the numerical results.

Keywords—Diffraction grating, interferences, amplitude modulation, laser.

I. Introduction

DIFFRACTION occurs to waves when they encounter obstacles or slits whose dimensions are comparable to the wavelength [1]-[3]. This property is common to all types of waves such as x rays, electromagnetic and acoustical ones. Starting from the position of the slit, the diffraction pattern in the far field is generally the Fourier transform of the slit's geometry.

The analysis of the diffraction pattern shows that, if we take two points from the slit, constructive interferences occur when the path difference between the two punctual sources is multiple of the wavelength, a simple example is the Bragg diffraction law where this relation enables the characterization of the crystal's structure [4]. An interesting effect takes place when the aperture is an array of slits which is known as diffraction grating [3]-[5], it is periodic structure of slits that, in general, have the same dimensions (width and height) and uniform spacing. Diffraction grating is used in many engineering fields such as electronics materials. When a planar wave is diffracted, the intensity profile in far field region has many characteristics. The resolution becomes sharper due to the presence of multiple slits, the central peak becomes narrower with large array and the intensity distrubution is modulated by single aperture diffraction pattern. This effect is also the same in antennas [6], given an array of multiple and identical antennas which are equally separated (in general the distance is half the wavelength), the total radiation pattern is simply the radiation pattern of single antenna multiplied by the array factor, which is the sum of the phase contributions from all antennas. If the distance between the apertures in the diffraction grating is not uniform, the intensity profile of the diffraction is affected. For example, we may notice the variation of half maximum beam width of the central peak, the amplitude modulation may also change and other properties such as the ratio of the first peak over the second one.

The purpose of this paper is to examine the intensity pattern of diffraction grating whose distance between consecutive slits is not uniform or simply the non periodic diffraction grating, we treat the problem in one dimension, we compare the intensity distributions of the periodic diffraction grating and that of the non periodic array where they both have the same length, the obtained results are based on computer simulation using Fraunhofer approximation [7].

In the next section we present the general framework of the problem and we explicit the theoretical expression of the intensity distribution for uniform diffraction grating. Next, we present the proposed model of the non periodic array. In the third section, we perform some computer simulations where we compare some characteristics of the periodic and aperiodic diffraction gratings.

II. Theory

We start this section by studying the diffraction pattern of single rectangular slit characterized by width a and height b, let us consider the position of the slit as the reference of Cartesian coordinates (x,y,z) as presented in Fig. 1.

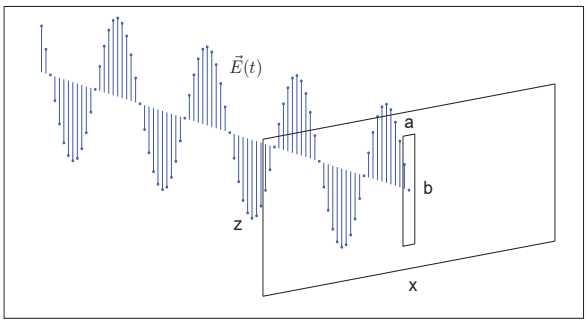

Fig. 1 Plane wave diffraction by rectangular aperture ($b \gg a$)

A plane monochromatic wave [8], polarized in z direction, is given by:

$$E(t) = E_0 e^{j(\omega t - \varphi)} \tag{1}$$

where E_0 is the amplitude of the wave in V/m, $\omega = 2\pi\nu$ and ν is the frequency in Hz, given the speed of wave $c \simeq 3 \times 10^8$ m/s, the wavelength is $\lambda = c\nu^{-1}$, $E(t)$ satisfies the equation of propagation (see appendix).

Youssef Khmou and Said Safi are with Department of Mathematics and Informatics, Sultan Moulay Slimane University, Morocco. (e-mail: khmou.y@gmail.com, safi.said@gmail.com).

Miloud Frikel is with GREYC Lab UMR 6072 CNRS, Equipe Automatique, Avenue 6 juin, 14053, Caen, France. (e-mail: mfrikel@greyc.ensicaen.fr).

To simplify the problem, we consider that $\varphi = 0$ and $b \gg a$ which reduces the problem to one dimensional system. A screen is placed at distance L from the slit where the Fraunhofer condition is verified by:

$$L \gg \frac{2a^2}{\lambda} \tag{2}$$

In this case of far field [9], [10], the radius of propagation r varies slowly, taking two rays r and r_2 from two points in the slit to a common point x on the screen, we have the approximation $r^{-1} \simeq r_2^{-1}$. The resulting electric field from diffraction is given by:

$$E(t) = \frac{E_0}{r} e^{j(\omega t - kr)} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-jkx\sin(\theta)} dx \tag{3}$$

$$= \frac{E_0 a}{r} e^{j(\omega t - kr)} sinc\Big(\frac{\pi a x}{\lambda L}\Big)$$

where $\sin(\theta) \simeq \tan(\theta) = x/L$ using the approximation of small angle θ, $k = 2\pi/\lambda$ is the wave number and $sinc(x) = sin(x)/x$ is the unnormalized sinc function with maximum value $sinc(0) = 1$.

We consider a diffraction grating of N identical slits with the same dimensions $b \gg a$ and uniform inter-element distance d, the total diffracted electric field is the sum of the contributions from all apertures, the field $E(t)$ is given by the linear superposition:

$$E(t) = \sum_{n=1}^{N} E_n(t) \tag{4}$$

Developing the above equation yields to the following expression:

$$E(t) = \frac{E_0 a}{r} e^{j(\omega t - kr)} sinc\Big(\frac{\pi a x}{\lambda L}\Big) \sum_{n=0}^{N-1} e^{-jknd\sin(\theta)} \tag{5}$$

Rearranging the right-hand side of the above equation using the sum of the geometric series, we obtain the following result:

$$E(t) = \frac{E_0 a}{r} e^{j(\omega t - kr)} sinc\Big(\frac{\pi a x}{\lambda L}\Big) e^{j\beta} \frac{sin\big(\frac{N\pi x d}{\lambda L}\big)}{sin\big(\frac{\pi x d}{\lambda L}\big)} \tag{6}$$

where the constant $e^{j\beta}$ is given as:

$$\beta = \frac{\pi x d(N-1)}{\lambda L} \tag{7}$$

Studying the diffraction pattern of $E(t)$ is performed over a time period T, the intensity distribution $I(x)$ is proportional to the first order correlation function as:

$$I(x) = \frac{1}{2}\epsilon_0 c \left\langle E(t)E^*(t)\right\rangle = \frac{\epsilon_0 c}{2T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} E(t)E^*(t)dt \tag{8}$$

$$= I_0 sinc^2\Big(\frac{\pi a x}{\lambda L}\Big) \left(\frac{sin\big(\frac{N\pi x d}{\lambda L}\big)}{sin\big(\frac{\pi x d}{\lambda L}\big)}\right)^2$$

where $I_0 = \epsilon_0 c E_0^2 a^2/2r^2$ in W/m^2, $\epsilon_0 \simeq 8.85 \times 10^{-12}$ F/m is the permittivity and $(.)^*$ is the conjugate operator. The maximum intensity at the origin $x = 0$ is given by:

$$\lim_{x\to 0} I(x) = I_0 N^2 \tag{9}$$

The diffraction resolution becomes sharper when the number of the apertures N increases. $I(x)$, in this case, is modulated by single slit diffraction pattern of width a where the intensity of the diffracted wave is $I_0 N^2$.

The contribution, in this paper, consists of studying the intensity distribution of diffraction grating where the spacing between the apertures is not uniform, we consider a case where the distance is modeled by square root function with fixed boundary conditions. It is assumed that the distance between two consecutive apertures is the length between theirs centers. The length of the uniform array where $d \gg a$ is given by the relation:

$$l = (N-1)d + a \tag{10}$$

Let us consider a vector $D \in \mathbb{R}^{N\times 1}$ that represents the coordinates of the apertures as:

$$D = \begin{pmatrix} 0 \\ d \\ 2d \\ \vdots \\ (N-1)d \end{pmatrix} \tag{11}$$

The length of the non periodic array is also l, with the same characteristics (a,b,N) except the coordinates of the apertures, the vector of coordinates is modeled by square root function. Given a vector $u \in \mathbb{R}^{N\times 1}$ with components:

$$u = \alpha D = \begin{pmatrix} 0 \\ (N-1)d^2 \\ 2(N-1)d^2 \\ \vdots \\ (N-1)^2 d^2 \end{pmatrix} \tag{12}$$

The new vector $D' \in \mathbb{R}^{N\times 1}$ that is the square root of u is given by:

$$D' = \sqrt{u} = \begin{pmatrix} 0 \\ \sqrt{(N-1)d} \\ 2\sqrt{(N-1)d} \\ \vdots \\ (N-1)d \end{pmatrix} \tag{13}$$

D' represents the coordinates of apertures in aperiodic diffraction grating, $D^{'} = (d'_1 = 0, d'_2, ..., d'_N = \alpha)$. For two slits $N = 2$, we have $D = D' = (0, \alpha)$, the two arrays are identical, however for $N > 3$, the coordinates are different, for example if we take $N = 3$ where $\alpha = 2(d+a)$, we obtain the following values for D:

$$D = \begin{pmatrix} 0 \\ d \\ 2d \end{pmatrix} \tag{14}$$

and for D' we have the values:

$$D = \begin{pmatrix} 0 \\ \sqrt{2}(d) \\ 2(d) \end{pmatrix} \quad (15)$$

To evaluate the diffraction pattern of the non periodic array, we calculate the intensity distribution $I'(x)$ as the following:

$$I'(x) = I_0 sinc^2\Big(\frac{\pi a x}{\lambda L}\Big) \sum_{n=1}^{N} e^{-jkxd'_n/L} \sum_{n=1}^{N} e^{jkxd'_n/L} \quad (16)$$

We can also remark from the above equation that at the origin $x = 0$, the intensity of the central peak is $N^2 I_0$, however for $x \neq 0$ the distributions of $I'(x)$ and $I(x)$ do not have the same pattern, to illustrate this difference, we present some numerical results in the next section.

III. Numerical Simulation

In this part, we conduct some computer simulations to compare the intensity distributions of uniform and non periodic diffraction gratings using monochromatic wave. We consider a HeNe laser with wavelength $\lambda = 0.633$ μm and initial intensity $I_0 = 0.04$ mW/m^2.

In the first part, we take a diffraction grating with $N = 20$ identical slits characterized by width $a = 20$ μm, height $b \gg a$ and uniform distance $d = 400$ μm, the length of the array is $l = 0.80$ cm. The screen is placed at distance $L = 1.2$ m from the array perpendicularly to the optical axis. Fig. 2 illustrates the intensity distribution $I(x)$.

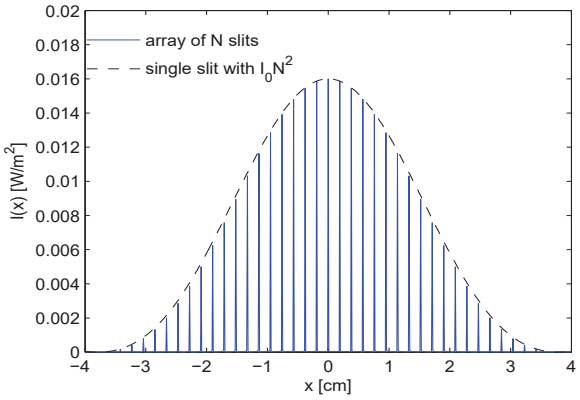

Fig. 2 Intensity pattern $I(x)$ of uniform diffraction grating, $N = 20$

The amplitude of $I(x)$ is modulated by single aperture diffraction pattern where the width of the central peak is given by:

$$\Delta x = \frac{2L\lambda}{a} \simeq 7.6cm \quad (17)$$

In the second part, we compare the two vectors of coordinates D and D' described by (11) and (13) respectively, the result is illustrated in Fig. 3.

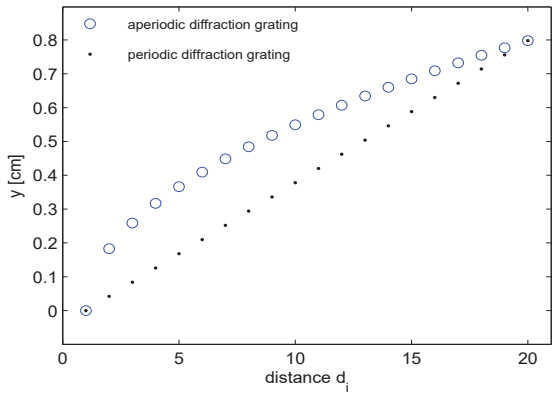

Fig. 3 Coordinates of apertures for periodic and nonperiodic diffraction gratings, $N = 20$

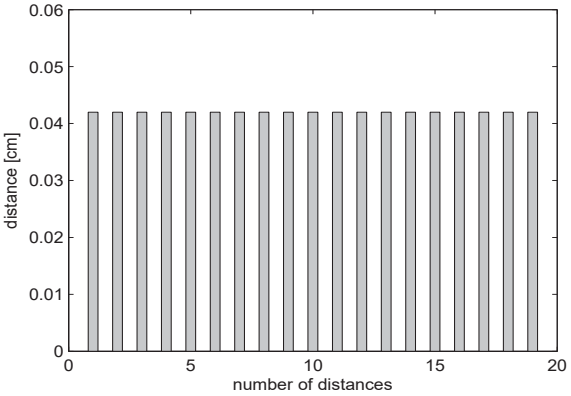

Fig. 4 Consecutive distance of periodic array, $N = 20$

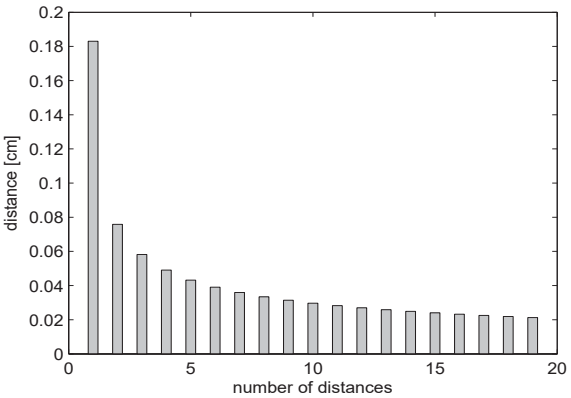

Fig. 5 Consecutive distance of aperiodic array, $N = 20$

For better illustration of the difference between the arrays, we present the consecutive distances d_i and d'_i in Figs. 4 and 5.

Next, we present, in Figs. 6 and 7, the prototypes of the arrays where $b \gg a$.

In the third part, we implement (16) for the intensity $I'(x)$ of the aperiodic array comparatively to the single aperture

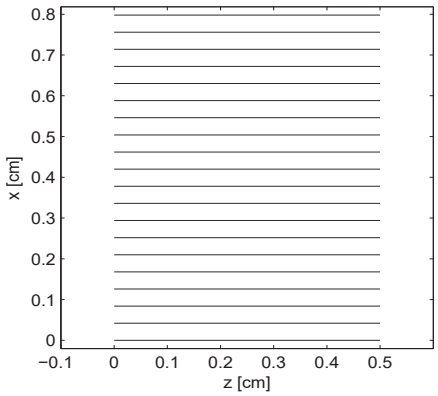

Fig. 6 Prototype of periodic diffraction grating, $N = 20$

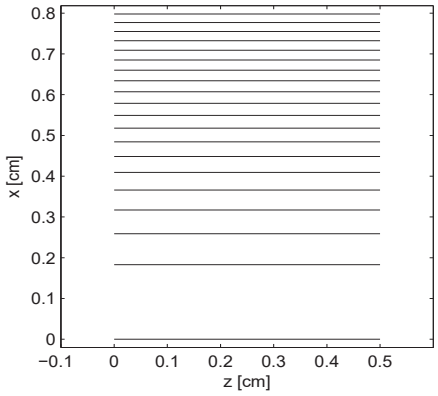

Fig. 7 Prototype of aperiodic diffraction grating, $N = 20$

diffraction pattern with intensity $N^2 I_0$, the result is presented in Fig. 8. We remark, from $I'(x)$, that for $x \neq 0$ the intensity decreases randomly and rapidly, the amplitude of $I'(x)$ is not a function of single slit diffraction pattern, which is the case for the periodic array as illustrated in Fig. 1.

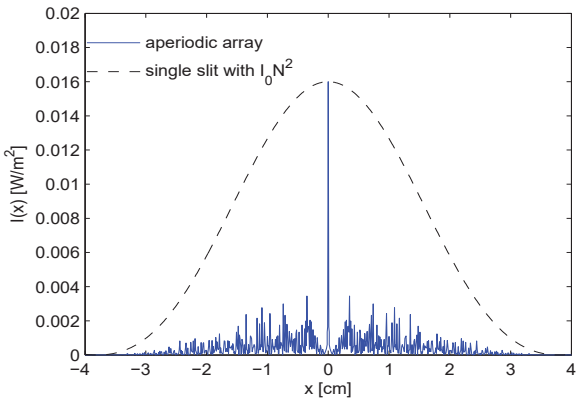

Fig. 8 Intensity pattern $I'(x)$ of aperiodic diffraction grating, $N = 20$

The two profiles have the same mean value $\langle I \rangle = \langle I' \rangle \simeq 0.4$ mW/m^2; however, they differ in terms of standard deviation $\Delta I \simeq 2$ mW/m^2, $\Delta I' \simeq 0.8$ mW/m^2. The amount of intensity in the range $\mathfrak{F}$ =[-4 cm,4 cm] is the same for both arrays:

$$\int_{\mathfrak{F}} I(x)dx = \int_{\mathfrak{F}} I'(x)dx \simeq 2.9 \times 10^{-5} \tag{18}$$

The intensity distributions are partially correlated, the degree of correlation in this simulation is:

$$\rho = \frac{\langle (I - \langle I \rangle)(I' - \langle I' \rangle) \rangle}{\sqrt{\langle (I - \langle I \rangle)^2 \rangle \langle (I' - \langle I' \rangle)^2 \rangle}} \simeq 0.22 \tag{19}$$

Next, we present in Fig. 9, the normalized correlation function.

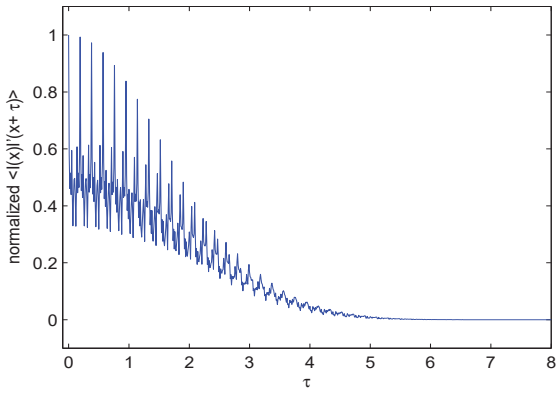

Fig. 9 normalized correlation function $\langle I(x)I'(x+\tau) \rangle$

This numerical study gives a multitude of differences of the intensity distributions, the vector of coordinates D' that is modeled by the square root function has an impact on the envelope of the diffraction pattern.

The presented case is based on spatial dimension where the positions of the apertures have an impact on diffraction pattern. For temporal dimension, we can interpret the problem differently, instead of considering single wave and N apertures, we can study the same effect in time dimension using a superposition of N waves with different frequencies. Let us consider a waveform $E_1(t)$ that consists of superposition of N frequencies ν_i with bandwidth $bw = [\nu_1, \nu_N]$ as:

$$E_1(t) = \sum_{i=1}^{N} e^{j2\pi\nu_i t} \tag{20}$$

where the frequencies are uniformly distributed with rate $d\nu$ as:

$$\nu_{i+1} = \nu_i + d\nu \tag{21}$$

$E_1(t)$ is equivalent to periodic diffraction grating in time dimension, similarly to the aperiodic diffraction grating, we consider a second waveform $E_2(t)$ with the same number of frequencies N, same bandwidth bw and different values of frequencies that are distributed according to:

$$\nu'_{i+1} = \sqrt{\nu_N(\nu_i + d\nu)} \tag{22}$$

Characterization of the frequency difference between the two waveforms can be obtained by ratio χ defined by:

$$\chi = \frac{\nu'_{i+1}}{\nu_{i+1}} = \frac{\sqrt{\nu_N(\nu_i + d\nu)}}{\nu_i + d\nu} = \sqrt{\frac{\nu_N}{\nu_i + d\nu}} \tag{23}$$

The equivalent of diffraction pattern in this case is simply the correlation function of the waveforms:

$$f(\tau) = \langle E(t)E^*(t+\tau)\rangle \tag{24}$$

Therefore, the correlation functions $f_1(\tau)$ and $f_2(\tau)$ do not have the same properties, similarly to the intensities $I(x)$ and $I'(x)$.

IV. Conclusion

In this paper, we have presented a geometric conception of non uniform diffraction grating of identical slits, where the coordinates of the apertures are modeled by the square root function, the effect of varying consecutive distance between the slits has an impact on the intensity distribution. To illustrate this effect, we have made a numerical comparison between the proposed and the periodic arrays where they have the same length. Simulation results demonstrated that the non periodic geometry changes the envelope of the diffracted intensity distribution.

V. Appendix

The waveform expression given in (1) is a function of the electric field that depends on time t and y coordinate, it is obtained by solving the Maxwell equations:

$$\vec{\nabla}.\vec{E} = 0 \tag{25}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{26}$$

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{27}$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{28}$$

where $\vec{B}$ is the magnetic field and μ_0 is the permeability $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}, it is related to the speed of propagation and permittivity by the equation:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{29}$$

The operator $\vec{\nabla}$ is defined by the relation:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \tag{30}$$

The vectorial expression $\vec{\nabla} \wedge \vec{E}$, is written as:

$$\vec{\nabla} \wedge \vec{E} = \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} \tag{31}$$

Since the field is only dependent on y, using (25) and (26) we get:

$$\frac{\partial E_y}{\partial y} = \frac{\partial B_y}{\partial y} = 0 \tag{32}$$

developing (27) yields to the following result:

$$\begin{cases} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \end{cases} \tag{33}$$

Further simplification gives:

$$\begin{cases} \frac{\partial E_z}{\partial y} = -\frac{\partial B_x}{\partial t} \\ \frac{\partial B_y}{\partial t} = 0 \\ \frac{\partial E_x}{\partial y} = \frac{\partial B_z}{\partial t} \end{cases} \tag{34}$$

Similarly, (28) is reduced into:

$$\begin{cases} \frac{\partial B_z}{\partial y} = \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \\ \frac{\partial E_y}{\partial t} = 0 \\ -\frac{\partial B_x}{\partial y} = \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t} \end{cases} \tag{35}$$

From (32), (34) and (35) we deduce that $E_y = B_y = 0$, using the first part of (34) and the last part of (35), we get:

$$-\frac{\partial}{\partial y}\int \frac{-\partial E_z}{\partial y}\partial t = \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t} \tag{36}$$

Next, the equation of propagation is:

$$\frac{\partial^2 E_z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \tag{37}$$

where the solution is $\vec{E} = E_0 e^{j(\omega t - ky)}$, since we are interested in diffraction, to simplify the expressions, we considered that $\varphi = ky = 0$.

REFERENCES

[1] Lord Rayleigh, "Note on the Remarkable Case of Diffraction Spectra Described by Prof. Wood," Philos. Mag. 14, 60, 1907.
[2] Kassemeyer, Stephan and Jafarpour, Aliakbar and Lomb, Lukas and Steinbrener, Jan and Martin, Andrew V. and Schlichting, Ilme, Optimal mapping of x-ray laser diffraction patterns into three dimensions using routing algorithms, Phys. Rev. E, 2013.
[3] Wood, R. W, Anomalous Diffraction Gratings, Phys. Rev., Vol. 48, P. 928–936, 1935.
[4] He, B.B., Introduction to two-dimensional X-ray diffraction, Powder Diffraction, 18(2), pp. 7185, 2003.
[5] I. A. Avrutsky, M. Fay and J. M. Xu, "Multiwavelength diffraction and apodization using binary superimposed gratings," in IEEE Photonics Technology Letters, vol. 10, no. 6, pp. 839-841, June 1998.
[6] A. Neto, S. Maci, G. Vecchi and M. Sabbadini, "A truncated Floquet wave diffraction method for the full wave analysis of large phased arrays. I. Basic principles and 2-D cases," in IEEE Transactions on Antennas and Propagation, vol. 48, no. 4, pp. 594-600, Apr 2000.
[7] J. P. Braud and P. L. Hagelstein, "Whispering-gallery laser resonators. I. Diffraction of whispering-gallery modes," in IEEE Journal of Quantum Electronics, vol. 27, no. 4, pp. 1069-1077, Apr 1991.
[8] G. B. Esmer and L. Onural, "Simulation of scalar optical diffraction between arbitrarily oriented planes," First International Symposium on Control, Communications and Signal Processing, 2004., 2004, pp. 225-228.
[9] N. Zareian, P. Abolghasem and A. S. Helmy, "Far Field of Bragg Reflection Waveguides: Characteristics and Closed-Form Approximation," in Journal of Lightwave Technology, vol. 29, no. 5, pp. 728-735.
[10] Liu Yongxin, Tao Hua, Pu Jixiong, L Baida, Detecting the topological charge of vortex beams using an annular triangle aperture, Optics & Laser Technology, Volume 43, Issue 7, October 2011, Pages 1233-1236.