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Complex Fuzzy Evolution Equation with Nonlocal Conditions

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Abstract—The objective of this paper is to study the existence and uniqueness of Mild solutions for a complex fuzzy evolution equation with nonlocal conditions that accommodates the notion of fuzzy sets defined by complex-valued membership functions. We first propose definition of complex fuzzy strongly continuous semigroups. We then give existence and uniqueness result relevant to the complex fuzzy evolution equation.

Keywords—Complex fuzzy evolution equations, nonlocal conditions, mild solution, complex fuzzy semigroups.

I. Introduction

The theory of evolution equations has become an important area of investigation in recent years, stimulated by their numerous applications to problems from mechanics, electrical engineering, medicine, biology, ecology, etc. The evolution equation with nonlocal conditions

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in [0, a], \\ x(0) + g(x) = x_0, \end{cases}$$

in a Banach space X,where A is a nondensely defined closed linear operator on X, has a mild solution under assumption some conditions [4].

Since D. Ramot introduced a new concept in the context of fuzzy sets theory (complex fuzzy set) and there are many examples of application namely solar activity, signal processing, etc. (see [10]). The Cauchy problem for complex fuzzy differential equations

$$\begin{cases} X'(t) = H(t, X(t)), & t \in [a, b], \\ X(a) = X_a, \end{cases}$$

is studied in [2], where the mapping $H : [a, b] \times \mathbb{E} \longrightarrow \mathbb{E}$ (fuzzy complex metric space) is Holder continuous and bounded.

The purpose of this paper is to study the existence and uniqueness of mild solution of the complex fuzzy evolution equation with nonlocal condition:

$$\begin{cases} X'(t) = \mathcal{A}X(t) + H(t, X(t)), & t \in [0, a], \\ X(0) = X_0 + G(X), \end{cases} \tag{1}$$

where $\mathcal{A}$ is the generator of a continuous semigroup $\{\pi(t),\ t \geq 0\}$ on $\mathbb{E}$ and $H : I \times \mathbb{E} \longrightarrow \mathbb{E}$ which we take to be continuous in both arguments, and $G : \mathcal{C}([0, a], \mathbb{E}) \longrightarrow \mathbb{E}$ is continuous.

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II. Preliminaries

Next, we review some basic concepts, notations and technical results that are necessary in our study.

Let $\mathcal{P}_K(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $\mathcal{P}_K(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorf metric,

$$d(A, B) = max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}$$

where $\|\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then it is clear that $(\mathcal{P}_K(\mathbb{R}^n), d)$ becomes a complete and separable metric space (see [9]).
Denote

$$E^n = \{u : \mathbb{R}^n \longrightarrow [0, 1] \ \mid \ \text{u satisfies (i)-(iv) below} \},$$

where

(i) u is normal i.e there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = cl\{x \in \mathbb{R}^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{t \in \mathbb{R}^n \mid u(t) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in \mathcal{P}_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha$$

where $u, v \in E^n$, $k \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to \mathbb{R}^+$ by

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d\Big([u]^\alpha\, ,\, [v]^\alpha\Big)$$

where d is the Hausdorff metric for non-empty compact sets in $\mathbb{R}^n$.Then it is easy to see that D is a metric in E^n.

Using the results in [9], we know that
(1) (E^n, D) is a complete metric space;
(2) $D(u + w, v + w) = D(u, v)$ for all $u,\ v,\ x \in E^n$;
(3) $D(k\, u, k\, v) = |k|\ D(u, v)$ for all $u,\ v \in E^n$ and $k \in \mathbb{R}^n$.
Note that, (E^n, D) is a complete metric space which can be embedded isomorphically as a cone in a Banach space (see [8]).

On E^n, we can define the substraction $\ominus$, called the H-difference (see [5]) as follows: $u \ominus v$ has sense if there exists $w \in E^n$ such that $u = v + w$.

Denote

$$\begin{aligned}\mathcal{C}_a &= \mathcal{C}([0,a],E^n)\\ &= \{f:[0,a]\longrightarrow E^n; f\text{is continuous on}[0,a]\},\end{aligned}$$

endowed with the metric

$$H(u,v)=\sup_{t\in[0,a]} D(u(t),v(t))$$

Then $(\mathcal{C}_a,H)$ is a complete metric space. We define the following set

$$\hat{E}^{2n}=\{(u,v)\in E^n\times E^n \mid \exists x_0\in\mathbb{R}^n\ s.t\ u(x_0)=v(x_0)=1\}$$

For $f=(u,v)\in\hat{E}^{2n}$, define $[f]^{(\alpha,\beta)}=[u]^\alpha\cap[v]^\beta$ for all $\alpha,\beta\in[0,1]$.

For $f=(u_f,v_f), g=(u_g,v_g)\in\hat{E}^{2n}$ and c is a scalar, let

$$f+g=(u_f+u_g,v_f+v_g),\qquad cf=(cu_f,cv_f).$$

Define $\hat{D}:\hat{E}^{2n}\times\hat{E}^{2n}\to\mathbb{R}^+$ by the equation

$$\begin{aligned}\hat{D}(f,g) &= \hat{D}\Big((u_f,v_f),(u_g,v_g)\Big)\\ &= \max\{D(u_f,u_g),D(v_f,v_g)\}.\end{aligned}$$

Then $\hat{D}$ is a metric in $\hat{E}^{2n}$ and $\left(\hat{E}^{2n},\hat{D}\right)$ is a complete metric space(see [2]).

There exists an embedding l from $\hat{E}^{2n}$ into a Banach space (see [1]). Recall from [8] that on E^n we define $\hat{0}\in E^n$ by $\hat{0}(x)=1$ when $x=0$ and $\hat{0}(x)=0$ otherwise. The zero element on E^{2n} then reads $\hat{0}_2(x)=(\hat{0}(x),\hat{0}(x))\in E^{2n}$. We have $\hat{0}_2(0)=(1,1)$, verifying that $\hat{0}_2\in\hat{E}^{2n}$.

The polar representation of the membership function as presented in [11]

$$u(V,z)=r(V)e^{i\sigma\phi(z)}.$$

For $x\in\mathbb{R}^n$, the polar form of f is defined as:

$$f(x)=r(x)e^{2\pi\phi(x)i},$$

where $r,\phi:\mathbb{R}^n\longrightarrow[0,1]$, and we denote f by (r,ϕ). The scaling factor is taken to be 2π, allowing the range of f to be the entire unit circle. Because $e^{2\pi i\phi}$ is periodic, we take the value of ϕ giving the maximum distance from e^0, $\phi=0.5$, to be the maximum membership value. Now, while $[r]^\alpha$ can be defined just as $[u]^\alpha$ above, the corresponding level sets for ϕ, denoted $[\phi]^{(\beta)}$, must be defined differently to account for the periodicity:

$[\phi]^{(\beta)}=\overline{\{x\in\mathbb{R}^n:\phi(x)\in[\beta,1-\beta]}, \beta\in(0,0.5]\}$
$[\phi]^{(0)}=\overline{\{x\in\mathbb{R}^n:0<\phi(x)<1\}}$
$[\phi]^{(\beta)}=[\phi]^{(1-\beta)}$, for all $\beta\in[0,1]$.

We can then define the level sets $[f]^{(\alpha,\beta)}$ as $[f]^{(\alpha,\beta)}=[r]^\alpha\cap[\phi]^\beta$, or by the relations

$[f]^{(\alpha,\beta)}=\overline{\{x\in\mathbb{R}^n:\ r(x)\geq\alpha,\ \phi(x)\in[\beta,1-\beta]\}}$
$[f]^{(\alpha,0)}=\overline{\{x\in\mathbb{R}^n:\ r(x)\geq\alpha>0,\ 0<\phi(x)<1\}}$
$[f]^{(0,\beta)}=\overline{\{x\in\mathbb{R}^n:\ r(x)>0,\ \phi(x)\in[\beta,1-\beta],\ \beta\in(0,0.5]\}}$
$[f]^{(0,0)}=\overline{\{x\in\mathbb{R}^n:\ r(x)>0,\ 0<\phi(x)<1\}}$

together with

$[f]^{(\alpha,\beta)}=[f]^{(\alpha,1-\beta)}$, for all α, $\beta\in[0,1]$.

Denote

$$F^n=\{w:\mathbb{R}^n\longrightarrow[0,1]\ \mid\ \text{w satisfies (i)-(iv) below}\},$$

where

(i) There exists an $x_0\in\mathbb{R}^n$ such that $w(x_0)=0.5$,
(ii) w is monotone,
(iii) w is upper semi-continuous on K_1 and lower semi-continuous on K_2 where,

$$K_1=\{x\in\mathbb{R}^n \mid 0<w(x)\leq 0.5\},$$

and

$$K_2=\{x\in\mathbb{R}^n \mid 0.5\leq w(x)<1\}.$$

(iv) $\overline{K_1\cup K_2}$ is compact.

Now, take

$$\hat{E}^{2n}_*=\{(r,\phi)\in E^n\times F^n \mid \exists x_0\in\mathbb{R}^n\ s.t\ r(x_0)=1\ and\ \phi(x_0)=0.5\}.$$

E^{2n}_* is embeddable into a Banach space (see [2]). Then the following results, apply equally to the space E^{2n} in the Cartesian case and to the space E^{2n}_* in the polar case. For brevity, we shall let $\mathbb{E}=\hat{E}^{2n}$ when dealing with the Cartesian complex form, and $\mathbb{E}=\hat{E}^{2n}_*$ when dealing with the polar complex form. We define differentiability as in [6] in terms of the Hukuhara difference. For $f,g\in\mathbb{E}$, if there exists $h\in\mathbb{E}$ such that $g+h=f$, we write $f-g=h$ and call h the difference of f and g. Let $I=[0,a]\subset\mathbb{R}$ be a compact interval.

Definition 1. *A mapping $F:I\longrightarrow\mathbb{E}$ is differentiable at $t_0\in I$ if there exists $F'(t_0)\in\mathbb{E}$ such that the following limits:*

$$\lim_{h\longrightarrow 0^+}\frac{F(t_0+h)-F(t_0)}{h}\qquad and\ \lim_{h\longrightarrow 0^+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

Let $F:I\longrightarrow\mathbb{E}$ be a continuous mapping. We define $G:I\longrightarrow\mathbb{E}$ by

$$G(t)=\int_0^a F(s)ds,\quad t\in I.$$

Note that

$$\frac{d}{dt}G(t)=G'(t)=F(t),\quad t\in I\quad (see[8]).$$

And we have

$$\hat{D}(F(a),F(0))\leq a\sup_{t\in I}\hat{D}(F'(t),\tilde{0}_2),$$

$$\hat{D}(G(a),G(0))\leq a\sup_{t\in I}\hat{D}(F(t),\tilde{0}_2).$$

III. Main Results

A. Complex Fuzzy Semigroups

We give here a definition of Semigroup on $\mathbb{E}$, which is similar to that given in [3], [7].

Definition 2. *A family $\{\pi(t),t\geq 0\}$ of operators from $\mathbb{E}$ into itself is a complex fuzzy strongly continuous semigroup on $\mathbb{E}$ if*

(i) *$\pi(0)=i$, the identity mapping on $\mathbb{E}$,*
(ii) *$\pi(t+s)=\pi(t)\pi(s)$ for all $t,s\geq 0$,*

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(iii) *the function* $H : [0,\infty[\to \mathbb{E}$*, defined by* $H(t) = \pi(t)f$ *is continuous at* $t = 0$ *for all* $f \in \mathbb{E}$ *i.e*

$$\lim_{t\to 0^+} \pi(t)f = f$$

(iv) *There exist two constants* $M > 0$ *and* ω *such that*

$$\hat{D}\Big(\pi(t)f, \pi(t)g\Big) \leq M\, e^{\omega t}\, \hat{D}(f,g), \quad for\, t \geq 0, f, g \in \mathbb{E}$$

In particular if $M = 1$ *and* $\omega = 0$*, we say that* $\{\pi(t), t \geq 0\}$ *is a contraction semigroup on* $\mathbb{E}$*.*

Remark 1. *The condition (iii) implies that the function* $t \longrightarrow \pi(t)x$ *is continuous on* $[0,\infty[$ *for all* $f \in \mathbb{E}$*.*

Definition 3. *Let* $\{\pi(t), t \geq 0\}$ *be a strongly continuous semigroup on* $\mathbb{E}$ *and* $f \in \mathbb{E}$*. If for* $h > 0$ *very small, the Hukuhara difference* $\pi(h)f - f$ *exits, we define*

$$\mathcal{A}f = \lim_{h\to 0^+} \frac{\pi(h)f - f}{h}$$

whenever this limit exists in the metric space $(\mathbb{E}, \hat{D})$*. Then the operator* $\mathcal{A}$ *defined on*

$$D(\mathcal{A}) = \left\{ f \in \mathbb{E} \,|\, \lim_{h\to 0^+} \frac{\pi(h)f - f}{h}\; exists \right\} \subset \mathbb{E}$$

is called the infinitesimal generator of the semigroup $\{\pi(t), t \geq 0\}$*.*

Lemma 1. *Let* $\mathcal{A}$ *be the generator of a semigroup* $\{\pi(t), t \geq 0\}$ *on* $\mathbb{E}$*, then for all* $f \in \mathbb{E}$ *such that* $\pi(t)f \in D(\mathcal{A})$ *for all* $t \geq 0$*, the mapping* $\mathcal{F} : t \to \pi(t)f$ *is differentiable and*

$$\mathcal{F}'(t) = \frac{d}{dt}\Big(\pi(t)f\Big) = \mathcal{A}\,\pi(t)f, \quad \forall t \geq 0$$

Proof:

For $f \in \mathbb{E}$, $t \geq 0$ and h very small, since $\pi(t)f \in D(\mathcal{A})$, so

$$\begin{aligned}
\lim_{h\longrightarrow 0^+} \frac{\mathcal{F}(t+h) - \mathcal{F}(t)}{h} &= \lim_{h\longrightarrow 0^+} \frac{\pi(t+h)f - \pi(t)f}{h} \\
&= \lim_{h\longrightarrow 0^+} \frac{\pi(h)\pi(t)f - \pi(t)f}{h} \\
&= \mathcal{A}\pi(t)f.
\end{aligned}$$

■

Example 1. *Define on* $\mathbb{E}$ *the family of linear operators* $\pi(t) : \mathbb{E} \longrightarrow \mathbb{E}$ *by*

$$\pi(t)(u,v) = (e^{nt}u\,,\, e^{mt}v), \quad (u,v) \in \mathbb{E},\, n, m \in \mathbb{N},\, t \geq 0.$$

1) $\{\pi(t)\,, t \geq 0\}$ *is a semigroup on* $\mathbb{E}$*. Indeed it is easy to see that* $\pi(t)$ *is well-defined.*

(i) *for* $(u,v) \in \mathbb{E}$*, we have*

$$\pi(0)(u,v) = (u,v).$$

(ii) *For* $(u,v) \in \mathbb{E}$*,* $t, s \geq 0$*, we have*

$$\begin{aligned}
\pi(t+s)(u,v) &= (e^{n(t+s)}u, e^{m(t+s)}v) \\
&= (e^{nt}e^{ns}u, e^{mt}e^{ms}v) \\
&= \pi(t)(e^{ns}u, e^{ms}v) \\
&= \pi(t)\pi(s)(u,v).
\end{aligned}$$

(iii) *For* $(u,v) \in \mathbb{E}$*,* $t \geq 0$ *and* $l = \max\{n, m\}$*, we have*

$$\begin{aligned}
&\hat{D}(\pi(t)(u,v), (u,v)) \\
&= \hat{D}\left((e^{nt}u, e^{mt}v), (u,v)\right) \\
&= \max\{D\left(e^{nt}u, u\right), D\left(e^{mt}v, v\right)\} \\
&= \max\{D\left((e^{nt}-1)u, \tilde{0}\right), D\left((e^{mt}-1)v, \tilde{0}\right)\} \\
&= \max\{(e^{nt}-1)D\left(u, \tilde{0}\right), (e^{mt}-1)D\left(v, \tilde{0}\right)\} \\
&\leq (e^{lt}-1)\max\{D\left(u, \tilde{0}\right), D\left(v, \tilde{0}\right)\} \\
&= (e^{lt}-1)\hat{D}\left((u,v), (\tilde{0}, \tilde{0})\right) \\
&= (e^{lt}-1)\hat{D}\left((u,v), \tilde{0}_2\right) \longrightarrow 0\, as\, t \longrightarrow 0.
\end{aligned}$$

Then, $\lim_{t\longrightarrow 0} \pi(t)(u,v) = (u,v)$*.*

(iv) *For* (u,v)*,* $(u',v') \in \mathbb{E}$ *and* $t \geq 0$*, we have*

$$\begin{aligned}
&\hat{D}\left(\pi(t)(u,v)\,,\, \pi(t)(u',v')\right) \\
&= \hat{D}\left((e^{nt}u, e^{mt}v)\,,\, (e^{nt}u', e^{mt}v')\right) \\
&= \max\left\{D\left(e^{nt}u, e^{nt}u'\right)\,,\, D\left(e^{mt}v, e^{mt}v'\right)\right\} \\
&= \max\left\{e^{nt}D\left(u, u'\right)\,,\, e^{mt}D\left(v, v'\right)\right\} \\
&\leq e^{lt}\max\left\{D\left(u, u'\right)\,,\, D\left(v, v'\right)\right\} \\
&\leq e^{lt}\hat{D}\left((u,v)\,,\, (u',v')\right).
\end{aligned}$$

2) *The linear operator* $\mathcal{A} : (u,v) \longrightarrow \mathcal{A}(u,v) = (nu, mv)$ *is the infinitesimal generator of the semigroup* $\{\pi(t), t \geq 0\}$*. Indeed, for* $(u,v) \in \mathbb{E}$ *and* $h \geq 0$ *very small, we have*

$$\begin{aligned}
&\left(\left(e^{nh}-1\right)u\,,\, \left(e^{mh}-1\right)v\right) + (u\,,\, v) \\
&= \left(\left(e^{nh}-1\right)u + u\,,\, \left(e^{mh}-1\right)v + v\right) \\
&= \left(e^{nh}u\,,\, e^{mh}v\right) \\
&= \pi(h)(u,v).
\end{aligned}$$

then the difference $\left(e^{nh}u\,,\, e^{mh}v\right) - (u,v)$ *exists and equal* $\left(\left(e^{nh}-1\right)u\,,\, \left(e^{mh}-1\right)v\right)$*.*
Therefore, we have

$$\begin{aligned}
\frac{\left(e^{nh}u\,,\, e^{mh}v\right) - (u,v)}{h} &= \frac{1}{h}\left(\left(e^{nh}-1\right)u\,,\, \left(e^{mh}-1\right)v\right) \\
&= \left(\left(\frac{e^{nh}-1}{h}\right)u\,,\, \left(\frac{e^{mh}-1}{h}\right)v\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \hat{D}\left(\frac{(e^{nh}u\,,\,e^{mh}v)-(u,v)}{h}\,,\,(nu,mv)\right)\\
= & \hat{D}\left(\left(\left(\frac{e^{nh}-1}{h}\right)u\,,\,\left(\frac{e^{mh}-1}{h}\right)v\right)\,,\,(nu,mv)\right)\\
= & \max\left\{D\left(\left(\frac{e^{nh}-1}{h}\right)u\,,\,nu\right)\,,\right.\\
& \left.D\left(\left(\frac{e^{mh}-1}{h}\right)v\,,\,mv\right)\right\}\\
= & \max\left\{D\left(nu+\left(\sum_{k=2}^{\infty}\frac{(nh)^k}{k!}\right)u\,,\,nu\right)\,,\right.\\
& \left.D\left(mv+\left(\sum_{k=2}^{\infty}\frac{(mh)^k}{k!}\right)v\,,\,mv\right)\right\}\\
= & \max\left\{D\left(\left(\sum_{k=2}^{\infty}\frac{(nh)^k}{hk!}\right)u\,,\,\tilde{0}\right)\,,\right.\\
& \left.D\left(\left(\sum_{k=2}^{\infty}\frac{(mh)^k}{hk!}\right)v\,,\,\tilde{0}\right)\right\}\\
= & \max\left\{\left(\sum_{k=2}^{\infty}\frac{(nh)^k}{hk!}\right)D\left(u\,,\,\tilde{0}\right)\,,\right.\\
& \left.\left(\sum_{k=2}^{\infty}\frac{(mh)^k}{hk!}\right)D\left(v\,,\,\tilde{0}\right)\right\}\\
\leq & \left(\sum_{k=2}^{\infty}\frac{(lh)^k}{hk!}\right)\max\left\{D\left(u\,,\,\tilde{0}\right)\,,\,D\left(v\,,\,\tilde{0}\right)\right\}\\
= & \left(\sum_{k=2}^{\infty}\frac{(lh)^k}{hk!}\right)\hat{D}\left((u\,,\,v)\,,\,(\tilde{0},\tilde{0})\right)\\
= & \frac{e^{lh}-1-lh}{h}\hat{D}\left((u\,,\,v)\,,\,\tilde{0}_2\right)\longrightarrow 0\,as\,t\longrightarrow 0.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{A}(u,v) &= \lim_{h\longrightarrow 0^+}\frac{\pi(h)(u,v)-(u,v)}{h}\\
&= \lim_{h\longrightarrow 0^+}\frac{\left(e^{nh}u\,,\,e^{mh}v\right)-(u,v)}{h}\\
&= (nu,mv).
\end{aligned}$$

B. Existence and Uniqueness

Let $\mathcal{C}(I,\mathbb{E})$ denote the set of all continuous maps from I to $\mathbb{E}$ and let $\hat{D}_{\mathcal{C}}$ denote a metric on $\mathcal{C}(I,\mathbb{E})$ defined as

$$\hat{D}_{\mathcal{C}}(\mathcal{F},\mathcal{G})=\sup_{t\in I}\hat{D}(\mathcal{F}(t),\mathcal{G}(t)),\qquad \mathcal{F},\mathcal{G}\in\mathcal{C}(I,\mathbb{E}),$$

It follows that $(\mathcal{C}(I,\mathbb{E}),\hat{D}_{\mathcal{C}})$ is a complete metric space.

Definition 4. *we say that $X:I\longrightarrow\mathbb{E}$ is a mild solution to the problem (1) if and only if $X\in\mathcal{C}(I,\mathbb{E})$, $X(t)\in D(\mathcal{A})$ for all $t\geq 0$ and X satisfies the integral equation*

$$X(t)=\pi(t)\Big(X_0+G(X)\Big)+\int_0^t\pi(t-s)H(s,X(s))ds,$$

for all $t\in I$.

Definition 5. *A mapping $H:\mathbb{E}\longrightarrow\mathbb{E}$ is Holder continuous if there exists a constant $L>0$ and a constant $0<\alpha\leq 1$ such that*

$$\hat{D}(H(X),H(Y))\leq L(\hat{D}(X,Y))^{\alpha},\qquad \forall X,Y\in\mathbb{E}.$$

Definition 6. *A mapping $H:I\times\mathbb{E}\longrightarrow\mathbb{E}$ is Lipschitzian with respect to the second argument if there exists a constant $N>0$ such that*

$$\hat{D}(H(t,X),H(t,Y))\leq N\hat{D}(X,Y),\qquad \forall X,Y\in\mathbb{E},\quad t\geq 0.$$

In the following it is assumed that all relevant Hukuhara differences taken exist. We first study the existence and uniqueness of mild solutions using the fixed point argument, under the following assumptions:
$(\mathbf{A}_0)$. $H:[0,a]\times\mathbb{E}\longrightarrow\mathbb{E}$ is continuous and Lipschitzian with respect to the second argument.i.e there exists a constants $L_1>0$, such that

$$\hat{D}(H(t,X),H(t,Y))\leq L_1\hat{D}(X,Y),\quad t\in[0,a],\quad X,Y\in\mathbb{E}$$

$(\mathbf{A}'_0)$. $H:[0,a]\times\mathbb{E}\longrightarrow\mathbb{E}$ is continuous and Holder continuous.
$(\mathbf{A}_1)$. $G:\mathcal{C}(I;\mathbb{E})\longrightarrow\mathbb{E}$ is continuous and there exists $k>0$, such that

$$\hat{D}(G(X),G(Y))\leq k\hat{D}_{\mathcal{C}}(X,Y),\quad X,Y\in\mathbb{E}$$

$(\mathbf{A}_2)$. Let M and ω given in the defintion 2,

$$Me^{\omega a}\big(L_1a+k\big)<1.$$

Under these assumptions, we can prove the following result.

Theorem 1. *Let Assumptions $(\boldsymbol{A}_0)$, $(\boldsymbol{A}_1)$ and $(\boldsymbol{A}_2)$ be satisfied. Then the problem (1) has a unique mild solution.*

Proof:
Define the operator $\mathcal{O}$ on $\mathcal{C}(I,\mathbb{E})$ by

$$\mathcal{O}X(t)=\pi(t)\Big(X_0+G(X)\Big)+\int_0^t\pi(t-s)H(s,X(s))ds,\quad t\in[0,a].$$

It is easy to see that $\mathcal{O}$ is well-defined.
For $X,Y\in\mathcal{C}(I,\mathbb{E})$ and $t\in I$, we have

$$\begin{aligned}
& \hat{D}(\mathcal{O}X(t)\,,\,\mathcal{O}Y(t))\\
= & \hat{D}\left(\mathcal{O}(X-Y)(t))\,,\,\tilde{0}_2\right)\\
= & \hat{D}\Bigg(\Big(\pi(t)\big(X_0+G(X)\big)-\pi(t)\big(X_0+G(Y)\big)\Big)+\\
& \int_0^t(\pi(t-s)H(s,X(s))-\pi(t-s)H(s,Y(s)))ds\,,\,\tilde{0}_2\Bigg)\\
\leq & \hat{D}\left(\int_0^t(\pi(t-s)H(s,X(s))-\pi(t-s)H(s,Y(s)))ds,\,\tilde{0}_2\right)+\\
& \hat{D}\Big(\pi(t)\big(X_0+G(X)\big)-\pi(t)\big(X_0+G(Y)\big),\,\tilde{0}_2\Big)
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\int_0^t \hat{D}\left(\pi(t-s)H(s,X(s)) - \pi(t-s)H(s,Y(s)),\, \tilde{0}_2\right) ds\right| + \\
&\quad \hat{D}\Big(\pi(t)\big(X_0+G(X)\big)\,,\, \pi(t)\big(X_0+G(Y)\big)\Big) \\
&\leq \int_0^t \left|\hat{D}\left(\pi(t-s)H(s,X(s)) - \pi(t-s)H(s,Y(s)),\, \tilde{0}_2\right)\right| ds + \\
&\quad \hat{D}\Big(\pi(t)\big(X_0+G(X)\big)\,,\, \pi(t)\big(X_0+G(Y)\big)\Big) \\
&= \int_0^t \hat{D}\left(\pi(t-s)H(s,X(s)),\, \pi(t-s)H(s,Y(s))\right) ds + \\
&\quad \hat{D}\Big(\pi(t)\big(X_0+G(X)\big)\,,\, \pi(t)\big(X_0+G(Y)\big)\Big) \\
&\leq Me^{\omega a}\int_0^t \hat{D}\left(H(s,X(s)),\, H(s,Y(s))\right) ds + \\
&\quad Me^{\omega a}\hat{D}\left(X_0+G(X)\,,\, X_0+G(Y)\right) \\
&\leq L_1 Me^{\omega a}\int_0^t \hat{D}\left(X(s),\, Y(s)\right) ds + kMe^{\omega a}\hat{D}_{\mathcal{C}}\left(X\,,\, Y\right) \\
&\leq Me^{\omega a}\big(L_1 a + k\big)\hat{D}_{\mathcal{C}}\left(X,\, Y\right).
\end{aligned}$$

which implies

$$\hat{D}_{\mathcal{C}}\left(\mathcal{O}X,\, \mathcal{O}Y\right) \leq Me^{\omega a}\big(L_1 a + k\big)\hat{D}_{\mathcal{C}}\left(X,\, Y\right).$$

Now from Assumption ($\mathbf{A}_2$), we find that $\mathcal{O}$ is a contraction operator on $\mathcal{C}_i$, and there exists a unique $X \in \mathcal{C}_i$ such that $\mathcal{O}X = X$.

So we conclude that X is the unique mild solution of (1). ∎

Corollary 1. *Let* $(\boldsymbol{A}_0')$, $(\boldsymbol{A}_1)$ *and* $(\boldsymbol{A}_2(\text{with}\, L_1$ *is the Holder constant in this case*$))$ *be satisfied. Then, there exists a unique mild solution to the problem (1) on* I.

We give an example to demonstrate the effectiveness of the proposed results.

Example 2. *Consider the evolution fuzzy equation for* $X = (u,v) \in \mathbb{E} = \hat{E}^2$ *on* $t \in [0,1]$ *given by:*

$$\begin{cases} X'(t) = \mathcal{A}X(t) + \dfrac{1}{24}\sqrt{|X(t)|} + \dfrac{\pi}{12}iX(t) + t, \\ X(0) = X_0 + \dfrac{\pi}{12}iX. \end{cases} \tag{2}$$

This problem can be abstracted into

$$\begin{cases} X'(t) = \mathcal{A}X(t) + H(t,X(t)), & t \in [0,1] \\ X(0) = X_0 + G(X). \end{cases}$$

where,

$$H(t,X(t)) = \frac{1}{24}\sqrt{|X(t)|} + \frac{\pi}{12}iX(t) + t,$$

$G(X) = \dfrac{\pi}{12}iX$ *and* $\mathcal{A}$ *is the infinitesimal generator of a complex fuzzy semigroups of contraction* $(M = 1,\, \omega = 0)$.

We have, H *is Holder continuous with* $L_1 = \dfrac{1+2\pi}{24}$, $\alpha = \dfrac{1}{2}$ *and* $a = 1$, $k = \dfrac{\pi}{12}$.
Hence, we have

$$Me^{\omega a}\big(L_1 a + k\big) = \frac{1+4\pi}{24} < 1.$$

by the previous corollary, the problem (2) has a unique mild solution.

IV. Conclusion

By using operator semigroup of fuzzy sets theory, we obtain existence results for Complex fuzzy evolution equation with nonlocal conditions. In addition, future work includes expanding the idea signalized in this work and introducing some examples of application. This is a fertile field with vast research projects, which can lead to numerous theories and applications. We plan to devote significant attention to this direction. And we intend to investigate the applications which are based on experimental data (real world problems)of the proposed theory.

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