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Mathematical Study for Traffic Flow and Traffic Density in Kigali Roads

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Abstract**—**This work investigates a mathematical study for traffic flow and traffic density in Kigali city roads and the data collected from the national police of Rwanda in 2012. While working on this topic, some mathematical models were used in order to analyze and compare traffic variables. This work has been carried out on Kigali roads specifically at roundabouts from Kigali Business Center (KBC) to Prince House as our study sites. In this project, we used some mathematical tools to analyze the data collected and to understand the relationship between traffic variables. We applied the Poisson distribution method to analyze and to know the number of accidents occurred in this section of the road which is from KBC to Prince House. The results show that the accidents that occurred in 2012 were at very high rates due to the fact that this section has a very narrow single lane on each side which leads to high congestion of vehicles, and consequently, accidents occur very frequently. Using the data of speeds and densities collected from this section of road, we found that the increment of the density results in a decrement of the speed of the vehicle. At the point where the density is equal to the jam density the speed becomes zero. The approach is promising in capturing sudden changes on flow patterns and is open to be utilized in a series of intelligent management strategies and especially in noncurrent congestion effect detection and control.

Keywords**—**Statistical methods, Poisson distribution, car moving techniques, traffic flow.

I. Introduction

RECENTLY, a lot of researchers have been attracted by traffic problems, and various models and method have been developed for complex traffic phenomena [7]-[9]. The most known are macroscopic models [10]-[15] and microscopic models [16]-[20]. These traffic phenomena become more complex and nonlinear if we take into considerations the interactions of a big number of vehicles. Vehicles do not easily interact following mechanical laws due to the individual reactions of the people who drive, and instead, they exhibit both cluster formation and shock wave propagation phenomena. Specifically, we can talk about traffic flow and traffic density.

Traffic flow can be interpreted as the study of the relationship between vehicles, drivers, and the built environments [1], i.e. highways, signage and traffic control devices, with the aim of understanding and developing a more favorable road system with efficient movement of traffic and the reduction of traffic congestion problems. Traffic density is defined as the average number of vehicles that occupy one mile or kilometer of road space, and expressed in vehicles per mile or per kilometer. Speed in traffic flow is defined as the distance covered per unit of time.

Traffic theory is a tool that helps transportation engineers understand and express the properties of traffic flow. At any given time, there are millions of vehicles on our roadways and the interaction of these vehicles with each other impact the overall movement of traffic, or the traffic flow [1].

To overcome the traffic jam, we can suggest these two measures: increasing the number of roads in the city of Kigali, Rwanda, and to modernize the existing roads so as to efficiently make use of the existing ones. Generally, our aim is to reduce traffic congestion and reduce number of accidents on Kigali roads depending on the existing roads and the number of vehicles travelling in a particular place.

Making efficient use of the existing facilities can be achieved by the application of efficient traffic control measures. The efficient traffic control measures include increasing the number of traffic lights and to setting serious policies on the accident makers.

In this work, we shall not work on all kinds of traffic problems, alternatively, we shall deal with problems which have already got a mathematical model such as how traffic flows along an unidirectional road rather than analyzing the behaviour of the individual car, here we shall make a study on traffic situations resulting from the complex interactions of many vehicles.

We discuss some of the fundamental traffic variables such as **flow**, **density** and **speed.** Here, speed flow and density are all related to each other; however, the relationship between the two is not easily seen except by doing a study on them, since the flow is the product of speed and density, it therefore follows that the flow is equal to zero in the case where either speed or density is zero. Also, there is another possibility of reducing that flow by maximizing it at some fixed combination of speed and density [21].

The mentioned points can be explained by the two following ordinary traffic conditions: the first is the modern traffic jam which occurs when there are a high number of vehicles moving at very low speeds due to insufficient lanes to accommodate the high traffic density. This combination leads to a very slow flow. However, the second condition occurs in cases where there are extremely low traffic densities and the drivers, motorists have the liberty to attain free flow speeds without any undue stress caused by other vehicles on the same roadway. The extremely low density culminates in high speeds, and in turn leads to very fast flow.

This project will consist of different parts, whereby the first part will be the introduction of the topic called mathematical

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study for traffic flow and traffic density in Kigali roads, the second part will present the mathematical methods, data analysis and resolutions and the third part will include the results, discussion, recommendations and conclusions.

A. Objectives of the Study

- ❖ To give a mathematical formulation of traffic variables.
- ❖ To determine the linear relationship between traffic variables.
- ❖ To interpret traffic variables using statistical methods.
- ❖ To know very well how to avoid traffic accidents.
- ❖ To modernize the existing roads and make the efficient usage of the existing ones.
- ❖ To reduce traffic congestion and the number of accidents on Kigali roads.

There is a mutual relationship between speed, flow, and density. The relationship between speed and density can easily be observed in the real world, albeit their effects being not as apparent.

Under undisturbed flow conditions, the relationship between speed, density, and flow is given by:

$$q = k * v$$

where q = Flow (vehicles/hour), v = Speed (miles/hour, kilometers/hour, k = Density (vehicles/mile, vehicles/ kilometer).

Since the flow is the product of speed and density, it therefore follows that the flow is equal to zero in the case where either speed or density is zero. In addition, it can be inferred that for a specific combination of speed and density, the flow is optimum. These aspects are illustrated by two typical traffic occurrences.

The first is the modern traffic jam, where traffic densities are very high and speeds are very low [3]. This combination produces a very low flow.

The second occurrence is when traffic densities are very low and motorists are able to obtain free flow speeds without any unnecessary stress caused by the presence of other vehicles on the motorway. The extremely low traffic densities allow for high speeds hence, the flow is rather high.

Traffic flow and traffic density theories are the tools that help mathematicians and engineers to understand and express the properties of traffic flow. At any given instant, there might thousands of thousands of vehicles on our roadways throughout the year and this leads to the loss of an infinite number of hours as a result of traffic jams every year. Frustratingly, most of all are phantom jams (jams with no apparent cause) i.e. no accident, no lanes closed under construction, no stalled vehicle. These phantom jams can only occur when there is a high congestion of cars on the road. In such high traffic density traffic, slight disturbances such as a driver hitting the brake too hard or getting too close to another car, can quickly lead into a full-blown and self-sustaining traffic jam [2].

II. Mathematical Methods and Analysis

A. Introduction

This part presents methods describing traffic flow and traffic density and these methods are Poisson distribution, green shield, and regression method [4], [5].

Three main traffic parameters are:

- ✓ Flow
- ✓ Density
- ✓ Speed

Relationship between three parameters:

- ➢ Green shield relationship: linear relationship
- ➢ Under wood relationship: exponential relationship
- ➢ Green berg relationship : logarithm relationship

Equipment used to measure traffic flow:

- PHANTOM speed gun
- HAND car counter
- PHANTOM speed gun: measures the speed of the vehicles
- HAND car counter: used to count the number of vehicles passing in 1 hour

B. Green Shield Equations

$q = u * k$ (Number of vehicles/hour), k = Number of vehicles/unit length), u = length/unit time (km/h).

$$\boldsymbol{q = u * k} \tag{1}$$

$$\frac{u}{u_f} + \frac{k}{k_j} = 1$$

where, $u =$ Speed, $u_f =$ Free flow speed, $k =$ Density, $k_j =$ Jam density.

$$u = \left(1 - \frac{k}{k_j}\right) u_f \tag{2}$$

Substitute (2) into (1) to get

$$q = \left[u_f - \frac{u_f}{k_j} k\right] k$$

$$q = u_f k - \frac{u_f}{k_j} k^2 \tag{3}$$

The flow will be the maximum if its first derivative is equal to zero. Then,

$$\frac{dq}{dk} = u_f - 2\frac{u_f}{k_j} k = 0 \tag{4}$$

$$= u_f \left(1 - \frac{2}{k_j} k\right) = 0 \tag{5}$$

$$= u_f\left(1-\frac{2}{k_j}k\right)=0$$

$$1-\frac{2}{k_j}k=0$$

$$\Rightarrow k=\frac{k_j}{2} \tag{6}$$

(The density is half its jam density)

If density and speed are maximums it means that the flow will be also the maximum, therefore:

$$q_{\max}=k_{\max}u_{\max}$$

Substitute (6) into (3) and obtain:

$$q_{\max}=\frac{k_j}{4}u_f$$

The maximum flow will be equal to quarter jam density times free flow speed, this is known as the **green shield relationship.**

C. Car Moving Average Method

This method shall also be used to count the number of vehicles passing through the section in unit time in one direction and is given by:

$$q=\left(x+\frac{y}{a}+w\right) \tag{7}$$

where, $q =$ The average number of vehicles passing through the section in unit time in one direction, $x =$ Average number of vehicles counted in this direction when the test car is travelling in opposite direction, $y =$ Average number of overtaking vehicles less the number overtaken when the test car is travelling in the direction of q, $w =$ Average journey time taken by the test car to travel over the section in the direction of q, $a =$ Average time taken in the opposite direction.

NOTE: the average journey time taken, t, of the stream, q, is given by:

$$t=w-\frac{v}{q} \tag{8}$$

The average journey time is then obtained by dividing the length of each section by the average journey time.

III. Results and Discussion

A. Introduction

This part presents the results and discusses the methods used to analyze the data collected from the site and data collected from national police of Rwanda.

B. Using Poisson Distribution Method [6]

Poisson distribution is a discrete distribution which is applicable when an outcome is the number of times an event occurs. It gives the probability that an outcome occurs in a specified number of times. This method can be used to determine the probability of rare events, whereby there are a large number of trials and the likelihood of the occurrence of any one of them is minimal. Therefore, the probability of exactly x occurrences is given by:

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!} \tag{9}$$

Poisson distribution is slightly positively skewed. The skew becomes more pronounced as λ becomes smaller, where λ (lambda) $=np$, is the value of both the mean and variance of Poisson distribution, and e is the base of natural logarithm equal to 2.718.

Using Poisson distribution, we found the probability of having the minimum number of accidents in 2012 by considering the number of accidents occurring per month. Note that the Poisson formula is given by:

$$p(x)=\frac{e^{-\lambda}\lambda^x}{x!}.$$

where P(x) is the probability of occurrence of x events. Then from the given data of the site, the probability of having minimum number of accidents, like 20, will be:

$$\lambda=\frac{84+87+94+90+66+88+116+117+99+29+31+42}{12}=78.58.$$

Then

$$P(20)=\frac{e^{-78.58}78.58^{20}}{20!}=2.47318*10^{-15}.$$

Based on the information available, it was found that the probability of having the minimum number of accidents is $2.47318*10^{-15}$, which is very small, since the average number of accidents in 2012 was 78.58, which is very high.

C. Using Least Square Method

Regression analysis is a statistical technique that uses observed data to relate the dependent variable to one or more independent variables.

Least square equation is given by:

$$y=ax+b \tag{10}$$

with,

$$a = \frac{SS_{xy}}{SS_{xx}}$$

$$SS_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}$$

$$SS_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}$$

From our study, we will assume that: $y = u$ (Speed), $x = k$ (Density). Then our equation will become:

$$u = ak + b \tag{11}$$

where **a** and **b** is defined as multiplying and adding constants, and is calculated as:

$$b = \bar{y} - a\bar{x}\ , \quad a = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2},$$

$$b = \bar{y} - a\bar{x} \qquad \bar{y} = \frac{\sum y_i}{n} \qquad \bar{x} = \frac{\sum x_i}{n}$$

D.Calculation of Proportionality

In regression, proportionality is used as simple correlation coefficient and is defined by:

Measures of the strength of the linear relationship between y and x and is denoted by r.

In traffic study r is calculated as:

$$r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}} \tag{12}$$

and is interpreted as:

- If r=0 No relationship
- If r=1 Perfect relationship
- If r=+ve Directly proportional
- If r=-ve Inversely proportional

TABLE I
CALCULATION OF PROPORTIONALITY

Correlations			
		speed	Density
Speed	Pearson Correlation	1	-0.996**
	Sig. (2-tailed)		0.000
	N	10	10
Density	Pearson Correlation	-0.996**	1
	Sig. (2-tailed)	0.000	
	N	10	10

**Correlation is significant at the 0.01 level (2-tailed).

Plugging the values of the parameter estimates into (11) we observe that $u = 64 - 0.591k$.

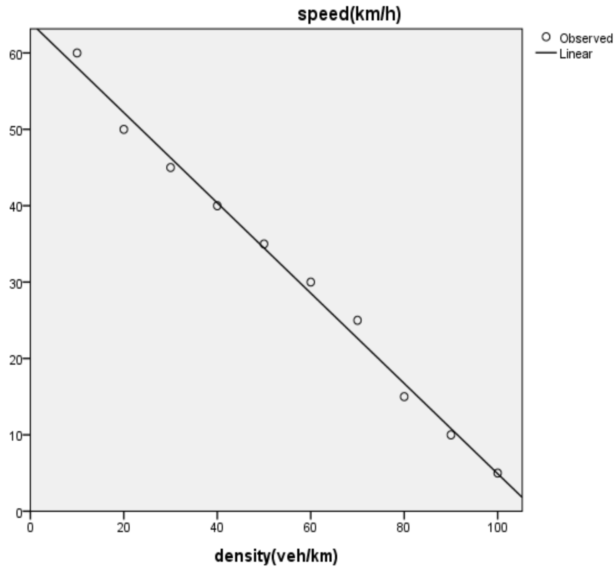

Fig. 1 Speed density plot

TABLE II
MODEL SUMMARY AND PARAMETER ESTIMATES

Dependent Variable: speed (km/h)							
Equation	Model Summary					Parameter Estimates	
	R Square	F	df1	df2	Sig.	Constant	b1
Linear	0.992	1056.250	1	8	0.000	64.000	-0.591

The independent variable is density (vehicles/km).

From our observation, we found that $r = -0.996 \approx -1$. Since our **r** is negative, it implies u and k are inversely proportional, which shows that speed and density are inversely proportional. Consequently, when density increases speed decreases rapidly. This negative regression coefficient means that the relationship between speed and density in that road is not perfect. Thus, the negative regression coefficient should be adjusted by decreasing the speed or density, or by increasing the number of lane per road. The equation:

$$u = 64 - 0.59k \tag{13}$$

where the independent variable is density and dependent variable is speed, is called ***speed-density equation.***

Using green shield relationship

$$q = u * k,$$

We can take the value of $u = \frac{q}{k}$ and replace it into (13) to get:

$$\frac{q}{k} = 64 - 0.59k$$

It is worth nothing that:

$$k^2 - 108k + 1.7q = 0 \tag{14}$$

Equation (14) is known as the **density-flow equation.**

Free speed: At free speed the density is zero and then

$$u = 64 - 0.59 * 0$$
$$= 64 km / h$$

Jam density: At jam density, the speed will be equal to zero implying that

$$u = o$$

meaning that:

$$64 - 0.59 k_j = 0$$
$$k_j = \frac{64}{0.59} = 108.5 veh / km$$
$$\approx 109 veh / km$$
$$k = \frac{k_j}{2} = \frac{108.5}{2} = 54.25 veh / km$$
$$\approx 54 veh / km$$

It shows that the density will be equal to half jam density.

IV. Interpretation and Conclusion

Accidents in this area occur due to the fact that the roads are very small and are constructed on highways with many junctions. According to the Poisson distribution, it was found that the probability of the occurrence of accidents in a year is very high. During this study, the Least Square Method was used on (13), where it was found that speed and density are inversely proportional; that is, when density increases the speed decreases. Considering Fig. 1, we do not have a good model since all data are not lying in the fitted line. The coefficient of proportionality is negative which means that speed and density are inversely proportional. Additionally, it was observed that the relationship between speed and density in this section of the road is not perfect. The above obtained results show that the velocity or speed is a function of density $u(k)$, which we assumed to be valid on strength of road for which the road variables such as the number of lanes and the smoothness of the road are constant. If $u = u(k)$ then a high speed of car as it approaches a slower line of traffic must itself slow down. This equation does not take into account the multi-lane highways where passing is not only permitted, but is a quite frequent event. In order for this equation to be a good approximation, the effect of a car passing must be small, as for example in one-line roads.

V. Conclusion

There is a link between the density of the traffic and the velocity of the vehicle. In a situation where the vehicles on the road are many, the velocity of the vehicles becomes slower. To relieve the over crowdedness and to stabilize the traffic flow, the rate of vehicles through the control zone must be less than or equal to the rate of vehicles leaving at a particular time. The state of the flow changes from stable to unstable at a critical traffic density and its critical velocity. In an unstable flow regime, the flow collapses when a vehicle brakes. Based on our study case, which is from the roundabout of KBC to Prince House, it was seen that the relationship between density and speed is not perfect according to the coefficient of proportionality, which gives negative correlation. Also based on the analysis of data of accidents obtained from the national police using Poisson distribution, it was shown that the probability of having more accidents on the specified roads in this study is very high.

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