Topological Sensitivity Analysis for Reconstruction of the Inverse Source Problem from Boundary Measurement

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Abstract—In this paper, we consider a geometric inverse source problem for the heat equation with Dirichlet and Neumann boundary data. We will reconstruct the exact form of the unknown source term from additional boundary conditions. Our motivation is to detect the location, the size and the shape of source support. We present a one-shot algorithm based on the Kohn-Vogelius formulation and the topological gradient method. The geometric inverse source problem is formulated as a topology optimization one. A topological sensitivity analysis is derived from a source function. Then, we present a non-iterative numerical method for the geometric reconstruction of the source term with unknown support using a level curve of the topological gradient. Finally, we give several examples to show the viability of our presented method.

Keywords—Geometric inverse source problem, heat equation, topological sensitivity, topological optimization, Kohn-Vogelius formulation.

I. Introduction

THE reconstruction of sources in physical systems represents a class of inverse problems in engineering, bioengineering and scientific applications. Among these, inverse source problems consist in determining an unknown measure with support in a domain Ω from a single measurement on the boundary $\partial\Omega$. The important practical examples of this inverse source problem include damage or defect identification from acoustic emission [33], [6], source identification in electromagnetics [9], odor or contaminant localization [2], [26], pollution in the environment [18], [29], optimal tomography [7], photo and thermo-acoustic tomography [5], [32] and bioluminescence tomography [34], [12], [22], [23] etc. The inverse source problem under consideration might be formulated as follows:

Let Ω be an open bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) with smooth boundary $\partial\Omega$, T be an arbitrary positive constant and $m \in \mathbb{N}^*$. We consider the following initial boundary value problem:

$$\frac{\partial u}{\partial t} - \Delta u + u = \sum_{i=1}^{m} b(t)\chi_{\mathscr{D}_i} \qquad \text{in } \Omega \times (0,T), \quad (1)$$

$$u(x,0) = \phi^* \ \forall x \in \Omega \text{ and } u(x,t) = 0 \ \forall(x,t) \in \partial\Omega \times (0,T) \quad (2)$$

where $\chi_{\mathscr{D}_i}$ denotes the characteristic function of a subdomain $\mathscr{D}_i$ of Ω with $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ for $i \neq j$ (see Fig. 1) and b is given

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function defined on $(0,T)$. The inverse source problem (1) and (2) describes the heat conduction procedure in a given medium Ω and $b(t)\chi_{\mathscr{D}_i}$ represents the discontinuous heat source. Some results on uniqueness and stability for this inverse problem can be found in [4], [8], [13], [14]. We want to detect the location and shape $\mathscr{D}_i$ by assigning the medium Ω at rest at initial time $t = 0$, the temperature on the boundary $\partial\Omega$ for an interval of time $(0,T)$, and measuring the flux of temperature, that is, the normal derivative of u on the exterior boundary $\partial\Omega$ for an interval time $(0,T)$.
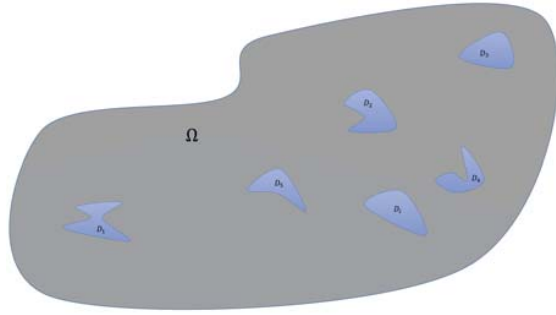

Fig. 1 Geometry of the problem

Over the years, there were different methods of identification and reconstruction of characteristic source functions from boundary data. There are methods which are based on providing as much a priori knowledge as possible about the source item. This priori knowledge may include permissible regions, prescribed forms, conditional stability (the source function is sought in a smoother set), stronger smoothness assumptions [16], [29]. The another approach for overcoming the ill-posedness is regularization. This regularization method includes truncated singular value decomposition, iterative regularization and Tikhonov regularization [27]-[29], [3], [17]. More recently, a decomposition method based on the Kirsch-Kress method and a Newton-type method, on the domain derivative, has been applied [31].

The majority of approaches used in the previous works for the recovery of the shape of the characteristic source term is centered around the concept of regularization technique or iterative method was implemented via domain derivative, using the boundary element method as numerical solver. In this context, we propose another simpler approach combining the advantages of the Kohn-Vogelius formulation [10], [1]

and the topological sensitivity method [24], [20]. One of the main advantages of this method is that, it provides fast and accurate results for reconstruction (location and shape). The Kohn-Vogelius formulation is a self regularization technique and rephrase the inverse problem as an optimization problem where the support of the unknown measure is variable.

The topological sensitivity analysis measures the sensitivity of a shape functional with respect to an infinitesimal singular domain perturbation, such as creation of a small holes, cavities, inclusions, cracks or source-terms [1], [30], [19], [11]. The basic idea, of this method, consists in approximating the subdomain $\mathscr{D}_i$ by a small geometrical region $\omega_{a_i,\varepsilon}$; having the form $\omega_{a_i,\varepsilon} = a_i + \varepsilon\omega$ where a_i determine the location of the source support $\mathscr{D}_i$, ε is the shared diameter and $\omega \subset \mathbb{R}^d$ are bounded and smooth domains containing the origin. We introduce a characteristic function defined by:

$$\delta f_{a,\varepsilon}(x,t) = \sum_{i=1}^{m} b(t)\chi_{\omega_{a_i,\varepsilon}} \quad \text{where} \quad a = (a_1,...,a_m).$$

We consider the source function $\mathscr{J}(f,T) = \int_0^T \mathrm{J}(u[f])\mathrm{d}t$ where f is a given source term and $u[f]$ is the solution of a given PDE in $\Omega \times (0,T)$ (in this case $u[f] = u$ solution of the problem 1 and 2). Then, the source functional $\mathscr{J}$ associated with the topological perturbation of the source if the following topological asymptotic expansion:

$$\mathscr{J}(f + \delta f_{a,\varepsilon},T) = \mathscr{J}(f,T) + \zeta(\varepsilon)\delta\mathscr{J}(a,T) + o(\zeta(\varepsilon)),$$

where ζ is a scalar positive function going to zero with ε. This expression is called the topological source asymptotic expansion and $\delta\mathscr{J}$ is the topological gradient.

In order to minimize the source function, the location of the source $\delta f_{a,\varepsilon}$ in $\Omega \times (0,T)$ is where $\delta\mathscr{J}$ is negative. In fact if $\delta\mathscr{J}(a,T) < 0$, we have $\mathscr{J}(f + \delta f_{a,\varepsilon},T) < \mathscr{J}(f,T)$ for small ε. The function $\delta\mathscr{J}$ can be used as a descent direction of the optimization process.

In this paper, we present a non-iterative method for source reconstruction. Theoretically, we present a non-iterative algorithm for the fast source reconstruction while in numerical parts, we are testing this algorithm for some parameters. Initially, we would like to discuss the theoretical part and then the numerical one.

The theoretical part of this work is to develop a general, simple, and robust approach that can be used in a wide variety of problems. Our method is based on the notion of the existence of a topological gradient field that represents perturbations to a cost functional caused by the appearance of hypothetical small sources. We have derived a topological sensitivity for the parabolic heat problem with respect to a small geometrical perturbation of the source. The obtained results are general and valid for a large class of source functions and arbitrary shaped sources.

After getting the theoretical results, we denote some numerical simulations in order to confirm and deepen our theoretical results by testing the influence of some parameters in our non iterative algorithm of reconstruction such as the location, the size and the shape of source. This algorithm allows us to obtain the number and qualitative location of the source. Moreover, it allows us to approximate the shape of this one.

This paper generalizes the combination of a topological sensitivity analysis and Kohn-Vogelius formulation for the reconstruction of surfaces supporting source function from boundary measurements.

This paper is summarized and roundup as follows: In Section II, we study the formulations and notations for the heat inverse source problem. In Section III, we present the main steps of the presented approach to solve this geometric inverse problem (to reconstruct the unknown source term), while Section IV is related to the topological asymptotic expansion of the Kohn-Vogelius source function. Whereas in Section V, we present a one-shot algorithm and the efficiency and accuracy of the presented algorithm are illustrated by some numerical results.

II. GEOMETRIC INVERSE SOURCE PROBLEM

Let $\Omega \subseteq \mathbb{R}^d$, $d = 2$ or 3 be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. Take a final time $T > 0$ and define

$$Q_T = \Omega \times (0,T), \quad \Sigma_T = \Gamma \times (0,T) \text{ and } \partial_t = \frac{\partial.}{\partial t}.$$

Let us the following boundary value problem:

$$(\mathscr{P}) \begin{cases} \partial_t u - \Delta u + u & = & \delta f^* & \text{in } Q_T \\ u & = & \phi^* & \text{on } \Sigma_T, \\ \nabla u.\mathbf{n} & = & \varphi^* & \text{on } \Sigma_T, \\ u(.,0) & = & 0 & \text{in } \Omega. \end{cases}$$

The geometric inverse source problem: given $\phi^* \in L^2(0,T;H^{1/2}(\Gamma))$ and $\varphi^* \in L^2(0,T;H^{-1/2}(\Gamma))$, to reconstruct the unknown source δf^* such that there exists $u[\delta f^*] \in H^1(\Omega)$ satisfying $(\mathscr{P})$, where source function δf^* having compact support within a finite number (the integer $m > 0$) of small subdomains $\mathscr{D}_i^* \subset \Omega$. Moreover, we assume that they are well separated (that is: $\overline{\mathscr{D}_i^*} \cap \overline{\mathscr{D}_j^*} = \emptyset$ for all $1 \leq i,j \leq m$ with $i \neq j$) and δf^* defined as:

$$\delta f^* = b^*(t)\chi_{\mathscr{D}^*} \quad \text{with} \quad \mathscr{D}^* := \bigcup_{i=1}^{m} \mathscr{D}_i^*. \tag{3}$$

The function b^* which is non-null belonging to the space $L^2(0,T)$

Remark 1. The theoretical aspect of this problem $(\mathscr{P})$ has been the subject of various researchers' works. One can see [18], [15] for some identification and stability results.

To reconstruct a discontinuous source, the majority of the developed approaches is based on the analytical approach (RGF) [18] and the method fundamental solutions (MFS) [31].

In this work, we combine the Kohn-Vogelius formulation and the topological sensitivity analysis method and we also propose a fast and efficient reconstruction numerical algorithm. The main ideas are presented in the next section.

III. THE PROPOSED APPROACH

We present, in this section, the main steps of the proposed approach. To this end, we introduce the Kohn-Vogelius formulations in Section III *A* and the topological sensitivity analysis method in Section III *B*. The proposed approach is described in Section IV.

A. Kohn-Vogelius Formulation

The Kohn-Vogelius formulation consists in splitting the overdetermined boundary valuer problem $(\mathscr{P})$ in two auxiliary problems. The first problem which will be called as the Neumann problem is associated to the Neumann datum φ^*:

$$(\mathscr{P}_N^{\delta f})\begin{cases} \partial_t u^N[\delta f] - \Delta u^N[\delta f] + u^N[\delta f] &=& \delta f &\text{in } Q_T, \\ \nabla u^N[\delta f].\mathbf{n} &=& \varphi^* &\text{on } \Sigma_T, \\ u^N[\delta f](.,0) &=& 0 &\text{in } \Omega. \end{cases}$$

The second is called the Dirichlet problem introduced in the following equation:

$$(\mathscr{P}_D^{\delta f})\begin{cases} \partial_t u^D[\delta f] - \Delta u^D[\delta f] + u^D[\delta f] &=& \delta f &\text{in } Q_T, \\ u^D[\delta f] &=& \phi^* &\text{on } \Sigma_T, \\ u^D[\delta f](.,0) &=& 0 &\text{in } \Omega, \end{cases}$$

where

$$\delta f \in \mathscr{S}_{ad} := \left\{ \delta h \in L^2((0,T) \times \Omega), \ \delta h = \sum_{i=1}^{m} b(t)\chi_{\mathscr{D}_i} \right\}, \quad (4)$$

with the subdomain $\mathscr{D}_i$ they are well separated (that is $\overline{\mathscr{D}_i} \cap \overline{\mathscr{D}_j} = \emptyset$ for $i \neq j$). We refer to [15], [21] for the results of existence and uniqueness of $(\mathscr{P}_D^{\delta f})$ and $(\mathscr{P}_N^{\delta f})$.

Remark that if $\delta f^* = \delta f$ (*e.i* $\mathscr{D}_i$ coincides with the actual subdomains $\mathscr{D}_i^*$, $i = 1, 2, ..., m$), then $u^N[\delta f] = u^D[\delta f]$ in Ω. According to this observation, the geometric inverse source problem can be formulated as a topological optimization one. The unknown subdomain $\mathscr{D}^* = \bigcup_{i=1}^{m} \mathscr{D}_i^*$ can be characterized as the solution to the topological optimization problem:

$$(\mathscr{P}_{op})\begin{cases} \text{Find } \mathscr{D}^* = \bigcup_{i=1}^{m} \mathscr{D}_i^* \subset \Omega \text{ such that} \\ \mathscr{K}(\delta f^*, T) = \min_{\delta f \in \mathscr{S}_{ad}} \mathscr{K}(\delta f, T) \end{cases}$$

where $\mathscr{S}_{ad}$ is a set of admissible source and $\mathscr{K}$ is a source function defined as

$$\mathscr{K}(\delta f, T) = \Psi(u^N[\delta f], u^D[\delta f]),$$

with Ψ is a given cost function defined on $L^2(0,T;H^1(\Omega))$, measuring the difference between the Neumann and Dirichlet solutions (see $(\mathscr{P}_N^{\delta f})$ and $(\mathscr{P}_D^{\delta f})$).

In this work, we will use the following Kohn-Vogelius type function

$$\mathscr{K}(\delta f, T) = \int_0^T \int_\Omega |\nabla(u^N[\delta f] - u^D[\delta f])|^2 \ dxdt \ \forall \delta f \in \mathscr{S}_{ad}.$$

B. Topological Sensitivity Method

To solve the topological optimization problem $(\mathscr{P}_{op})$, we propose an alternative approach based on the topological sensitivity analysis method. The main step of this approach consists of studying the variation of a source function $\mathscr{K}$ with respect to a small geometrical subdomain of the source term inside in the domain Ω. More precisely, for a given source term f, we study the variation of the Kohn-Vogelius functional $\mathscr{K}$ with respect to a topological perturbation $\delta f_{z,\varepsilon}$ of f of the form,

$$\delta f_{z,\varepsilon} = \begin{cases} b(t) & \text{in } \omega_{z,\varepsilon} \times (0,T) \\ 0 & \text{in } \Omega \backslash \overline{\omega_{z,\varepsilon}} \times (0,T) \end{cases} \quad (5)$$

where $\omega_{z,\varepsilon}$ is an small geometrical perturbation and has the geometry form $\omega_{z,\varepsilon} = z + \varepsilon\omega \subset \Omega$ where ε; $0 < \varepsilon \ll diam(\Omega)$; is the shared diameter with $diam(\Omega)$ which is the diameter of Ω and $\omega \subset \mathbb{R}^d$ is a fixed open and bounded set containing the origin, whose boundary $\partial\omega$ is smooth. The point $a \in \Omega$ determines the location of the source support.

The topological sensitivity analysis leads to an asymptotic expansion of the functional $\mathscr{K}$ on the form,

$$\mathscr{K}(f + \delta f_{z,\varepsilon}, T) = \mathscr{K}(f,T) + \zeta(\varepsilon)\delta\mathscr{K}(z) + o(\zeta(\varepsilon)) \quad z \in \Omega,$$

where

- ζ is a positive scalar function verifying $\lim_{\varepsilon \to 0} \zeta(\varepsilon) = 0$.
- $\delta\mathscr{K}$ is the leading term of the variation $\mathscr{K}(f + \delta f_{z,\varepsilon}) - \mathscr{K}(f)$ called the topological gradient.

In the next section, we will define the topological gradient for the Kohn-Vogelius functional $\mathscr{K}$.

IV. TOPOLOGICAL SENSITIVITY FOR KOHN-VOGELIUS FUNCTIONAL

In this section, we establish the asymptotic expansion of the Kohn-Vogelius functional $\mathscr{K}$. It consists in studying the variation of $\mathscr{K}$ with respect to the presence of a tolpological perturbation $\delta f_{z,\varepsilon}$ of the source term f. The Kohn-Vogelius function $\mathscr{K}$ is defined by

$$\mathscr{K}(\delta f_{z,\varepsilon}, T) = \int_0^T \int_\Omega |\nabla(u^N[\delta f_{z,\varepsilon}] - u^D[\delta f_{z,\varepsilon}])|^2 dxdt.$$

where $u^N[\delta f_{z,\varepsilon}]$ is solution to the Neumann boundary value problem:

$$(\mathscr{P}_N^{\delta f_\varepsilon})\begin{cases} \partial_t u^N[\delta f_{z,\varepsilon}] - \Delta u^N[\delta f_{z,\varepsilon}] + u^N[\delta f_{z,\varepsilon}] &=& \delta f_{z,\varepsilon} &\text{in } Q_T, \\ \nabla u^N[\delta f_{z,\varepsilon}].\mathbf{n} &=& \varphi &\text{on } \Sigma_T, \\ u^N[\delta f_{z,\varepsilon}](.,0) &=& 0 &\text{in } \Omega, \end{cases}$$

and $u^D[\delta f_{z,\varepsilon}]$ is the solution of the Dirichlet problem

$$(\mathscr{P}_D^{\delta f_\varepsilon})\begin{cases} \partial_t u^D[\delta f_{z,\varepsilon}] - \Delta u^D[\delta f_{z,\varepsilon}] + u^D[\delta f_{z,\varepsilon}] &=& \delta f_{z,\varepsilon} &\text{in } Q_T, \\ u^D[\delta f_{z,\varepsilon}] &=& \phi &\text{on } \Sigma_T, \\ u^D[\delta f_{z,\varepsilon}](.,0) &=& 0 &\text{in } \Omega, \end{cases}$$

We define two adjoint states $v^N[f]$ and $v^D[f]$ respectively solutions to:

$$(\mathscr{P}_{v^N[f]})\begin{cases} \text{find } v^N[f] \in L^2(0,T;H_0^1(\Omega)), \\ \mathscr{A}(w, v^N[f]) = \mathscr{L}(w) \ \forall w \in L^2(0,T;H_0^1(\Omega)), \\ v^N[f](.,T) = 0 \text{ in } \Omega, \end{cases}$$

$$(\mathscr{P}_{v^D[f]})\begin{cases} \text{find } v^D[f] \in L^2(0,T;H^1(\Omega)), \\ \mathscr{A}(w, v^D[f]) = \mathscr{L}(-w) \ \forall w \in L^2(0,T;H^1(\Omega)), \\ v^D[f](.,T) = 0 \text{ in } \Omega, \end{cases}$$

where the bilinear form $\mathscr{A}$, and the linear form $\mathscr{L}$ defined by

$$
\begin{aligned}
\mathscr{A}(v,u) &= \int_0^T \int_\Omega \partial_t u \ v \mathrm{d}x \mathrm{d}t + \int_0^T \int_\Omega \nabla u . \nabla v \mathrm{d}x \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega u \ v \mathrm{d}x \mathrm{d}t, \\
\mathscr{L}(w) &= 2 \int_0^T \int_\Omega \nabla (u^N[f] - u^D[f]) \nabla w \mathrm{d}x \mathrm{d}t.
\end{aligned}
$$

The following theorem gives us the expression of the topological gradient of the Kohn-Vogelius functional $\mathscr{K}$:

Theorem 1. *Let $\delta f_{z,\varepsilon} := b(t)\chi_{\omega_{z,\varepsilon}}$ where $b \in L^2(0,T)$ and $\omega_{z,\varepsilon}$ a small support in Ω of the form $\omega_{z,\varepsilon} = z + \varepsilon \omega \subset \Omega$. Then, the Kohn-Vogelius source functional $\mathscr{K}$ admits the following asymptotic expansion*

$$
\mathscr{K}(f + \delta f_{z,\varepsilon}, T) = \mathscr{K}(f,T) + \rho(\varepsilon)\delta\mathscr{K}(z) + o(\rho(\varepsilon)),
$$

where $\rho = meas(\omega_{z,\varepsilon})$ and the function $\delta\mathscr{K}$ is called the source topological gradient defined by

$$
\delta\mathscr{K}(z) = \int_0^T b(t)(v^D[f] + v^N[f])(z,t) \ dt.
$$

To prove this theorem see [25].

V. Numerical Results

In this section, we present several numerical examples that demonstrate the efficiency of the proposed method in two-dimensional case. The aim is the detection and location of the unknown support source term using a level curve of the topological gradient. We propose a fast and efficient identification procedure. Our numerical algorithm is based on the asymptotic expansion established in the previous theorem 1 to detect and locate the support source term with the help of boundary measurements.

A. Identification Procedure

Let us now describe a fast and very simple one-iteration identification algorithm. It is based on the following steps:
One-shot algorithm:

- Solve the two direct problems $(\mathscr{P}_N^{\delta f = 0})$ and $(\mathscr{P}_D^{\delta f = 0})$,

- Solve the two adjoint problems $(\mathscr{P}_{v^N[0]})$ and $(\mathscr{P}_{v^D[0]})$,

- Compute the topological gradient $\delta\mathscr{K}(x)$, $x \in \Omega$,

- Determine the support source term $\mathscr{D}^*$;
  $\mathscr{D}^* = \{x \in \Omega; \text{ such that } \delta\mathscr{K}(x) \leq c < 0\}$,
  where c is a constant chosen in such a way that the cost function $\mathscr{J}_K$ decreases as most as possible. The topological gradient $\delta\mathscr{K}$ defined as

$$
\delta\mathscr{K}(x) = \int_0^T b(t)(v^D[0] + v^N[0])(x,t) \ dt \ \forall x \in \Omega. \quad (6)
$$

B. Approximation Method

In this subsection, we present the method used for the numerical resolution of the non stationary parabolic equation.

a) Time discretization.: We consider a subdivision of the interval $[0,T]$ in small interval $I_n = [t_n, t_{n+1}]$ of length $\Delta t = t_{n+1} - t_n$. We assume that the time step Δt is defined by $\Delta t = \frac{T}{NT-1}$ where $NT \in \mathbb{N}^*$ is given. For all $n \in \{0,1,...,NT\}$, we denote u^n the approximated velocity at the time $t_n = n\Delta t$.

The time discretization is based on the Euler implicit schema. Then, we have the following approximation:

$$
\left[\frac{\partial u}{\partial t}\right]_{t=t_n} \approx \frac{u^{n+1} - u^n}{\Delta t}, n \in \{1,2,...,NT\}.
$$

The time discretization of the parabolic equation reads,

$$
\begin{cases}
u^n - \Delta t \Delta u^n + \Delta t u^n &= \Delta t f^{*n} + u^{n-1} & \text{in } \Omega \\
\nabla u^n . n &= \varphi^{*n} & \text{on } \Gamma \\
u^n &= \phi^{*n} & \text{on } \Gamma \\
u^n(.,0) &= 0 & \text{in } \Omega,
\end{cases} \quad (7)
$$

where φ^{*n} and ϕ^{*n} are respectively the approximations of φ^* and ϕ^* on the time t^n. Then, at each time step, we have to solve a steady state generalized parabolic equation.

b) Spatial discretization.: For the spacial discretization, we use a $\mathbb{P}2$ finite element discretization. Let $\mathscr{T}_h$ a triangulation of the domain $\overline{\Omega}$, where h refers to the size of the mesh defined by

$$
h = \max_{T \in \mathscr{T}_h} h_T,
$$

where h_T is the diameter of the triangle T.
Let $\mathscr{X}_h$ be the space discretization

$$
\mathscr{X}_h = \{v_h \in C^0(\Omega)^2; v_{h_{|T}} \in \mathbb{P}_2^2, \forall T \in \mathscr{T}_h\},
$$

Concerning the mesh, we impose a fixed number of discretization points for the exterior boundary Γ, that is 40 points for $[0,1]$ in order to have a uniform mesh.

The numerical algorithm is implemented using *FreeFem++*.

C. Reconstruction Results

In this subsection, we present some numerical results showing the efficiency and accuracy of our proposed one-iteration algorithm. In Fig. 2, we test our algorithm on circular shape. In Fig. 3, we consider the case of an elliptical shape. As one can observe, the sub-domain (support source) to be detected is located at zone where the topological gradient is negative and it is approximated by a level set curve of the topological gradient $\delta\mathscr{K}$. The result is quite efficient. In Fig. 4, we aim to detect shape with corners. More precisely, we want to detect a small square. As one can see, we detect the location of the unknown sub-domain and we give a good approximation of its shape. The obtained result can serve as a good initial guess for an iterative optimization process based on the shape derivative.

Finally, we want to detect three circles $\mathscr{D}_1^*$, $\mathscr{D}_2^*$ and $\mathscr{D}_3^*$ centered respectively in $(0.3,0.7)$, $(0.35,0.35)$ and $(0.65,0.65)$ with shared radius $r = 0.05$. The detection is quite efficient (see Fig. 5).
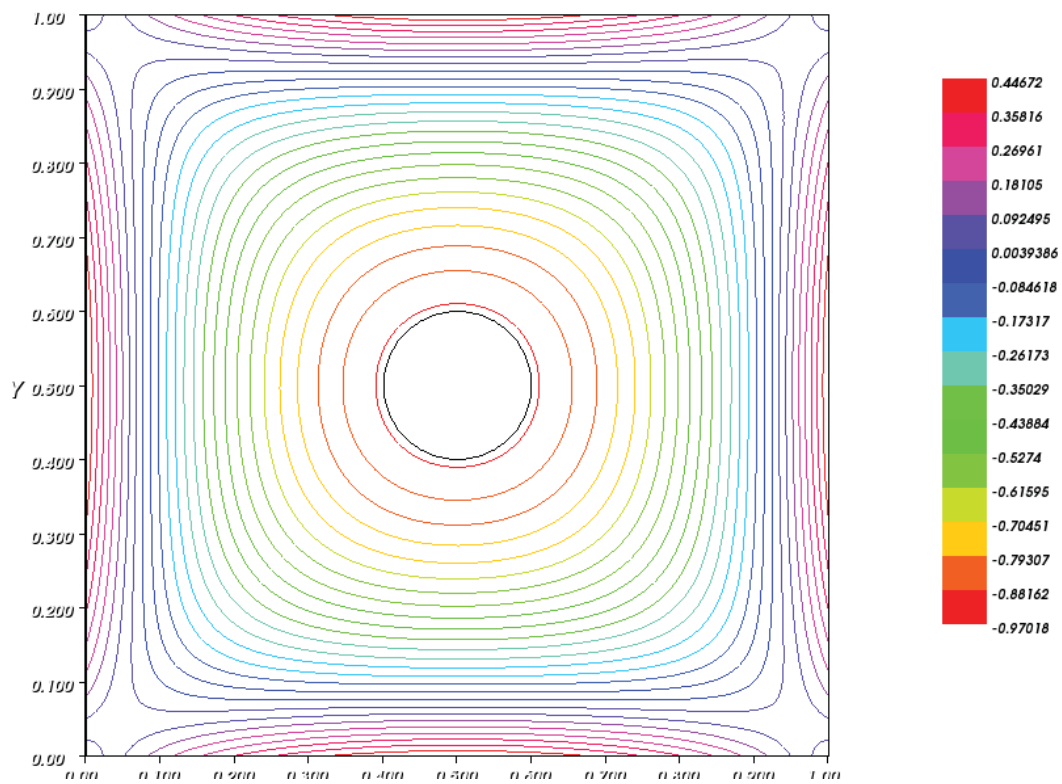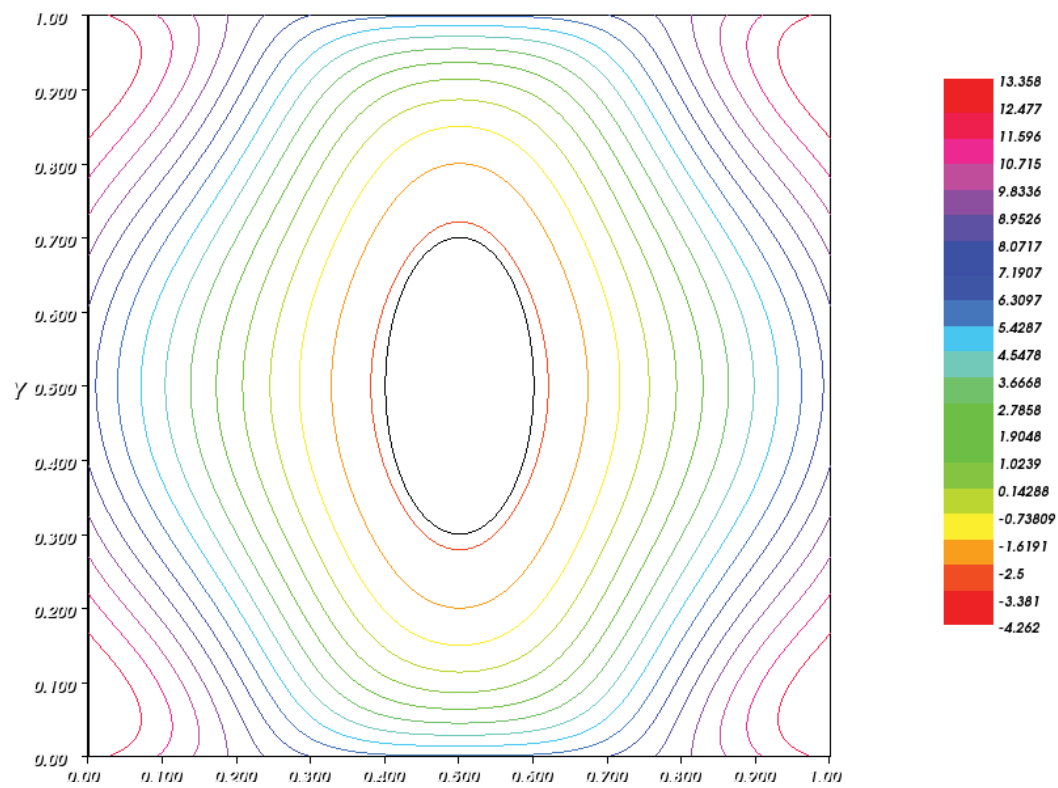
Fig. 2 Reconstruction of an circle shape

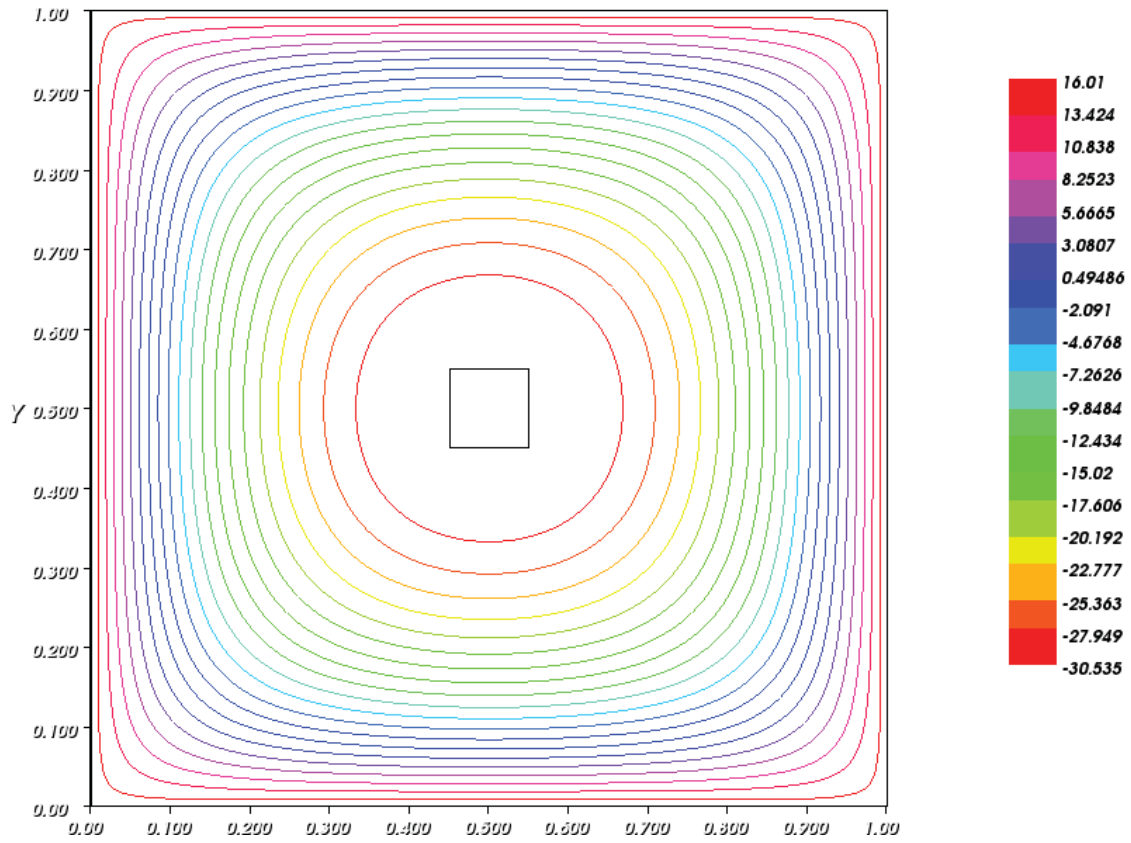Fig. 3 Reconstruction of an ellipse shape

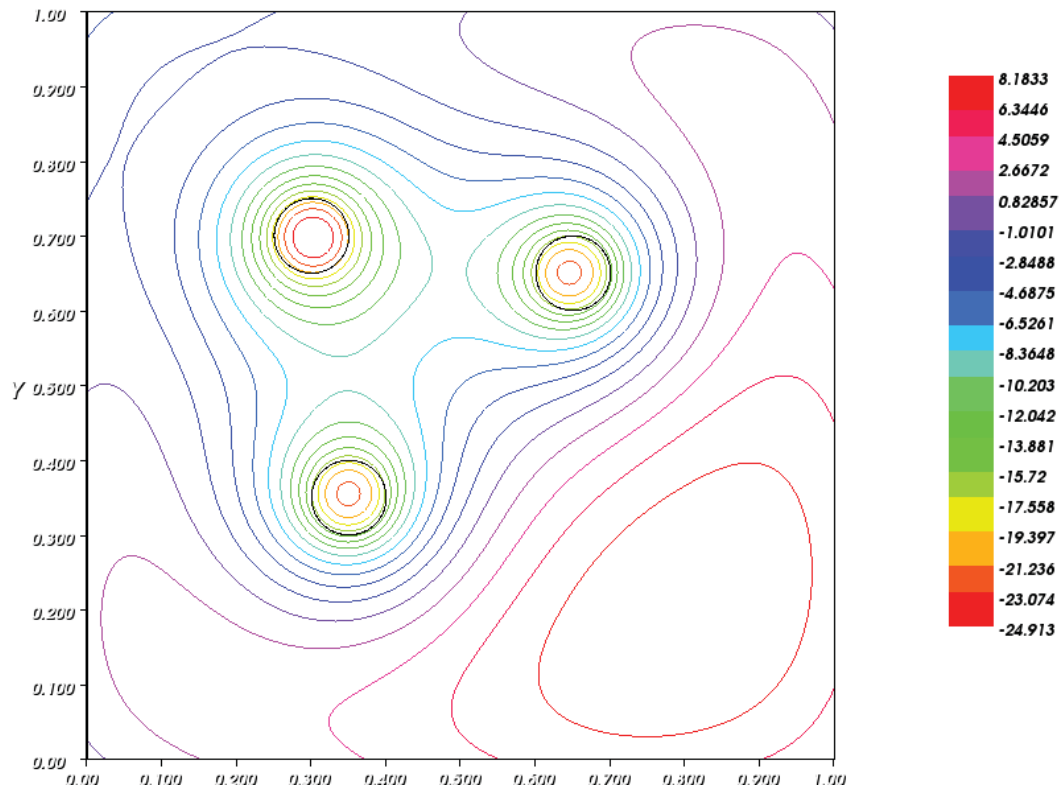Fig. 4 Reconstruction of a non trivial shape

Fig. 5 Reconstruction of multi-sources

VI. Conclusion

We have used a one-shot topological optimization algorithm. Our aim is to reconstruct the form of an unknown source support by detecting its location, size and shape from additional boundary conditions.

The presented method has two main features. The first one devoted to the advantages of the Kohn-Vogelius formulation and the topological sensitivity method for the heat equation with Dirichlet and Neumann boundary data. The obtained result is general and valid for large class of shape functions. The second interesting feature of the approach is that it leads to a simple and fast numerical reconstruction algorithm. It has a goal of identification for the source term with unknown support using a level curve of the topological gradient. The efficiency of the proposed approach is performed by some numerical results in the bidirectional case.

References

[1] A. B. Abda, M. Hassine, M. Jaoua, and M. Masmoudi, Topological sensitivity analysis for the location of small cavities in stokes flow, *SIAM Journal on Control and Optimization.* 48(2009), 2871-2900.
[2] V. Akçelik, G. Biros, O. Ghattas, K. R. Long, and B. van BloemenWaanders, A variational finite element method for source inversion for convectivediffusive transport, *Finite Elements in Analysis and Design.* 39(2003), 683-705.
[3] Y. Alber and I. Ryazantseva, *Nonlinear ill-posed problems of monotone type*. Springer, 2006.
[4] G. Alessandrini and V. Isakov, *Analicity and uniqueness for the inverse conductivity problem*. 1996.
[5] M. A. Anastasio, J. Zhang, D. Modgil, and P. J. La Rivière, Application of inverse source concepts to photoacoustic tomography, *Inverse Problems.* 23(2007), S21.
[6] O. Andreikiv, O. Serhienko, et al, Acoustic-emission criteria for rapid analysis of internal defects in composite materials, *Materials Science.* 37(2001), 106-117.
[7] S. R. Arridge, Optical tomography in medical imaging, *Inverse problems.* 15(1999), R41.
[8] H. Bellout, A. Friedman, and V. Isakov, Stability for an inverse problem in potential theory, *Transactions of the American Mathematical Society.* 332(1992), 271-296.
[9] J. Benoit, C. Chauvière, and P. Bonnet, Source identification in time domain electromagnetics, *Journal of Computational Physics.* 231(2012), 3446-3456.
[10] A. Canelas, A. Laurain, and A. A. Novotny, A new reconstruction method for the inverse potential problem, *Journal of Computational Physics.* 268(2012), 417-431.
[11] A. Canelas, A. Laurain, and A. A. Novotny, A new reconstruction method for the inverse source problem from partial boundary measurements, *Inverse Problems.* 31(2015), 075009.
[12] X. Cheng, R. Gong, and W. Han, A new general mathematical framework for bioluminescence tomography, *Computer Methods in Applied Mechanics and Engineering.* 197(2008), 524-535.
[13] M. Choulli, Local stability estimate for an inverse conductivity problem, *Inverse problems.* 19(2003), 895.
[14] M. Choulli, On the determination of an inhomogeneity in an elliptic equation, *Applicable Analysis.* 85(2006), 693-699.
[15] M. Choulli, *Une introduction aux probl'emes inverses elliptiques et paraboliques,* Berlin, Springer, 2009.
[16] M. Choulli and M. Yamamoto, Conditional stability in determining a heat source, *Journal of inverse and ill-posed problems.* 12(2004), 233244.
[17] A. Doicu, T. Trautmann, and F. Schreier, *Numerical regularization for atmospheric inverse problems*, Springer Science & Business Media, 2010.
[18] A. El Badia and T. Ha-Duong, On an inverse source problem for the heat equation. application to a pollution detection problem, *Journal of inverse and ill-posed problems.* 10(2002), 585-599.
[19] J. Ferchichi, M. Hassine, and H. Khenous, Detection of point-forces location using topological algorithm in stokes flows, *Applied Mathematics and Computation.* 219(2013), 7056-7074.
[20] S. Garreau, P. Guillaume, and M. Masmoudi, The topological asymptotic for pde systems: the elasticity case, *SIAM journal on control and optimization.* 39(2001), 1756-1778.
[21] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, springer, 2015.
[22] W. Guo, K. Jia, D. Han, Q. Zhang, X. Liu, J. Feng, C. Qin, X. Ma, and J. Tian, Efficient sparse reconstruction algorithm for bioluminescence tomography based on duality and variable splitting, *Applied optics.* 51(2012), 5676-5685.
[23] W. Han, W. Cong, and G. Wang, Mathematical theory and numerical analysis of bioluminescence tomography, *Inverse Problems.* 22(2006), 1659.
[24] M. Hassine, S. Jan, and M. Masmoudi, From differential calculus to 01 topological optimization, *SIAM Journal on Control and Optimization.* 45(2007), 1965-1987.
[25] M. Hassine, M. Hrizi, One-iteration reconstruction algorithm for geometric inverse source problem, *Submitted.*
[26] A. T. Hayes, A. Martinoli, and R. M. Goodman, Swarm robotic odor localization, *Off-line optimization and validation with real robots. Robotica.* 21(2003), 427-441.
[27] F. Hettlich and W. Rundell, The determination of a discontinuity in a conductivity from a single boundary measurement, *Inverse Problems.* 14(1998), 67.
[28] F. Hettlich and W. Rundell, Identification of a discontinuous source in the heat equation, *Inverse Problems.* 17(2001), 1465.
[29] V. Isakov. *Inverse problems for partial differential equations*, volume 127. Springer Science & Business Media, 2006.
[30] S. Larnier and M. Masmoudi, The extended adjoint method, *ESAIM: Mathematical Modelling and Numerical Analysis.* 47(2013), 83-108.
[31] N. F. Martins, An iterative shape reconstruction of source functions in a potential problem using the mfs, *Inverse Problems in Science and Engineering.* 20(2012), 1175-1193.
[32] P. Stefanov and G. Uhlmann, Thermoacoustic tomography with variable sound speed, *Inverse Problems.* 25(2009), 075011.
[33] R. Unnthorsson, T. P. Runarsson, and M. T. Jonsson. Acoustic emission based fatigue failure criterion for cfrp, *International Journal of Fatigue.* 30(2008), 11-20.
[34] G. Wang, Y. Li, and M. Jiang, Uniqueness theorems in bioluminescence tomography, *Medical physics.* 31(2004), 2289-2299.