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Discovering Liouville-Type Problems for p-Energy Minimizing Maps in Closed Half-Ellipsoids by Calculus Variation Method

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Abstract**—**The goal of this project is to investigate constant properties (called the Liouville-type Problem) for a p-stable map as a local or global minimum of a p-energy functional where the domain is a Euclidean space and the target space is a closed half-ellipsoid. The First and Second Variation Formulas for a p-energy functional has been applied in the Calculus Variation Method as computation techniques. Stokes' Theorem, Cauchy-Schwarz Inequality, Hardy-Sobolev type Inequalities, and the Bochner Formula as estimation techniques have been used to estimate the lower bound and the upper bound of the derived p-Harmonic Stability Inequality. One challenging point in this project is to construct a family of variation maps such that the images of variation maps must be guaranteed in a closed half-ellipsoid. The other challenging point is to find a contradiction between the lower bound and the upper bound in an analysis of p-Harmonic Stability Inequality when a p-energy minimizing map is not constant. Therefore, the possibility of a non-constant p-energy minimizing map has been ruled out and the constant property for a p-energy minimizing map has been obtained. Our research finding is to explore the constant property for a p-stable map from a Euclidean space into a closed half-ellipsoid in a certain range of p. The certain range of p is determined by the dimension values of a Euclidean space (the domain) and an ellipsoid (the target space). The certain range of p is also bounded by the curvature values on an ellipsoid (that is, the ratio of the longest axis to the shortest axis). Regarding Liouville-type results for a p-stable map, our research finding on an ellipsoid is a generalization of mathematicians' results on a sphere. Our result is also an extension of mathematicians' Liouville-type results from a special ellipsoid with only one parameter to any ellipsoid with $(n+1)$ parameters in the general setting.

Keywords**—**Bochner Formula, Stokes' Theorem, Cauchy-Schwarz Inequality, first and second variation formulas, Hardy-Sobolev type inequalities, Liouville-type problem, p-harmonic map.

I. INTRODUCTION

THE Calculus Variation Method has been the most important and useful tool to deal with optimization of quantities for functionals. The optimization of quantities is to find the maxima or minima (which are collectively known as extrema) for functionals. The most classical way to obtain the extrema is to solve the Euler-Lagrange equation, which is derived from the functional derivative equal to zero. In this project, we apply the Calculus Variation Method for the p-energy functionals. The p-energy functional $E_p(u)$ is defined as the definite integral for the differential of a map u from the domain to the target space (cf. Definition 1). We are interested in an optimization of quantity as minima of the p-energy functional. Here, we study the extrema of both local and global minima of a p-energy functional defined as p-stable maps (cf. Definition 3). Solutions to the Euler-Lagrange equation associated with the p-energy functional derivative defined as p-harmonic maps (cf. Definition 2) are topics in this project.

The Calculus Variation Method has been vastly used to solve many problems such as Iso-Perimetric Problems, Geodesics on Surfaces, Minimal Surfaces, Plateau's Problem, and others. Moreover, this variation method has been used to establish new theories or to create new principles. In 1925, Morse [6] developed the Calculus of Variation to Equilibrium Problems in The Morse Theory. In 1956, Bellman [1] created his dynamic programming principle as an alternative way of Calculus Variation Method. During the 1960s and early 1970s, Pontryagin [8], Rockafellar [9], and Clarke [3] generalized the Calculus Variation Method to develop new mathematical tools in Optimal Control Theory. Many famous mathematicians have made significant contributions to applications of the Calculus Variation Method.

In this research project, we focus on applying the Calculus Variation Method for Liouville-type Problems (i.e. the constant properties) in p-Harmonic Theory. In particular, Liouville-type Problems in p-Harmonic Theory have been studied extensively throughout mathematical history. In 1976, Schoen and Yau [10] studied The Liouville Theorem for p-harmonic maps u for $p = 2$ in the L^q space where the domain of u was a Riemannian manifold M with non-negative Ricci curvature value (i.e. $Ricci^M \geq 0$). In 1995, Cheung and Leung [2] obtained a Liouville-type result for p-harmonic maps u when $p \geq 2$ in the L^q space where the target space for u was a Cartan-Hadamand Riemannian manifold. In 1999, Kawai [5] proved a Liouville Theorem for p-harmonic maps when $p \geq 2$ in the L^q space where the domain of u was a p-parabolic Riemannian manifold and the target space for u was a non-positively curved Riemannian manifold. In 2008, S.Pigola, M.Rigoli, and A.G.Setti [7] studied the Liouville Theorem for p-harmonic maps u for $p \geq 2$ where the domain of u was a Riemannian manifold with the support of Poincaré-Sobolev Inequality and the target space for u was a non-positively curved Riemannian manifold.

In particular, we apply the Calculus Variation Method to investigate Liouville-type problems for a p-energy minimizing map (also called a p-stable map) from a Euclidean space to

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a closed half-ellipsoid. In order to use the variation method, we construct a family of variational maps with images in the closed half-ellipsoid. For this family of variational maps, we apply the variational formulas to derive p-Harmonic Stability Inequality [12]. To estimate the lower bound and the upper bound of the derived p-Harmonic Stability Inequality, we use the Bochner Formula [11], Stokes' Theorem, Cauchy-Schwarz Inequality, and Hardy-Sobolev type Inequalities [4]. Our research finding is to obtain a Liouville-type result by exploring the constant properties for a p-energy minimizing map in a certain range of p. The certain range of p is determined by the dimension values of a Euclidean Space and an ellipsoid. The value of p is also determined by the curvature values on an ellipsoid, which is estimated by the longest axis and the shortest axis [14]. The research findings and calculation techniques in this project could lead to the future research study of the Calculus Variation Method for Liouville-type Problem and its related problems such as Regularity Problems [13] on convex compact hyper-surfaces in the general setting.

II. Preliminary and Main Result

In this section, we recall the definitions of a p-energy functional, a p-harmonic map, a p-stable map, and a p-minimizing map between two Riemannian manifolds. In order to apply the Calculus Variation Method to solve Liouville-type Problems for a p-harmonic map, we need to recall the First Variational Formula and Second Variational Formula for a p-energy functional. The most useful estimation techniques in computation to obtain the Liouville-type result are the Bochner Formula and Hardy-Sobolev type Inequalities. We recall the Bochner Formula and Hardy-Sobolev type Inequalities at the end of this section.

Let M^m be a compact Riemannian manifold with possibly nonempty boundary, and let N^n be isometrically immersed in R^q. $L_1^p(M,N)$ denotes the set of maps $u : M \to R^q$ whose component functions have first order weak derivatives in L^p and $u(x) \in N$ a.e. on M.

Definition 1. *The p-energy ($p > 1$) functional for $u \in L_1^p(M,N)$ is given by*

$$E_p(u) = \frac{1}{p}\int_M |du|^p dv$$

where du denotes the differential of u, and dv is the volume element of M.

Definition 2. *A map $u \in L_1^p(M,N)$ is said to be weakly p-harmonic ($p > 1$) if it is a weak solution to the following Euler-Lagrange equation for E_p on $L_1^p(M,N)$:*

$$div(|\nabla u|^{p-2}\nabla u) = 0$$

Definition 3. *A p-harmonic map u is called p-stable (resp. p-minimizing) if u is a local (resp. global) minimum of p-energy functional E_p within a homotopy class of $L_1^p(M,N)$ having the same trace on ∂M.*

Definition 4. *A C^2 map $u : M \to N$ is said to be p-superharmonic (resp. p-subharmonic) if*

$$div(|\nabla u|^{p-2}\nabla u) \leq (resp. \geq) 0$$

Consider a smooth map $u : M \to N$ where M is compact. Denote the pull-back tangent bundle of N by $u^{-1}TN$, and the pull-back connection by ∇^u. Let v be smooth sections in $u^{-1}TN$, vanishing identically on a neighborhood of ∂M.

Choose a one-parameter family of smooth maps u_t such that $u_0 = u$ and $\frac{du_t}{dt}|_{t=0} = v$. We shall also denote by $F : M \times R \to N$ the smooth map defined by $F(x,t) = u_t(x)$ for one-parameter variational maps.

1) First Variation Formula for a One-Parameter Family of p-Harmonic Maps:

$$\begin{aligned} &\frac{d}{dt}E_p(u_t)|_{t=0} \\ &= \int_M |du|^{p-2}\sum_i \langle \nabla^u_{e_i} v, du(e_i)\rangle dv \end{aligned} \tag{1}$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on M.

2) Second Variation Formula for a One-Parameter Family of p-Harmonic Maps:

$$\begin{aligned} &\frac{d^2}{dt^2}E_p(u_t)|_{t=0} \\ &= \int_M (p-2)|du|^{p-4}\big(\sum_i \langle \nabla^u_{e_i} v, du(e_i)\rangle\big)^2 \\ &\quad + |du|^{p-2}[\sum_i \langle R^N(v, du(e_i))v, du(e_i)\rangle \\ &\quad - \sum_i \langle \nabla^u_{e_i}\nabla^u_{e_i} v - \nabla^u_{\nabla_{e_i}e_i} v, v\rangle + div\theta^\nu]dv \end{aligned} \tag{2}$$

where θ^ν is a differential 1-form given by $\theta^\nu(x) = \langle \nabla^u_x v, v\rangle$, for a local vector field on M.

Lemma 1. *(Bochner Formula) Let $u : M \to N$ be a p-harmonic map ($p > 1$) when M is compact, then:*

$$\begin{aligned} &\int_M -K|du|^{p-2+2\beta}dv \\ &\leq -(p-1)(2\beta-1)\int_M |du|^{p-4+2\beta}|\nabla|du||^2 dv \\ &\quad + \frac{m-1}{m}\int_M S|du|^{p+2\beta}dv \end{aligned} \tag{3}$$

where $-K$ = lower bound of the Ricci curvature on M (i.e. $Ricci^M$) and S = upper bound of the sectional curvature on N (i.e. $Riem^N$) and $m = dim(M)$ for every positive constant $\beta \geq 1$.

Proof: cf. Lemma 1.3 [11] ∎

Lemma 2. *(Hardy-Sobolev type Inequalities) For all $u \in C_0^\infty(R^N)$, it holds:*

$$\int_{R^N} |x|^{-2a}|\nabla u|^2 dx \geq S(a,b)\big\{\int_{R^N} |x|^{-bp}|u|^p dx\big\}^{\frac{2}{p}} \tag{4}$$

where

$$\begin{cases} a < -\frac{1}{2}, a + \frac{1}{2} < b \leq a+1, p = \frac{2}{-1+2(b-a)} & \text{when } N=1, \\ a < 0, a < b \leq a+1, p = \frac{2}{b-a} & \text{when } N=2, \\ a < \frac{N-2}{2}, a \leq b \leq a+1, p = \frac{2N}{N-2+2(b-a)} & \text{when } N \geq 3. \end{cases}$$

and $S(a,b)$ is the best constant. In particular, $S(a,a+1) = (\frac{N-2-2a}{2})^2$.

Proof: cf. [4] ■

Here and throughout this paper, we denote an ellipsoid

$$E^n = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1,$$
$$a_i > 0, \forall 1 \leq i \leq n+1\}$$

We define a closed upper-half ellipsoid

$$\bar{E}_+^n = \{(x_1, \cdots, x_{n+1}) \in E^n : x_{n+1} \geq 0\}$$

We also give the notations of $\max(a_i) = \max_{1 \leq i \leq n+1}\{a_i\}$, $\min(a_i) = \min_{1 \leq i \leq n+1}\{a_i\}$, and $\mu = \frac{\max(a_i)}{\min(a_i)}$. In addition, we denote a sphere

$$S^n(r) = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{r^2} + \cdots + \frac{x_{n+1}^2}{r^2} = 1$$
$$r > 0, \forall 1 \leq i \leq n+1\}$$

We define a unit sphere

$$S^n = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

The following is the main theorem and our Liouville-type result:

Theorem 1. *(Liouville-type Theorem for p-Minimizing Maps from a Euclidean Space to a Closed Half-Ellipsoid) Every p-stable or p-minimizing tangent map ($p > 1$) from R^l ($l > 2$) into $\bar{E}_+^n$ is constant for*

$$\max\{1, l - \frac{2}{\mu^2}\sqrt{\frac{l-1}{4n+2l-2}}\} < p < l$$

III. Methodology and Proof of Main Result

In this section, we focus on a p-harmonic map ($p > 1$) $u : M \to N$ from the domain of a Euclidean space $M = R^l$ ($l > 2$) to the target space of a closed half-ellipsoid $N = \bar{E}_+^n$ with $(n+1)$ parameters. We provide the detailed proof in the main theorem to verify our Liouville-type result.

Lemma 3. *(Application of Bochner Formula for a p-Harmonic Map from a Unit Sphere to a Closed Half-Ellipsoid) For a p-harmonic map ($p > 1$) $u : S^{l-1} \to \bar{E}_+^n$, we have:*

$$\begin{aligned} &\int_{S^{l-1}} (l-2)|du|^p d\omega \\ \leq &-(p-1)\int_{S^{l-1}} |du|^{p-2}|\nabla|du||^2 d\omega \\ &+ \frac{l-2}{l-1}\int_{S^{l-1}} \frac{(\max(a_i))^2}{(\min(a_i))^4}|du|^{p+2} d\omega \end{aligned} \tag{5}$$

where $d\omega$ is the volume element of a unit sphere on S^{l-1}.

Proof: Via Lemma 1, we apply the Bochner Formula for a p-harmonic map from a unit Sphere to an Ellipsoid by setting $\beta = 1$. Here, we know that $-K = l-2$ is the lower bound of the Ricci curvature on a unit sphere S^{l-1}, $m = dim(S^{l-1}) = l-1$, and $S = \frac{(\max(a_i))^2}{(\min(a_i))^4}$ is the upper bound of the sectional curvature $Riem$ on an ellipsoid $\bar{E}_+^n$ (cf. Lemma 2.1 and Lemma 2.2 [14]). ■

Lemma 4. *(Application of Hardy-Sobolev type Inequalities for a Radial Function) For a radial function $f = f(r) \in L_1^2(R)$ where $r = |x|$, we have:*

$$\inf_{f \in L_1^2(R)\setminus\{0\}} \frac{\int_0^\infty (f')^2 r^{l-p+1} dr}{\int_0^\infty f^2 r^{l-p-1} dr} = \frac{(l-p)^2}{4} \tag{6}$$

for $l > p$.

Proof: Via Lemma 2, we apply Hardy-Sobolev type Inequalities by setting $N = 1$, $a = -\frac{l-p+1}{2}$, and $b = a+1 = -\frac{l-p-1}{2}$ for a radial function $u = f(r)$ where $r = |x|$. Therefore, we obtain $p = \frac{2}{-1+2(b-a)} = 2$ and

$$\begin{aligned} &S(a,b) = S(a,a+1) \\ =&(\frac{N-2-2a}{2})^2 = \frac{(l-p)^2}{4} \\ =&\inf_{f \in L_1^2(R)\setminus\{0\}} \frac{\int_0^\infty (f')^2 r^{l-p+1} dr}{\int_0^\infty f^2 r^{l-p-1} dr} \end{aligned} \tag{7}$$

for $l > p$ due to the fact $a = -\frac{l-p+1}{2} < -\frac{1}{2}$. We are able to consider a radial function f in the space of $L_1^2(R)$ since the $L_1^2(R)$ space is the completion of the $C_0^\infty(R)$ space. ■

Here is the proof of our main theorem.

Proof of Theorem 1: By the First Variation Formula for a p-energy functional, any p-harmonic map $u \in L_1^p(M, N \hookrightarrow R^q)$ is a weak solution of

$$div(|du|^{p-2} \bigtriangledown u) + A(\bigtriangledown u, \bigtriangledown u)|du|^{p-2} = 0 \tag{8}$$

where A is the second fundamental form of N into R^q. From the elementary calculus, E^n is the level set of

$$\frac{u_1^2}{a_1^2} + \cdots + \frac{u_{n+1}^2}{a_{n+1}^2} - 1$$

whose gradient vector field $(\frac{u_1}{a_1^2}, \cdots, \frac{u_{n+1}}{a_{n+1}^2})$ is normal to the convex hypersurface E^n in R^{n+1}. It follows that any p-harmonic map $u = (u_1, \cdots, u_{n+1}) : S^{l-1} \to \bar{E}_+^n \subset R^{n+1}$ of class C^1 is a weak solution to

$$div(|du|^{p-2} \bigtriangledown u_i) + |du|^p u_i \varrho_i = 0 \tag{9}$$

for some function $\varrho_i > 0$ and $1 \leq i \leq (n+1)$. Taking $i = n+1$ in this equation, integrating it over S^{l-1} and using Stokes' theorem, we have

$$\int_{S^{l-1}} |du|^p u_{n+1} \varrho_{n+1} = 0 \tag{10}$$

Define $\Sigma = \{x \in S^{l-1}; |du|(x) > 0\}$.

- If $\Sigma = \emptyset$, we have $|du|(x) \equiv 0$ on S^{l-1}. In other words, we know $u \equiv$constant because $du \equiv 0$. Therefore, we already solve the Liouville-type problem and obtain the constant property for u.

- If $\Sigma \neq \emptyset$, we know that Σ is open and nonempty. Hence, via (10), we have $u_{n+1} \equiv 0$ on Σ. Therefore, we can assume $u(S^{l-1}) \subset \partial(\bar{E}^n_+)$ where $\partial(\bar{E}^n_+) = \{(x_1, \cdots, x_{n+1}) \in \bar{E}^n_+, x_{n+1} = 0\}$.

Now we only consider the second case. Here, we deform its p-minimizing homogeneous extension of u ($\bar{u}(x) = u(\frac{x}{\|x\|})$), that is, $\bar{u}(x) = (\bar{u}_1, \cdots, \bar{u}_n, 0) : R^l\backslash\{0\} \to \partial(\bar{E}^n_+)$. Furthermore, along $(0, 0, \cdots, 0, \varphi)$ for a positive function $\varphi \in C^1(R^l\backslash\{0\})$, we construct a family of variational maps $\bar{u}_t(x) : R^l\backslash\{0\} \to \bar{E}^n_+$ given by

$$\bar{u}_t(x) = \frac{1}{\sqrt{1+t^2\varphi^2}}(\bar{u}_1(x), \cdots, \bar{u}_n(x), ta_{n+1}\varphi) \tag{11}$$

for $t \geq 0$. Here we claim the images of this family of variational maps must be in the upper closed half-ellipsoid (that is, $\bar{u}_t(x) \in \bar{E}^n_+$ for $\forall t > 0$ and $\forall x \in R^l\backslash\{0\}$). The reason is the following:

1) For $t = 0$, we know that $\bar{u}_0(x) = \bar{u}(x)$ by (11). So, we have $\bar{u}_0(x) = \bar{u}(x) \in \partial(\bar{E}^n_+)$ since $\bar{u}(x) = u(\frac{x}{\|x\|})$ and $u(S^{l-1}) \subset \partial(\bar{E}^n_+)$.
2) For $\forall t > 0$, via (11), we verify $\bar{u}_t(x) \in \bar{E}^n_+$ due to two facts below:

$$\frac{(\bar{u}_1)^2}{a_1^2} + \cdots + \frac{(\bar{u}_n)^2}{a_n^2} = 1$$

$$\frac{(\frac{\bar{u}_1(x)}{\sqrt{1+t^2\varphi^2}})^2}{a_1^2} + \cdots + \frac{(\frac{\bar{u}_n(x)}{\sqrt{1+t^2\varphi^2}})^2}{a_n^2} + \frac{(\frac{ta_{n+1}\varphi}{\sqrt{1+t^2\varphi^2}})^2}{a_{n+1}^2} = 1$$

Let $\dot{\bar{u}}$ and $\ddot{\bar{u}}$ denote the first and second derivatives of $\bar{u}_t$ with respect to t evaluated at $t = 0$. More precisely, we have:

$$\begin{aligned}
\dot{\bar{u}} =& \frac{d\bar{u}_t}{dt}|_{t=0}\\
=&(\frac{-t\varphi^2\bar{u}_1(x)}{\sqrt{(1+t^2\varphi^2)^3}}, \cdots, \frac{-t\varphi^2\bar{u}_n(x)}{\sqrt{(1+t^2\varphi^2)^3}}, \frac{a_{n+1}\varphi}{\sqrt{(1+t^2\varphi^2)^3}})|_{t=0}\\
=&(0, \cdots, 0, a_{n+1}\varphi)\\
\ddot{\bar{u}} =& \frac{d^2\bar{u}_t}{dt^2}|_{t=0}\\
=&(\frac{-\varphi^2\bar{u}_1(x)}{\sqrt{(1+t^2\varphi^2)^3}} + \frac{3t^2\varphi^4\bar{u}_1(x)}{\sqrt{(1+t^2\varphi^2)^5}}, \cdots,\\
&\frac{-\varphi^2\bar{u}_n(x)}{\sqrt{(1+t^2\varphi^2)^3}} + \frac{3t^2\varphi^4\bar{u}_n(x)}{\sqrt{(1+t^2\varphi^2)^5}}, \frac{-3t\varphi^3a_{n+1}}{\sqrt{(1+t^2\varphi^2)^5}})|_{t=0}\\
=& -\varphi^2(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n, 0) = -\varphi^2\bar{u}\\
d\dot{\bar{u}} =& \frac{d}{dx}(\dot{\bar{u}}) = (0, \cdots, 0, a_{n+1}d\varphi)\\
d\ddot{\bar{u}} =& \frac{d}{dx}(\ddot{\bar{u}}) = -2\varphi\bar{u}d\varphi - \varphi^2 d\bar{u}
\end{aligned} \tag{12}$$

where d means the derivative with respect to x. In a family of equations (12), we are taking derivative with respect to x and also taking derivative with respect to t.

$$\begin{aligned}
\frac{d}{dt}(\frac{1}{p}|d\bar{u}_t|^p) =& \frac{d}{dt}(\frac{1}{p}(|d\bar{u}_t|^2)^{\frac{p}{2}})\\
=& \frac{1}{p}\frac{p}{2}(|d\bar{u}_t|^2)^{\frac{p-2}{2}}\frac{d}{dt}(|d\bar{u}_t|^2)\\
=& \frac{1}{2}(|d\bar{u}_t|^2)^{\frac{p-2}{2}}\frac{d}{dt}\langle d\bar{u}_t, d\bar{u}_t\rangle\\
=& (|d\bar{u}_t|^2)^{\frac{p-2}{2}}\langle\frac{d}{dt}(d\bar{u}_t), d\bar{u}_t\rangle
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\frac{d^2}{dt^2}(\frac{1}{p}|d\bar{u}_t|^p)\\
=&\frac{d}{dt}[(|d\bar{u}_t|^2)^{\frac{p-2}{2}}\langle\frac{d}{dt}(d\bar{u}_t), d\bar{u}_t\rangle]\\
=&(|d\bar{u}_t|^2)^{\frac{p-2}{2}}[\langle\frac{d}{dt}(d\bar{u}_t), \frac{d}{dt}(d\bar{u}_t)\rangle\\
&+ \langle\frac{d^2}{dt^2}(d\bar{u}_t), d\bar{u}_t\rangle]\\
&+ \frac{p-2}{2}(|d\bar{u}_t|^2)^{\frac{p-4}{2}}\frac{d}{dt}(|d\bar{u}_t|^2)\langle\frac{d}{dt}(d\bar{u}_t), d\bar{u}_t\rangle\\
=&|d\bar{u}_t|^{p-2}[|\frac{d}{dt}(d\bar{u}_t)|^2 + \langle\frac{d^2}{dt^2}(d\bar{u}_t), d\bar{u}_t\rangle]\\
&+ (p-2)|d\bar{u}_t|^{p-4}\langle\frac{d}{dt}(d\bar{u}_t), d\bar{u}_t\rangle^2
\end{aligned} \tag{14}$$

Here, we also observe the following facts:

$$\begin{aligned}
d\bar{u}_t|_{t=0} &= d\bar{u}_0 = d\bar{u}\\
\frac{d}{dt}(d\bar{u}_t)|_{t=0} &= d(\frac{d\bar{u}_t}{dt})|_{t=0} = d\dot{\bar{u}}\\
\frac{d^2}{dt^2}(d\bar{u}_t)|_{t=0} &= d(\frac{d^2\bar{u}_t}{dt^2})|_{t=0} = d\ddot{\bar{u}}
\end{aligned}$$

Actually we can obtain the following inequality:

$$\begin{aligned}
&\frac{d^2}{dt^2}(\frac{1}{p}|d\bar{u}_t|^p)|_{t=0}\\
=&\{|d\bar{u}_t|^{p-2}[|\frac{d}{dt}(d\bar{u}_t)|^2 + \langle\frac{d^2}{dt^2}(d\bar{u}_t), d\bar{u}_t\rangle]\\
&+ (p-2)|d\bar{u}_t|^{p-4}\langle\frac{d}{dt}(d\bar{u}_t), d\bar{u}_t\rangle^2\}|_{t=0}\\
=&|d\bar{u}|^{p-2}(|d\dot{\bar{u}}|^2 + \langle d\bar{u}, d\ddot{\bar{u}}\rangle)\\
&+ (p-2)|d\bar{u}|^{p-4}\langle d\bar{u}, d\dot{\bar{u}}\rangle^2
\end{aligned} \tag{15}$$

Moreover, we have for $\bar{u}(R^l\backslash\{0\}) \subset \partial(\bar{E}^n_+)$,

$$\begin{aligned}
|\bar{u}|^2 &= (\bar{u}_1)^2 + \cdots + (\bar{u}_n)^2\\
&\leq a_1^2 + \cdots + a_n^2\\
&= \sum_{i=1}^n a_i^2
\end{aligned}$$

We apply the Cauchy-Schwarz Inequality for $-2\bar{u}\varphi\langle d\varphi, d\bar{u}\rangle$ to get:

$$\begin{aligned}
-2\bar{u}\varphi\langle d\varphi, d\bar{u}\rangle &\leq 2|\frac{\varphi|d\bar{u}|}{\sqrt{2}}||\sqrt{2}|\bar{u}||d\varphi||\\
&= \frac{\varphi^2|d\bar{u}|^2}{2} + 2|\bar{u}|^2|d\varphi|^2
\end{aligned}$$

So, we have

$$\begin{aligned}\langle d\bar{u}, d\ddot{\bar{u}}\rangle &= -2\bar{u}\varphi\langle d\varphi, d\bar{u}\rangle - \varphi^2|d\bar{u}|^2 \\ &\leq -\frac{1}{2}\varphi^2|d\bar{u}|^2 + 2|\bar{u}|^2|d\varphi|^2 \\ &\leq -\frac{1}{2}\varphi^2|d\bar{u}|^2 + 2(\sum_{i=1}^{n} a_i^2)|d\varphi|^2\end{aligned} \tag{16}$$

For a p-minimizing map $\bar{u}$, via (15) and (16) and

$$|d\dot{\bar{u}}| = a_{n+1}|d\varphi|$$

we have

$$\begin{aligned}0 \leq & \frac{d^2}{dt^2}E_p(\bar{u}_t)|_{t=0} \\ = & \int_{R^l\backslash\{0\}} \frac{d^2}{dt^2}(\frac{1}{p}|d\bar{u}_t|^p)|_{t=0}dv \\ = & \int_{R^l\backslash\{0\}} |d\bar{u}|^{p-2}(|d\dot{\bar{u}}|^2 + \langle d\bar{u}, d\ddot{\bar{u}}\rangle) \\ & + (p-2)|d\bar{u}|^{p-4}\langle d\bar{u}, d\dot{\bar{u}}\rangle^2 dv \\ \leq & \int_{R^l\backslash\{0\}} (p-1)|d\bar{u}|^{p-2}a_{n+1}^2|d\varphi|^2 \\ & + |d\bar{u}|^{p-2}\{\frac{-1}{2}\varphi^2|d\bar{u}|^2 + 2(\sum_{i=1}^{n} a_i^2)|d\varphi|^2\}dv \\ = & \int_{R^l\backslash\{0\}} |d\bar{u}|^{p-2}|d\varphi|^2((p-1)a_{n+1}^2 + 2\sum_{i=1}^{n} a_i^2) \\ & - \frac{1}{2}|d\bar{u}|^p\varphi^2 dv\end{aligned} \tag{17}$$

So we have

$$\textstyle\int_{R^l\backslash\{0\}} C_a\varphi^2|d\bar{u}|^p dv \leq \int_{R^l\backslash\{0\}} |d\varphi|^2|d\bar{u}|^{p-2}dv \tag{18}$$

where

$$C_a = \frac{1}{2(p-1)a_{n+1}^2 + 4\sum_{i=1}^{n} a_i^2}$$

Here we split the integrals (18) over $R^l\backslash\{0\}$ into spherical and radial directions and choose $\varphi = f(r)g(\omega)$ where $r = |x|$ and $\omega = \frac{x}{|x|} \in S^{l-1}$. By substituting φ into fg in (18), via the following facts:

$$\begin{aligned}& dx^2 = dr^2 + \frac{1}{r^2}d\omega^2 \\ & |d\varphi|^2 = (f')^2g^2(\omega) + \frac{1}{r^2}f^2(r)|\nabla g|^2, \quad f' = \frac{df}{dr} \\ & |d\bar{u}(x)|^2 = \frac{1}{r^2}|du(\omega)|^2 \\ & dv = dx_1dx_2\cdots dx_l = r^{l-1}drd\omega\end{aligned} \tag{19}$$

we have the inequality:

$$\begin{aligned}& \int_0^\infty \int_{S^{l-1}} C_a f^2(r)g^2(\omega)\frac{1}{r^p}|du(\omega)|^p r^{l-1}drd\omega \\ \leq & \int_0^\infty \int_{S^{l-1}} [(f')^2g^2 + \frac{1}{r^2}f^2|\nabla g|^2]\frac{1}{r^{p-2}}|du(\omega)|^{p-2}r^{l-1}drd\omega\end{aligned}$$

that is

$$\begin{aligned}& \int_0^\infty \int_{S^{l-1}} C_a f^2g^2|du|^p r^{l-p-1}drd\omega \\ \leq & \int_0^\infty \int_{S^{l-1}} (f')^2g^2|du|^{p-2}r^{l-p+1}drd\omega \\ & + \int_0^\infty \int_{S^{l-1}} f^2|\nabla g|^2|du|^{p-2}r^{l-p-1}drd\omega\end{aligned}$$

By combining the alike integrals, we have

$$\begin{aligned}& \int_0^\infty \int_{S^{l-1}} C_a f^2g^2|du|^p r^{l-p-1}drd\omega \\ & - \int_0^\infty \int_{S^{l-1}} f^2|\nabla g|^2|du|^{p-2}r^{l-p-1}drd\omega \\ \leq & \int_0^\infty \int_{S^{l-1}} (f')^2g^2|du|^{p-2}r^{l-p+1}drd\omega\end{aligned}$$

that is

$$\begin{aligned}& \int_0^\infty \int_{S^{l-1}} f^2r^{l-p-1}(C_ag^2|du|^p - |\nabla g|^2|du|^{p-2})drd\omega \\ \leq & \int_0^\infty \int_{S^{l-1}} (f')^2g^2|du|^{p-2}r^{l-p+1}drd\omega\end{aligned}$$

In other words, we have

$$\begin{aligned}& \int_0^\infty f^2r^{l-p-1}dr \int_{S^{l-1}} (C_ag^2|du|^p - |\nabla g|^2|du|^{p-2})d\omega \\ \leq & \int_0^\infty (f')^2r^{l-p+1}dr \int_{S^{l-1}} g^2|du|^{p-2}d\omega\end{aligned}$$

Therefore, we obtain the simplified inequality:

$$\frac{\int_{S^{l-1}}(C_ag^2|du|^p - |\nabla g|^2|du|^{p-2})d\omega}{\int_{S^{l-1}} g^2|du|^{p-2}d\omega} \leq \frac{\int_0^\infty (f')^2 r^{l-p+1}dr}{\int_0^\infty f^2r^{l-p-1}dr} \tag{20}$$

Via Lemma 4

$$\inf \frac{\int_0^\infty (f')^2r^{l-p+1}dr}{\int_0^\infty f^2r^{l-p-1}dr} = \frac{1}{4}(l-p)^2$$

for $l > p$ and by choosing $g = |du|$, we can get:

$$\begin{aligned}& \int_{S^{l-1}} C_a|du|^{p+2}d\omega - \int_{S^{l-1}} |\nabla|du||^2|du|^{p-2}d\omega \\ \leq & \frac{1}{4}(l-p)^2 \int_{S^{l-1}} |du|^p d\omega\end{aligned} \tag{21}$$

On the other hand, by applying Lemma 3 the Bochner Formula for a p-harmonic map ($p > 1$) $u : S^{l-1} \to \bar{E}_+^n$ ($l > 2$), we have

$$\begin{aligned}& \int_{S^{l-1}} (l-2)|du|^p d\omega \\ \leq & -(p-1)\int_{S^{l-1}} |du|^{p-2}|\nabla|du||^2 d\omega \\ & + \frac{l-2}{l-1}\int_{S^{l-1}} \frac{(max(a_i))^2}{(min(a_i))^4}|du|^{p+2}d\omega\end{aligned} \tag{22}$$

Combining (21) and (22), we can get the following inequality:

$$\begin{aligned}& (\frac{l-2}{p-1} - \frac{(l-p)^2}{4})\int_{S^{l-1}} |du|^p d\omega \\ \leq & (\frac{l-2}{(p-1)(l-1)}\frac{(max(a_i))^2}{(min(a_i))^4} - C_a)\int_{S^{l-1}} |du|^{p+2}d\omega\end{aligned} \tag{23}$$

Then u must be constant (i.e.$|du| = 0$) if

$$\begin{cases} \frac{l-2}{p-1} - \frac{(l-p)^2}{4} > 0, \\ \frac{l-2}{(p-1)(l-1)} \frac{(max(a_i))^2}{(min(a_i))^4} - C_a < 0. \end{cases}$$

Therefore, we need to find a solution to

$$\frac{(l-p)^2}{4} < C_a(l-1)\frac{(\min(a_i))^4}{(\max(a_i))^2} \tag{24}$$

Via the fact $C_a > \frac{1}{(4n+2p-2)(\max(a_i))^2}$, we claim solutions to the following inequality must be the solutions to inequality (24):

$$\frac{(l-p)^2}{4} < \frac{1}{(4n+2p-2)(\max(a_i))^2}(l-1)\frac{(\min(a_i))^4}{(\max(a_i))^2}$$

that is:

$$(l-p)^2 < \frac{4(l-1)}{(4n+2p-2)\mu^4} \tag{25}$$

where $\mu = \frac{\max(a_i)}{\min(a_i)}$. For $1 < p < l$ and $l > 2$, we claim that a solution to

$$(l-p)^2 < \frac{4(l-1)}{(4n+2l-2)\mu^4}$$

must be the solution to inequality (25) by the fact

$$\frac{1}{(4n+2p-2)} > \frac{1}{(4n+2l-2)}$$

So, we find the solution

$$\max\Big\{1, l - \frac{2}{\mu^2}\sqrt{\frac{l-1}{4n+2l-2}}\Big\} < p < l$$

Finally, we solve the Liouville-type problem and obtain the constant property of u for

$$\max\Big\{1, l - \frac{2}{\mu^2}\sqrt{\frac{l-1}{4n+2l-2}}\Big\} < p < l$$

■

Remark 1. One of the most important step in proof is to obtain a p-Stability Inequality via (18). Generally speaking, we can define a p-Stability Inequality for a map w from M to N given by

$$\int_M \alpha\eta^2|dw|^p dv \le \int_M |d\eta|^2|dw|^{p-2}dv$$

where η is any test function $\eta \in C_0^\infty(M)$ and α is a constant. The existence of a p-Stability Inequality derived from the Variation Method is one of the crucial achievements to solve the Liouville-type problem. S. W. Wei proved the existence of a p-Stability Inequality for a p-stable map on a special ellipsoid with only one parameter $E^n = E^n_a$, (i.e. $E^n_a = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{a^2} + x_2^2 + \cdots + x_{n+1}^2 = 1,\ a > 0\}$) (cf. sections 7, 8 [12]). Our result is an extension of S. W. Wei's result to any ellipsoid with $(n+1)$ parameters.

Remark 2. A sphere ($S^n(r) = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{r^2} + \cdots + \frac{x_{n+1}^2}{r^2} = 1,\ r > 0\}$) is considered as a simple case of an ellipsoid (i.e. $S^n(r) = \{(x_1, \cdots, x_{n+1}) \in E^n | a_1 = a_2 = \cdots = a_{n+1} = r > 0$). Regarding Liouville-type Problems for a p-stable map, our result on an ellipsoid is an generalization of mathematicians' results on a sphere.

IV. Conclusion

In this paper, we solve the Liouville-type problem for a p-energy minimizing map from the domain of a Euclidean space to the target space of a closed half-ellipsoid. The most important method is the Calculus Variation Method. The most challenging point for us to apply the Calculus Variation Method is to construct a family of variational maps such that the image of every variational map must be in the closed half-ellipsoid. The most crucial step in the use of variational formulas is to derive the p-Stability Inequality. The most useful estimation techniques in computation to estimate the lower bound and the upper bound of p-Stability Inequality are the Bochner Formula, Stokes' Theorem, Cauchy-Schwarz Inequality, and Hardy-Sobolev type Inequalities. The most fundamental idea for us to analyze the p-Stability Inequality is to find a contradiction between the lower bound and the upper bound in the p-Stability Inequality when the p-energy minimizing map u is not constant. Therefore, we are able to rule out a possibility of a non-constant p-energy minimizing map. We obtain the constant property for a p-energy minimizing map as our Liouville-type result.

Our research finding reveals the constant property of a p-energy minimizing map for p in a certain range. The certain range of p is determined by the dimension values of a Euclidean space (the domain) and an ellipsoid (the target space). The certain range of p is also determined by the curvature value on the target space of an ellipsoid indicated by the ratio of the longest axis to the shortest axis.

Regarding Liouville-type results for a p-stable map as a local or global p-energy minimizing map, our research result generalizes mathematicians' results from a sphere to an ellipsoid where a sphere is a simple case of an ellipsoid (i.e. All the axes in each direction are equal.) Our result also extends Liouville-type results from a special ellipsoid with only one parameter (i.e. Except for one axis, all the other axes are equal to 1.) to any ellipsoid with $(n+1)$ parameters in the general setting.

In this paper, we demonstrate a successful way to work on the Liouville-type problems for solutions to p-harmonic equations (called p-harmonic maps). Along with this research direction, we can continue to study Liouville-type problems for solutions to p-harmonic inequalities (that is, p-super-harmonic maps and p-sub-harmonic maps). As for our ongoing research, we will work on studying the Liouville-type problems for solutions to various partial differential equations or various partial differential inequalities. Research topics related with Liouville-type Problems such as Regularity Problems [13] will be our interests in the future.

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