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# (λ, μ)-Intuitionistic Fuzzy Subgroups of Groups with Operators

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***Abstract*—**The aim of this paper is to introduce the concepts of the (λ, μ)-intuitionistic fuzzy subgroups and (λ, μ)-intuitionistic fuzzy normal subgroups of groups with operators, and to investigate their properties and characterizations based on M-group homomorphism.

***Keywords*—**Intuitionistic fuzzy group, (λ, μ)-intuitionistic fuzzy subgroup of groups with operators, (λ, μ)-intuitionistic fuzzy normal subgroup of groups with operators, M-group homomorphism.

## I. Introduction

THE fuzzy set is an effective generalization of the classical set. In 1965, Zadeh [1] first raised the fuzzy set. In 1986, the Bulgarian Scholar K. Atannassov [2] introduced the intuitionistic fuzzy sets (IFS). After that, the two theories were extensively applied to many mathematical fields. Based on the two theories, W. X. Gu [3] raised the definition of fuzzy group with operators; [4]-[6] researched intuitionistic fuzzy relations, martingale theory and topological spaces; [7]-[10] studied intuitionistic fuzzy subgroups and some properties are discussed; [11] gave the definition of $(\lambda,\mu)-$ intuitionistic fuzzy subgroups; [12] defined the $(\lambda,\mu)-$ intuitionistic fuzzy implicative ideals of BCI-algebras.

At first, this paper gives the concepts of the (λ, μ)-intuitionistic fuzzy subgroups and (λ, μ)-intuitionistic fuzzy normal subgroups of groups with operators. Secondly, it is proven that A is a (λ, μ)-intuitionistic fuzzy subgroup or (λ, μ)-intuitionistic fuzzy normal subgroup of a group G with operators if and only if cut sets of A are subgroup or normal subgroup of G. Thirdly, some properties are discussed. Finally, in the sense of M-group homomorphism between two classical groups, the image and the preimage of the (λ, μ)-intuitionistic fuzzy subgroups and (λ, μ)-intuitionistic fuzzy normal subgroups of groups with operators are studied, which enriches and expands the theory of the IFS and group.

## II. Preliminaries

In this paper, we always assume $0 \le \lambda < \mu \le 1$.

Let $IFG[G]$ and $IFNG[G]$ be the intuitionistic fuzzy subgroups and intuitionistic fuzzy normal subgroups of $G$.

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**Definition 1.** [13] Let $A: X \to [0,1]$ be a mapping. If there exist $a \in (0,1]$ and $x \in A$ such that

$$A(y)=\begin{cases} a, & y = x; \\ 0, & y \ne x. \end{cases}$$

Then $A$ is called a fuzzy point, and denoted by $x_a$.

**Definition 2.** [2] Let $X$ be any nonempty set. An intuitionistic fuzzy subset $A$ of $X$ is an object of the following form

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}.$$

where $\mu_A: X \to [0,1]$ and $\nu_A: X\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$,

$$0 \le \mu_A(x)+\nu_A(x) \le 1.$$

Let $IFS[X]$ be the IFS of $X$.

**Definition 3.** [2] Let $X$ be any nonempty set, $A, B \in IFS[X]$ and

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\},$$
$$B=\{\langle x,\mu_B(x),\nu_B(x)\rangle \mid x\in X\}.$$

The rules and operations are as follows:

1. $A\cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\rangle \mid x\in X\}$;
2. $A\cup B=\{\langle x,\mu_A(x)\vee\mu_B(x),\nu_A(x)\wedge\nu_B(x)\rangle \mid x\in X\}$;
3. $\bigcap_{j\in J} A_j=\left\{\left\langle x,\bigwedge_{j\in J}\mu_{A_j}(x),\bigvee_{j\in J}\nu_{A_j}(x)\right\rangle \mid x\in X\right\}$;
4. $\bigcup_{j\in J} A_j=\left\{\left\langle x,\bigvee_{j\in J}\mu_{A_j}(x),\bigwedge_{j\in J}\nu_{A_j}(x)\right\rangle \mid x\in X\right\}$.

where $A_j=\{\langle x,\mu_{A_j}(x),\nu_{A_j}(x)\rangle \mid x\in X\}\in IFS[X]$, $j=1,2,\cdots$, $J$ is the index sets.

**Definition 4.** [8] Let $X$, $Y$ be any two nonempty sets and $f: X\to Y$ be a mapping. Let $A\in IFS[X]$ and

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}.$$

Then $F_f : IFS[X] \to IFS[Y]$ and $F_f(A)$ are also the IFS of $Y$, and

$$F_f(A) = \{\langle y, F_f(\mu_A)(y), F_f(\nu_A)(y)\rangle \, y \in Y\}.$$

where

$$F_f(\mu_A)(y) = \begin{cases} \sup\{\mu_A(x) \mid f(x) = y, x \in X\}, & f^{-1}(y) \neq \varnothing; \\ 0, & f^{-1}(y) = \varnothing. \end{cases}$$

$$F_f(\nu_A)(y) = \begin{cases} \inf\{\nu_A(x) \mid f(x) = y, x \in X\}, & f^{-1}(y) \neq \varnothing; \\ 1, & f^{-1}(y) = \varnothing. \end{cases}$$

**Definition 5.** [8] Let $X$, $Y$ be any two nonempty sets and $f : X \to Y$ be a mapping. Let $B \in IFS[Y]$ and

$$B = \{\langle y, \mu_B(y), \nu_B(y)\rangle \mid y \in Y\}.$$

Then $F_f^{-1} : IFS[Y] \to IFS[X]$ and $F_f^{-1}(B)$ are also the IFS of $X$, and

$$F_f^{-1}(B) = \{\langle x, F_f^{-1}(\mu_B)(x), F_f^{-1}(\nu_B)(x)\rangle \mid x \in X\}.$$

Definitions 4 and 5 are called the extension principle of IFS. Denote $\langle I\rangle = \{\langle a,b\rangle : a,b \in [0,1]\}$.

**Definition 6.** [12] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in S\}$ be an IFS in a set $S$. For $\langle \alpha, \beta\rangle \in \langle I\rangle$, the set $A_{\langle \alpha,\beta\rangle} = \{x \in S : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is called a cut set of $A$.

**Definition 7.** [8] Let $G$ be a group and

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\} \in IFS[G].$$

If for $\forall x, y \in G$,

1. $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$,
2. $\mu_A(x^{-1}) \geq \mu_A(x)$, $\nu_A(x^{-1}) \leq \nu_A(x)$.

Then $A$ is called the intuitionistic fuzzy subgroup of $G$.

**Definition 8.** [11] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in G\}$ be the intuitionistic fuzzy subgroup of $G$. If for $\forall x, y \in G$,

1. $\mu_A(xy) \vee \lambda \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \mu$,
   $\nu_A(xy) \wedge \mu \leq (\nu_A(x) \vee \nu_A(y)) \vee \lambda$,
2. $\mu_A(x^{-1}) \vee \lambda \geq \mu_A(x) \wedge \mu$, $\nu_A(x^{-1}) \wedge \mu \leq \nu_A(x) \vee \lambda$.

Then $A$ is called the $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of $G$.

**Definition 9.** [11] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in G\}$ be the $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of $G$. If for $\forall x, y \in G$,

$$\mu_A(xyx^{-1}) \vee \lambda \geq \mu_A(y) \wedge \mu, \; \nu_A(xyx^{-1}) \wedge \mu \leq \nu_A(y) \vee \lambda.$$

Then $A$ is called the $(\lambda, \mu)-$ intuitionistic fuzzy normal subgroup of $G$.

**Definition 10.** [3] A group with operators in an algebraic system consisting of a group, a set $M$ and a function defined in the product set $M \times G$ and having values in $G$ such that, if $ma$ denotes the element in $G$ determined by the element $a$ of $G$ and the element $m$ of $M$, then

$$m(ab) = (ma)(mb),$$

Holds for any $a, b$ in $G$, $m$ in $M$. We shall usually use the phrase "$G$ is an $M-$ group" to a group with operators.

**Definition 11.** [14] A subgroup $A$ of $M-$ group is said to be an $M-$ subgroup if $mx$ in $A$ for every $m$ in $M$ and $x$ in $A$.

**Definition 12.** [14] Let $G_1$ and $G_2$ both be $M-$ groups, $f$ be a homomorphism from $G_1$ onto $G_2$

$$f(mx) = mf(x), m \in M, x \in G,$$

Then $f$ is called a $M-$ homomorphism.

**Proposition 1.** [10] Let $G$ be a $M-$ group, $e$ be the identity element of $G$, and $A \in IFG[G]$. Then for $\forall x \in G$,

$$\mu_A(x) \leq \mu_A(e), \; \nu_A(x) \geq \nu_A(e).$$

**Proposition 2.** [11] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in G\}$ be the $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of $G$ and $e$ be the identity element. Then

$$\mu_A(e) \vee \lambda \geq \mu_A(x) \wedge \mu, \; \nu_A(e) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

**Proposition 3.** [11] Let $A$ be the intuitionistic fuzzy subset. Then $A$ is a $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of $G$ iff for $\forall x, y \in G$,

$$\mu_A(x^{-1}y) \vee \lambda \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \mu,$$
$$\nu_A(x^{-1}y) \wedge \mu \leq (\nu_A(x) \vee \nu_A(y)) \vee \lambda.$$

**Proposition 4.** [11] Let $A$ be the intuitionistic fuzzy subset. Then $A$ is a $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of $G$ iff for $\forall \alpha, \beta \in (\lambda, \mu)$, $A_{\langle \alpha,\beta\rangle}$ is the subgroup when $A_{\langle \alpha,\beta\rangle} \neq \varnothing$, where $\langle \alpha, \beta\rangle \in \langle I\rangle$.

**Proposition 5.** [11] Let $f: G_1 \to G_2$ be a surjective homomorphism of groups. If $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $G_1$, then $f(A)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $G_2$.

**Proposition 6.** [11] Let $f: G_1 \to G_2$ be a homomorphism of groups. If $B$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $G_2$, then $f^{-1}(B)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $G_1$.

**Proposition 7.** [11] Let $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $G$ .Then $A$ is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of $G$ iff for $\forall x, y \in G$,

$$\mu_A(xy) \vee \lambda \geq \mu_A(yx) \wedge \mu,\ \nu_A(xy) \wedge \mu \leq \nu_A(yx) \vee \lambda.$$

**Proposition 8.** [11] Let $f: G_1 \to G_2$ be a surjective homomorphism of groups. If $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of $G_1$, then $f(A)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of $G_2$.

**Proposition 9.** [11] Let $f: G_1 \to G_2$ be a homomorphism of groups. If $B$ be a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of $G_2$, then $f^{-1}(B)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of $G_1$.

## III. $(\lambda,\mu)-$ Intuitionistic Fuzzy Subgroups of Groups with Operators

**Definition 13.** Let $G$ be a $M-$ group and $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup. If for $\forall x \in G,\ m \in M$,

$$\mu_A(mx) \vee \lambda \geq \mu_A(x) \wedge \mu,\ \nu_A(mx) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Then $A$ is called a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of a group $G$ with operators, and denoted by a $(\lambda,\mu)-\ M-$ intuitionistic fuzzy subgroup.

Let $(\lambda,\mu)-\ IFMG[G]$ be the $(\lambda,\mu)-\ M-$ intuitionistic fuzzy subgroups of $M-$ group $G$.

**Proposition 10.** Let $G$ be a $M-$ group, $A \in (\lambda,\mu)-$ $IFMG[G]$ and $e$ be the identity element of $G$. Then

$$\mu_A(me) \vee \lambda \geq \mu_A(x) \wedge \mu,\ \nu_A(me) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

**Proof.** For $\forall x \in G,\ m \in M$,

$$\mu_A(me) \vee \lambda \geq \mu_A(e) \wedge \mu \geq \mu_A(x) \wedge \mu,$$
$$\nu_A(me) \wedge \mu \leq \nu_A(e) \vee \lambda \leq \nu_A(x) \vee \lambda.$$

**Proposition 11.** Let $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group $G$. Then $A \in (\lambda,\mu)-\ IFMG[G]$ iff for $\forall x, y \in G,\ m \in M$,

$$\mu_A\left(m\left(x^{-1}y\right)\right) \vee \lambda \geq \left(\mu_A(mx) \wedge \mu_A(my)\right) \wedge \mu,$$
$$\nu_A\left(m\left(x^{-1}y\right)\right) \wedge \mu \leq \left(\nu_A(mx) \vee \nu_A(my)\right) \vee \lambda.$$

**Proof.** For $\forall x, y \in G,\ m \in M$,

$$\begin{aligned}
\mu_A\left(m\left(x^{-1}y\right)\right) \vee \lambda &= \left(\mu_A\left(\left(mx^{-1}\right)(my)\right) \vee \lambda\right) \vee \lambda \\
&\geq \left(\left(\mu_A\left(mx^{-1}\right) \wedge \mu_A(my)\right) \wedge \mu\right) \vee \lambda \\
&= \left(\mu_A\left(mx^{-1}\right) \vee \lambda\right) \wedge \mu_A(my) \wedge \mu \\
&= \left(\mu_A(mx)^{-1} \vee \lambda\right) \wedge \mu_A(my) \wedge \mu \\
&\geq \left(\mu_A(mx) \wedge \mu\right) \wedge \mu_A(my) \wedge \mu \\
&= \left(\mu_A(mx) \wedge \mu_A(my)\right) \wedge \mu,
\end{aligned}$$

$$\begin{aligned}
\nu_A\left(m\left(x^{-1}y\right)\right) \wedge \mu &= \left(\nu_A\left(\left(mx^{-1}\right)(my)\right) \wedge \mu\right) \wedge \mu \\
&\leq \left(\left(\nu_A\left(mx^{-1}\right) \vee \nu_A(my)\right) \vee \lambda\right) \wedge \mu \\
&= \left(\nu_A\left(mx^{-1}\right) \wedge \mu\right) \vee \nu_A(my) \vee \lambda \\
&= \left(\nu_A(mx)^{-1} \wedge \mu\right) \vee \nu_A(my) \vee \lambda \\
&\leq \left(\nu_A(mx) \vee \lambda\right) \vee \nu_A(my) \vee \lambda \\
&= \left(\nu_A(mx) \vee \nu_A(my)\right) \vee \lambda.
\end{aligned}$$

Conversely, for $\forall x \in G,\ m \in M$, let $y = e$,

$$\begin{aligned}
\mu_A\left(m(xe)\right) \vee \lambda &= \left(\mu_A\left(m\left(\left(x^{-1}\right)^{-1} e\right)\right) \vee \lambda\right) \vee \lambda \\
&\geq \left(\left(\mu_A\left(mx^{-1}\right) \wedge \mu_A(me)\right) \wedge \mu\right) \vee \lambda \\
&= \left(\mu_A(me) \vee \lambda\right) \wedge \left(\mu_A\left(mx^{-1}\right) \wedge \mu\right) \\
&\geq \left(\mu_A(e) \wedge \mu\right) \wedge \left(\mu_A\left(x^{-1}\right) \wedge \mu\right) \\
&\geq \left(\mu_A(e) \wedge \mu\right) \wedge \left(\mu_A(x) \wedge \mu\right) \\
&= \mu_A(x) \wedge \mu,
\end{aligned}$$

$$\begin{aligned}\nu_A\left(m(xe)\right)\wedge\mu &= \left(\nu_A\left(m\left(\left(x^{-1}\right)^{-1}e\right)\right)\wedge\mu\right)\wedge\mu\\ &\le \left(\left(\nu_A\left(mx^{-1}\right)\wedge\nu_A\left(me\right)\right)\vee\lambda\right)\wedge\mu\\ &= \left(\nu_A\left(me\right)\wedge\mu\right)\vee\left(\nu_A\left(mx^{-1}\right)\vee\lambda\right)\\ &\le \left(\nu_A\left(e\right)\vee\lambda\right)\vee\left(\nu_A\left(x^{-1}\right)\vee\lambda\right)\\ &\le \left(\nu_A\left(e\right)\vee\lambda\right)\vee\left(\nu_A\left(x\right)\vee\lambda\right)\\ &= \nu_A(x)\vee\lambda.\end{aligned}$$

Thus, $A\in(\lambda,\mu)-IFMG[G]$.

**Proposition 12.** Let $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group $G$. Then $A\in(\lambda,\mu)-IFMG[G]$ iff for $\forall\alpha,\beta\in(\lambda,\mu)$, $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of $G$ when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$, where $\langle\alpha,\beta\rangle\in\langle I\rangle$.

**Proof.** For $\forall\alpha,\beta\in(\lambda,\mu)$, $x\in A_{\langle\alpha,\beta\rangle}$ when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$. Therefore, $\mu_A(x)\ge\alpha$, $\nu_A(x)\le\beta$. Then,

$$\mu_A(mx)\vee\lambda\ge\mu_A(x)\wedge\mu\ge\alpha>\lambda,$$
$$\nu_A(mx)\wedge\mu\le\nu_A(x)\vee\lambda\le\beta<\mu.$$

We have $\mu_A(mx)\ge\alpha$, $\nu_A(mx)\le\beta$. Therefore, $mx\in A_{\langle\alpha,\beta\rangle}$. Thus, $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of $G$ when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$.

Conversely, for $\forall\alpha,\beta\in(\lambda,\mu)$, we get the information from Proposition 4 that $A$ is a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of group $G$. Besides, for $\forall x\in G$, let $\alpha=\mu_A(x)\wedge\mu$, $\beta=\nu_A(x)\vee\lambda$. Therefore, $\mu_A(x)\ge\alpha$, $\nu_A(x)\le\beta$, and $x\in A_{\langle\alpha,\beta\rangle}$. And $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of $G$ when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$. Thus, $mx\in A_{\langle\alpha,\beta\rangle}$. We have $\mu_A(mx)\ge\alpha$, $\nu_A(mx)\le\beta$. And

$$\mu_A(mx)\vee\lambda\ge\alpha=\mu_A(x)\wedge\mu,$$
$$\nu_A(mx)\wedge\mu\le\beta=\nu_A(x)\vee\lambda.$$

Thus, $A\in(\lambda,\mu)-IFMG[G]$.

**Proposition 13.** Let $G$ be a $M-$ group and $A,B\in(\lambda,\mu)-IFMG[G]$. Then $A\cap B\in(\lambda,\mu)-IFMG[G]$.

**Proof.** Let

$$A=\left\{\left\langle x,\mu_A(x),\nu_A(x)\right\rangle \middle| x\in G\right\},$$
$$B=\left\{\left\langle x,\mu_B(x),\nu_B(x)\right\rangle \middle| x\in G\right\}.$$

Then

$$A\cap B=\left\{\left\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\right\rangle \middle| x\in G\right\}.$$

Let

$$\mu_{A\cap B}(x)=\mu_A(x)\wedge\mu_B(x),\ \nu_{A\cap B}(x)=\nu_A(x)\vee\nu_B(x).$$

First, we provide that $A\cap B$ is $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group $G$. For $\forall x,y\in G$, on the one hand,

$$\begin{aligned}\mu_{A\cap B}(xy)\vee\lambda &= \left(\mu_A(xy)\wedge\mu_B(xy)\right)\vee\lambda\\ &= \left(\left(\mu_A(xy)\wedge\mu_B(xy)\right)\vee\lambda\right)\vee\lambda\\ &= \left(\mu_A(xy)\vee\lambda\right)\wedge\left(\mu_B(xy)\vee\lambda\right)\\ &\ge \left(\left(\mu_A(x)\wedge\mu_A(y)\right)\wedge\mu\right)\wedge\left(\left(\mu_B(x)\wedge\mu_B(y)\right)\wedge\mu\right)\\ &= \left(\mu_A(x)\wedge\mu_B(x)\right)\wedge\left(\mu_A(y)\wedge\mu_B(y)\right)\wedge\mu\\ &= \left(\mu_{A\cap B}(x)\wedge\mu_{A\cap B}(y)\right)\wedge\mu.\end{aligned}$$

Similarly, $\nu_{A\cap B}(xy)\wedge\mu\le\left(\nu_{A\cap B}(x)\vee\nu_{A\cap B}(y)\right)\vee\lambda.$ On the other hand,

$$\begin{aligned}\mu_{A\cap B}\left(x^{-1}\right)\vee\lambda &= \left(\mu_A\left(x^{-1}\right)\wedge\mu_B\left(x^{-1}\right)\right)\vee\lambda\\ &= \left(\left(\mu_A\left(x^{-1}\right)\wedge\mu_B\left(x^{-1}\right)\right)\vee\lambda\right)\vee\lambda\\ &= \left(\mu_A\left(x^{-1}\right)\vee\lambda\right)\wedge\left(\mu_B\left(x^{-1}\right)\vee\lambda\right)\\ &\ge \left(\mu_A(x)\wedge\mu\right)\wedge\left(\mu_B(x)\wedge\mu\right)\\ &= \left(\mu_A(x)\wedge\mu_B(x)\right)\wedge\mu\\ &= \mu_{A\cap B}(x)\wedge\mu.\end{aligned}$$

Similarly, $\nu_{A\cap B}\left(x^{-1}\right)\wedge\mu\le\nu_{A\cap B}(x)\vee\lambda$. Thus, $A\cap B$ is $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group $G$. Then for $\forall x\in G$, $m\in M$, $A,B\in(\lambda,\mu)-IFMG[G]$.

The following can be obtained from Definition 15. On the one hand,

$$\begin{aligned}\mu_{A\cap B}(mx)\vee\lambda &= \left(\left(\mu_A(mx)\wedge\mu_B(mx)\right)\vee\lambda\right)\vee\lambda\\ &= \left(\mu_A(mx)\vee\lambda\right)\wedge\left(\mu_B(mx)\vee\lambda\right)\\ &\ge \left(\mu_A(x)\wedge\mu\right)\wedge\left(\mu_B(x)\wedge\mu\right)\\ &= \left(\mu_A(x)\wedge\mu_B(x)\right)\wedge\mu\\ &= \mu_{A\cap B}(x)\wedge\mu.\end{aligned}$$

On the other hand,

$$\begin{aligned}\nu_{A\cap B}(mx)\wedge\mu &= \left(\left(\nu_A(mx)\vee\nu_B(mx)\right)\wedge\mu\right)\wedge\mu\\ &= \left(\nu_A(mx)\wedge\mu\right)\vee\left(\nu_B(mx)\wedge\mu\right)\\ &\le \left(\nu_A(x)\vee\lambda\right)\vee\left(\nu_B(x)\vee\lambda\right)\\ &= \left(\nu_A(x)\vee\nu_B(x)\right)\vee\lambda\\ &= \nu_{A\cap B}(x)\vee\lambda.\end{aligned}$$

Thus, $A\cap B\in(\lambda,\mu)-IFMG[G]$.

**Proposition 14.** Let $G_1$, $G_2$ be $M-$group, $f:G_1\to G_2$ be a $M-$surjective homomorphism of groups, and $A\in(\lambda,\mu)-IFMG[G_1]$. Then $f(A)\in(\lambda,\mu)-IFMG[G_2]$.

**Proof.** Let $A=\left\{\left\langle x,\mu_A(x),\nu_A(x)\right\rangle \middle| x\in G_1\right\}$. We get the information from Definition 4 that

$$f(A)=\left\{<y,\mu_{f(A)}(y),\nu_{f(A)}(y)>y\in G_2\right\}.$$

It is can be obtained from Proposition 5 that $f(A)$ is $(\lambda,\mu)-$intuitionistic fuzzy subgroup of $G_2$.

Because $f$ is $M-$surjective homomorphism, we have $f^{-1}(y)\neq\varnothing$ for $\forall y\in G_2$, $m\in M$. For $\forall x\in f^{-1}(y)$, then

$$x^{-1}\in f^{-1}\left(y^{-1}\right),\ \ f(mx)=mf(x)=my,\ \ x\in f^{-1}(y).$$

The following can be obtained from Definition 15. On the one hand,

$$\mu_A(mx)\vee\lambda\ge\mu_A(x)\wedge\mu,$$

and

$$\begin{aligned}\mu_{f(A)}(my)\vee\lambda &= \sup_{x\in f^{-1}(my)}\mu_A(x)\vee\lambda\\ &= \sup_{f(x)=my}\mu_A(x)\vee\lambda\\ &\ge \sup_{f(mx')=my}\mu_A(mx')\vee\lambda\\ &= \sup_{\substack{mf(x')=my\\ mx'\in G_1}}\mu_A(mx')\vee\lambda\\ &\ge \sup_{\substack{f(x')=y\\ x'\in G_1}}\mu_A(x')\wedge\mu\\ &= \mu_{f(A)}(y)\wedge\mu.\end{aligned}$$

On the other hand,

$$\nu_A(mx)\wedge\mu\le\nu_A(x)\vee\lambda,$$

and

$$\begin{aligned}\nu_{f(A)}(my)\wedge\mu &= \inf_{x\in f^{-1}(my)}\nu_A(x)\wedge\mu\\ &= \inf_{f(x)=my}\nu_A(x)\wedge\mu\\ &\le \inf_{f(mx')=my}\nu_A(mx')\wedge\mu\\ &= \inf_{\substack{mf(x')=my\\ mx'\in G_1}}\nu_A(mx')\wedge\mu\\ &\le \inf_{\substack{f(x')=y\\ x'\in G_1}}\nu_A(x')\vee\lambda\\ &= \nu_{f(A)}(y)\vee\lambda.\end{aligned}$$

Thus, $f(A)\in(\lambda,\mu)-IFMG[G_2]$.

**Proposition 15.** Let $G_1$, $G_2$ be $M-$group, $f:G_1\to G_2$ be a $M-$surjective homomorphism of groups, and $B\in(\lambda,\mu)-IFMG[G_2]$. Then $f^{-1}(B)\in(\lambda,\mu)-IFMG[G_1]$.

**Proof.** Let $B=\left\{\left\langle y,\mu_B(y),\nu_B(y)\right\rangle \middle| y\in G_2\right\}$. We get the information from Definition 5 that

$$f^{-1}(B)=\left\{\left\langle x,\mu_{f^{-1}(B)}(x),\nu_{f^{-1}(B)}(x)\right\rangle \middle| x\in G_1\right\}.$$

It is can be obtained from Proposition 6 that $f^{-1}(B)$ is $(\lambda,\mu)-$intuitionistic fuzzy subgroup of $G_1$.

Because $B\in(\lambda,\mu)-IFMG[G_2]$, we have for $\forall x\in G_1$, $m\in M$,

$$\begin{aligned}\mu_{f^{-1}(B)}(mx)\vee\lambda &= \mu_B\left(f(mx)\right)\vee\lambda\\ &= \mu_B\left(mf(x)\right)\vee\lambda\\ &\ge \mu_B\left(f(x)\right)\wedge\mu\\ &= \mu_{f^{-1}(B)}(x)\wedge\mu.\end{aligned}$$

$$\begin{aligned}\nu_{f^{-1}(B)}(mx)\wedge\mu &= \nu_B\left(f(mx)\right)\wedge\mu\\ &= \nu_B\left(mf(x)\right)\wedge\mu\\ &\le \nu_B\left(f(x)\right)\vee\lambda\\ &= \nu_{f^{-1}(B)}(x)\vee\lambda.\end{aligned}$$

Thus, $f^{-1}(B) \in (\lambda,\mu)-IFMG[G_1]$.

## IV. $(\lambda,\mu)-$ Intuitionistic Fuzzy Normal Subgroups of Groups with Operators

**Definition 14.** Let $G$ be a $M-$ group, $A \in (\lambda,\mu)-IFMG[G]$ and $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup. Then $A$ is called a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of a group $G$ with operators, denoted by a $(\lambda,\mu)-M-$ intuitionistic fuzzy normal subgroup.

Let $(\lambda,\mu)-IFMNG[G]$ be the $(\lambda,\mu)-M-$ intuitionistic fuzzy normal subgroups of $M-$ group $G$.

**Proposition 16.** Let $G$ be a $M-$ group and $A \in (\lambda,\mu)-IFMG[G]$. Then $A \in (\lambda,\mu)-IFMNG[G]$ iff for $\forall x,y \in G$, $m \in M$,

$$\mu_A(m(xy)) \vee \lambda \geq \mu_A(m(yx)) \wedge \mu,$$
$$\nu_A(m(xy)) \wedge \mu \leq \nu_A(m(yx)) \vee \lambda.$$

**Proof.** For $\forall x,y \in G$, $m \in M$,

$$\begin{aligned}\mu_A(m(xy)) \vee \lambda &= \mu_A\left(m\left(x(yx)x^{-1}\right)\right) \vee \lambda \\ &= \mu_A\left((mx)(m(yx))(mx)^{-1}\right) \vee \lambda \\ &\geq \mu_A(m(yx)) \wedge \mu,\end{aligned}$$

$$\begin{aligned}\nu_A(m(xy)) \wedge \mu &= \nu_A\left(m\left(x(yx)x^{-1}\right)\right) \wedge \mu \\ &= \nu_A\left((mx)(m(yx))(mx)^{-1}\right) \wedge \mu \\ &\leq \nu_A(m(yx)) \vee \lambda.\end{aligned}$$

Conversely, for $\forall x \in G$, $m \in M$, let $y = e$. We get the information from Proposition 7 that $A$ is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of $G$. $A \in (\lambda,\mu)-IFMG[G]$, thus $A \in (\lambda,\mu)-IFMNG[G]$.

**Proposition 17.** Let $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group $G$. Then $A \in (\lambda,\mu)-IFMNG[G]$ iff for $\forall x,y \in G$, $m \in M$,

1. $\mu_A\left(m\left(x^{-1}y\right)\right) \vee \lambda \geq \left(\mu_A(mx) \wedge \mu_A(my)\right) \wedge \mu,$
   $\nu_A\left(m\left(x^{-1}y\right)\right) \wedge \mu \leq \left(\nu_A(mx) \vee \nu_A(my)\right) \vee \lambda.$
2. $\mu_A(m(xy)) \vee \lambda \geq \mu_A(m(yx)) \wedge \mu,$
   $\nu_A(m(xy)) \wedge \mu \leq \nu_A(m(yx)) \vee \lambda.$

**Proposition 18.** Let $A$ be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group $G$. Then $A \in (\lambda,\mu)-IFMNG[G]$ iff for $\forall \alpha,\beta \in (\lambda,\mu)$, $A_{\langle\alpha,\beta\rangle}$ is $M-$ normal subgroup of $G$ when $A_{\langle\alpha,\beta\rangle} \neq \varnothing$, where $\langle\alpha,\beta\rangle \in \langle I\rangle$.

**Proof.** We get the information from Proposition 12 that for $\forall \alpha,\beta \in (\lambda,\mu)$, $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of $G$ when $A_{\langle\alpha,\beta\rangle} \neq \varnothing$. For $\forall x \in G$, $\forall y \in A_{\langle\alpha,\beta\rangle}$, we have $\mu_A(y) \geq \alpha$, $\nu_A(y) \leq \beta$ .Therefore,

$$\mu_A\left(xyx^{-1}\right) \vee \lambda \geq \mu_A(y) \wedge \mu \geq \alpha \wedge \mu > \lambda,$$
$$\nu_A\left(xyx^{-1}\right) \wedge \mu \leq \nu_A(y) \vee \lambda \leq \beta < \mu.$$

Then, $\mu_A\left(xyx^{-1}\right) \geq \alpha$, $\nu_A\left(xyx^{-1}\right) \leq \beta$, and $xyx^{-1} \in A_{\langle\alpha,\beta\rangle}$. Thus, $A_{\langle\alpha,\beta\rangle}$ is $M-$ normal subgroup of $G$ when $A_{\langle\alpha,\beta\rangle} \neq \varnothing$.

Conversely, we get the information from Proposition 12 that $A \in (\lambda,\mu)-IFMG[G]$. And for $\forall x,y \in G$, $m \in M$, we have

$$\mu_A\left(m\left(x^{-1}y\right)\right) \vee \lambda \geq \left(\mu_A(mx) \wedge \mu_A(my)\right) \wedge \mu,$$
$$\nu_A\left(m\left(x^{-1}y\right)\right) \wedge \mu \leq \left(\nu_A(mx) \vee \nu_A(my)\right) \vee \lambda.$$

If exist $x_0, y_0 \in G$, satisfying

$$\mu_A\left(x_0y_0x_0^{-1}\right) \vee \lambda \leq \mu_A(y_0) \wedge \mu,$$
$$\nu_A\left(x_0y_0x_0^{-1}\right) \wedge \mu \geq \nu_A(y_0) \vee \lambda.$$

Let $\alpha = \mu_A(y_0) \wedge \mu$, $\beta = \nu_A(y_0) \vee \lambda$. Then $y_0 \in A_{\langle\alpha,\beta\rangle}$. But $x_0y_0x_0^{-1} \notin A_{\langle\alpha,\beta\rangle}$, it is in contradiction with $A_{\langle\alpha,\beta\rangle}$ is $M-$ normal subgroup of $G$. Therefore, for $\forall x,y \in G$,

$$\mu_A\left(xyx^{-1}\right) \vee \lambda \geq \mu_A(y) \wedge \mu,$$
$$\nu_A\left(xyx^{-1}\right) \wedge \mu \leq \nu_A(y) \vee \lambda.$$

Thus, $A \in (\lambda,\mu)-IFMNG[G]$.

The following proposition can be easily proved.

**Proposition 19.** Let $G_1, G_2$ be $M-$ group, $f: G_1 \to G_2$ be a $M-$ surjective homomorphism of groups, and $A \in (\lambda,\mu)-IFMNG[G_1]$. Then $f(A) \in (\lambda,\mu)-IFMNG[G_2]$.

**Proposition 20.** Let $G_1, G_2$ be $M-$ group, $f: G_1 \to G_2$ be a $M-$ surjective homomorphism of groups, and $B \in (\lambda,\mu)-IFMNG[G_2]$. Then $f^{-1}(B) \in (\lambda,\mu)-IFMNG[G_1]$.

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