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Module and Comodule Structures on Path Space

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***Abstract*—**On path space kQ, there is a trivial kQ^a-module structure determined by the multiplication of path algebra kQ^a and a trivial kQ^c-comodule structure determined by the comultiplication of path coalgebra kQ^c. In this paper, on path space kQ, a nontrivial kQ^a-module structure is defined, and it is proved that this nontrivial left kQ^a-module structure is isomorphic to the dual module structure of trivial right kQ^c-comodule. Dually, on path space kQ, a nontrivial kQ^c-comodule structure is defined, and it is proved that this nontrivial right kQ^c-comodule structure is isomorphic to the dual comodule structure of trivial left kQ^a-module. Finally, the trivial and nontrivial module structures on path space are compared from the aspect of submodule, and the trivial and nontrivial comodule structures on path space are compared from the aspect of subcomodule.

***Keywords*—**Quiver, path space, module, comodule, dual.

I. Introduction and Preliminaries

It is well known that, for a given quiver, there is an algebra structure and a coalgebra structure on path space, called path algebra and path coalgebra respectively. References [1] and [2] proved that, over an algebraically closed field, any finite dimensional algebra is Morita equivalent to one factor algebra of a path algebra. So path algebra occupies an important position in the representation of finite dimensional algebras. Reference [3] proved that, over an algebraically closed field, any coalgebra is Morita-Takeuchi equivalent to one large subcoalgebra of a path colagebra. And so path coalgebra plays an equally important role in the representation of coalgebras. Since the structures of path algebra and path coalgebra are based on path space, we choose path space as our research object, and we will study its module and comodule structures.

In the following, we do some preparation for main results of this paper.

First, k denotes a field. Spaces, algebras and coalgebras in this paper are all defined over k. For the space V, V^* denotes the dual space $Hom(V,k)$.

Given a quiver $Q=(Q_0,Q_1)$, Q_0 and Q_1 denote the vertex set and arrow set respectively. For a path p in Q, $s(p)$ and $t(p)$ denote the starting and terminating vertex of p respectively, $l(p)$ denotes the length of p. A vertex $i\in Q_0$ determines a trivial path of length 0, denoted by v_i. With all paths as a basis, it can generate a k-space kQ, called path space. In this paper, Q is a finite quiver, i.e. Q_0 and Q_1 are both finite sets.

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On path space kQ, define multiplication and unit as

$$m(p\otimes q)=\begin{cases} pq, & if \quad t(q)=s(p),\\ 0, & otherwise,\end{cases}$$
$$\mu(1)=\sum_{i\in Q_0} v_i.$$

Thus, kQ becomes an algebra, called path algebra and denoted by kQ^a. Meanwhile, under this algebra structure path space kQ holds a trivial kQ^a-module structure, denoted by $(kQ,*)$. Also on path space kQ, define comultiplication and counit as

$$\Delta(p)=\sum_{p_1p_2=p} p_1\otimes p_2,$$
$$\varepsilon(p)=\begin{cases} 1, & if \quad l(p)=0,\\ 0, & otherwise.\end{cases}$$

Thus, kQ becomes a coalgebra, called path coalgebra and denoted by kQ^c. Meanwhile, under this coalgebra structure, path space kQ holds a trivial kQ^c-comodule structure, denoted by (kQ,Δ).

Lemma 1. [4] Let A be a finite dimensional algebra, and C a finite dimensional coalgebra, then

(1) on dual space C^*, there is an algebra structure $(C^*,\Delta^*,\varepsilon^*)$;

(2) on dual space A^*, there is a coalgebra structure (A^*,m^*,μ^*).

Lemma 2. [5] Let Q be a finite quiver, then

(1) the dual algebra of path coalgebra kQ^c is isomorphic to path algebra kQ^a;

(2) the dual coalgebra of path algebra kQ^a is isomorphic to path coalgebra kQ^c.

Proof. It is easy to show that the linear map defined by $\varphi(p^*)=p$ is not only an isomorphism of algebras from $(kQ^c)^*$ to kQ^a, but also an isomorphism of coalgebras from $(kQ^a)^*$ to kQ^c.

Lemma 3. [4] Let A be a finite dimensional algebra, and C a finite dimensional coalgebra, then

(1) any finite dimensional right C-comodule has a left C^*-module structure (called dual module structure of the original right C-comodule structure);
(2) any finite dimensional left A-module has a right A^*-comodule structure (called dual comodule structure of the original left A-module structure).

Proof.
(1) Let M be a finite dimensional right C-comodule, with the comodule structure map $\rho(m)=\sum m_0\otimes m_1$. Then under the left action $f\cdot m=\sum f(m_1)m_0$, M becomes a left C^*-module.
(2) Let M be a finite dimensional left A-module. For $m\in M$, choose a basis $\{m_1,m_2,\cdots,m_n\}$ of $A\cdot m$. Then for any $a\in A$, there are some $f_i(a)\in k$, such that $a\cdot m=\sum_{i=1}^{n} f_i(a)m_i$. So under the map $\rho(m)=\sum_{i=1}^{n} m_i\otimes f_i$, M becomes a right A^*-comodule.

II. kQ^a-Module Structures on Path Space

On path space kQ, the trivial kQ^a-module action is connection of two paths, which is similar to operation of addition. In this section, we will give a new kQ^a-module structure on path space kQ based on the idea of subtraction.

Let Q be a finite quiver, for any two paths p and q, if there is a path p' such that $q=p'p$, then the path p' must be unique. Therefore, we can define a k-linear left kQ^a-action on kQ as:

$$p\rhd q=\begin{cases}p', & if\quad q=p'p;\\ 0, & otherwise.\end{cases}$$

Proposition 1. Under the left action $\rhd$, path space kQ becomes a left kQ^a-module.

Proof. For any three paths p,q and r in quiver Q, it is only need to show that $p\rhd(q\rhd r)=(pq)\rhd r$ and $1\rhd r=r$, where $1=\sum_{i\in Q_0} v_i$ is the identity.

Indeed,

$$p\rhd(q\rhd r)=\begin{cases}p\rhd q', & if\quad r=q'q,\\ 0, & otherwise,\end{cases}=\begin{cases}p', & if\quad r=q'q,\quad q'=p'p,\\ 0, & otherwise,\end{cases}$$
$$=\begin{cases}p', & if\quad r=p'pq,\\ 0, & otherwise,\end{cases}=(pq)\rhd r,$$

$$1\rhd r=\sum_{i\in Q_0} v_i\rhd r=v_{s(r)}\rhd r=r.$$

Therefore, the left action $\rhd$ can make kQ become a left kQ^a-module. Denote this nontrivial left kQ^a-module structure on kQ by $(kQ,\rhd)$. Next, we will show that under isomorphism $(kQ,\rhd)$ is just happened to be the dual module structure of trivial right kQ^c-comodule on kQ.

Theorem 1. Let Q be a finite quiver without oriented cycles, then on path space left kQ^a-module structure $(kQ,\rhd)$ is isomorphic to the dual module structure of trivial right kQ^c-comodule structure (kQ,Δ).

Proof. Firstly, since Q is a finite quiver without oriented cycles, then kQ is a finite dimensional space, and so both path algebra and path coalgebra over kQ are of finite dimension.

By Lemma 3 (1), when the trivial right kQ^c-comodule structure (kQ,Δ) is dual, kQ becomes left $(kQ^c)^*$-module. And for any two paths p,q in Q, if $\Delta(q)=\sum_{q_1q_2=q} q_1\otimes q_2$, then the left $(kQ^c)^*$-module action on kQ is given by

$$p^*\cdot q=\sum_{q_1q_2=q} p^*(q_2)q_1 \qquad =\begin{cases}p', & if\quad q=p'p;\\ 0, & otherwise.\end{cases}$$

By Lemma 2 (1), linear map $\varphi(p^*)=p$ is an isomorphism of algebras from $(kQ^c)^*$ to kQ^a. So, under isomorphism, kQ also becomes a left kQ^a-module, and the module action is just happened to be the nontrivial left kQ^a-module action on kQ given by

$$p\rhd q=\begin{cases}p', & if\quad q=p'p;\\ 0, & otherwise.\end{cases}$$

Example 1. Let Q be a finite quiver as $1\xrightarrow{\alpha}2\xrightarrow{\beta}3$, then path space $kQ=k\{v_1,v_2,v_3,\alpha,\beta,\beta\alpha\}$, the nontrivial left kQ^a-module action $\rhd$ on kQ is given by:

$$v_1\rhd v_1=v_1,\ \ v_2\rhd v_1=v_3\rhd v_1=\alpha\rhd v_1=\beta\rhd v_1=\beta\alpha\rhd v_1=0;$$
$$v_2\rhd v_2=v_2,\ \ v_1\rhd v_2=v_3\rhd v_2=\alpha\rhd v_2=\beta\rhd v_2=\beta\alpha\rhd v_2=0;$$
$$v_3\rhd v_3=v_3,\ \ v_1\rhd v_3=v_2\rhd v_3=\alpha\rhd v_3=\beta\rhd v_3=\beta\alpha\rhd v_3=0;$$
$$v_1\rhd\alpha=\alpha,\alpha\rhd\alpha=v_2,\ \ v_2\rhd\alpha=v_3\rhd\alpha=\beta\rhd\alpha=\beta\alpha\rhd\alpha=0;$$
$$v_2\rhd\beta=\beta,\beta\rhd\beta=v_3,\ \ v_1\rhd\beta=v_3\rhd\beta=\alpha\rhd\beta=\beta\alpha\rhd\beta=0;$$
$$v_1\rhd\beta\alpha=\beta\alpha,\alpha\rhd\beta\alpha=\beta,\beta\alpha\rhd\beta\alpha=v_3,$$
$$v_2\rhd\beta\alpha=v_3\rhd\beta\alpha=\beta\rhd\beta\alpha=0.$$

Remark 1. Similarly for a finite quiver Q, we can define a nontrivial right kQ^a-module action on path space kQ as

$$q \triangleleft p = \begin{cases} p', & if \quad q = pp'; \\ 0, & otherwise. \end{cases}$$

It can also be checked that, when Q has no oriented cycles, this nontrivial right module structure is isomorphic to the dual module structure of the trivial left kQ^c-comodule structure on kQ.

In the following of this section, we will compare module structures $(kQ,*)$ and $(kQ,\triangleright)$ in term of submodule. For clearness, we give a definition as follows.

Definition 1. Let p be a path in quiver Q,

(1) if there is no arrow α such that $s(\alpha)=t(p)$, p is called a sink path in Q;

(2) if there is no arrow α such that $t(\alpha)=s(p)$, p is called a source path in Q;

(3) if $p=p_2p_1$, p_1 is called a starting subpath of p and p_2 is called a terminal subpath of p.

Theorem 2. Let Q be a finite quiver, then

(1) M is a simple submodule of $(kQ,*)$ if and only if $M=k\{p\}$, where p is a sink path in Q;

(2) M is a simple submodule of $(kQ,\triangleright)$ if and only if $M=k\{v_i\}$, where v_i is a trivial path of length 0 corresponding to vertex i in Q.

Proof.

(1) **Sufficiency.** Let $M=k\{p\}$, where p is a sink path in Q. Then for any path q in Q, we have

$$q * p = \begin{cases} p, & if \quad q = v_{t(p)}; \\ 0, & otherwise. \end{cases}$$

So $q*p \in M$, and $M=k\{p\}$ is a submodule of $(kQ,*)$. Since $M=k\{p\}$ is a space of dimension 1, therefore it is a simple submodule.

Necessity. Let M be a simple submodule of $(kQ,*)$.

First, in M there is no path being a subpath of some oriented cycles. If not, suppose that in M there is a path q which is a subpath of some an oriented cycle, then in this oriented cycle all paths which have q as a starting subpath are in M. Thus, in this oriented cycle all paths with length greater than $l(q)$ and meanwhile having q as a starting subpath can generate a submodule of M. Since this submodule is not M, it is contradictory to the fact that M is a simple module. Then, in M there must be a sink path. Indeed, in M choose any one path q, if q is a sink path, the end. If not, there must be a path q_1 with length greater than 0, such that $q_1q \in M$. If q_1q is a sink path, the end. Otherwise, there also must be a path q_2 with length greater than 0, such that $q_2q_1q \in M$. Since Q is a finite quiver and in M there is no path being a subpath of some oriented cycles, so in M there must be a sink path.

In M, choose any one sink path p, from sufficiency, $k\{p\}$ is a simple submodule of M. Since M is also a simple module, therefore, $M=k\{p\}$.

(2) **Sufficiency.** Suppose $M=k\{v_i\}$, where v_i is a trivial path of length 0 corresponding to vertex i in Q. Then for any path p in Q, we have

$$p \triangleright v_i = \begin{cases} v_i, & if \quad p = v_i; \\ 0, & otherwise. \end{cases}$$

So $M=k\{v_i\}$ is a submodule of $(kQ,\triangleright)$. Since $M=k\{v_i\}$ is a space of dimension 1, therefore it is a simple submodule.

Necessity. Let M be a simple submodule of $(kQ,\triangleright)$. For any path p in M, since $p \triangleright p = v_{t(p)} \in M$, then in M there must be a path of length 0. Choose any one path v_i of length 0 in M, from sufficiency, $k\{v_i\}$ is a simple submodule of M. Since M is also a simple submodule, so $M=k\{v_i\}$.

From Theorem 2, we can get some common properties held by the two module structures on path space.

Corollary 1. Let Q be a finite quiver, then

(1) the simple submodule of $(kQ,*)$ is of dimension 1;

(1') the simple submodule of $(kQ,\triangleright)$ is of dimension 1;

(2) $(kQ,*)$ is a simple module if and only if Q contains only one vertex without loops;

(2') $(kQ,\triangleright)$ is a simple module if and only if Q contains only one vertex without loops.

III. kQ^c-Comodule Structures on Path Space

In this section, also based on the idea of subtraction, we will give a new kQ^c-comodule structure on path space kQ.

Given a finite quiver Q, define a k-linear right kQ^c-coaction on kQ as:

$$\rho_r(p) = \sum_{p'p=q} q \otimes p'.$$

Proposition 2. Under the right coaction ρ_r, path space kQ becomes a right kQ^c-comodule.

Proof. For any path p in quiver Q, it is only need to show that $(\rho_r \otimes id)\rho_r(p) = (id \otimes \Delta)\rho_r(p)$ and $(id \otimes \varepsilon)\rho_r(p) = p \otimes 1$. Indeed,

$$(\rho_r \otimes id)\rho_r(p) = (\rho_r \otimes id)(\sum_{p'p=q} q \otimes p') = \sum_{p'p=q}\sum_{q'q=r} r \otimes q' \otimes p'$$
$$= \sum_{q'p'p=r} r \otimes q' \otimes p' = (id \otimes \Delta)(\sum_{r'p=r} r \otimes r') = (id \otimes \Delta)\rho_r(p)$$
$$(id \otimes \varepsilon)\rho_r(p) = (id \otimes \varepsilon)(\sum_{p'p=q} q \otimes p') = \sum_{p'p=q} q \otimes \varepsilon(p') = p \otimes 1$$

So the right coaction ρ_r can make kQ become a right kQ^c -comodule. Denote this nontrivial right kQ^c -comodule structure on kQ by (kQ,ρ_r). Next, we will show that under isomorphism (kQ,ρ_r) is just happened to be the dual comodule structure of trivial left kQ^a-module on kQ.

Theorem 3. Let Q be a finite quiver without oriented cycles, then on path space right kQ^c -comodule structure (kQ,ρ_r) is isomorphic to the dual comodule structure of trivial left kQ^a -module structure $(kQ,*)$.

Proof. Firstly, since Q is a finite quiver without oriented cycles, then kQ is a finite dimensional space, and so both path algebra and path coalgebra over kQ are of finite dimension.

By Lemma 3 (2), when the trivial left kQ^a -module structure $(kQ,*)$ is dual, kQ becomes right $(kQ^a)^*$ -comodule. Its comodule structure map ρ is given as follows.

For a path p in Q, let $\left\{p_i \middle| p_i = p_i' p, i = 1,2,\cdots,n\right\}$ be a k -basis of $kQ * p$. Then for any path q in Q,

$$q * p = \begin{cases} qp, & if \quad t(p) = s(q), \\ 0, & otherwise, \end{cases} \qquad = \sum_{i=1}^{n} (p_i')^*(q) p_i$$

Hence,

$$\rho(p) = \sum_{i=1}^{n} p_i \otimes (p_i')^* = \sum_{p'p=q} q \otimes (p')^*.$$

By Lemma 2 (2), linear map $\varphi(p^*) = p$ is an isomorphism of coalgebras from $(kQ^a)^*$ to kQ^c. So, under isomorphism, kQ also becomes a right kQ^c -comodule, and the comodule structure map is just happened to be the nontrivial right kQ^c -comodule structure map ρ_r given by

$$\rho_r(p) = \sum_{p'p=q} q \otimes p'.$$

Example 2. Let Q be a finite quiver as $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, then path space $kQ = k\{v_1, v_2, v_3, \alpha, \beta, \beta\alpha\}$, the nontrivial right kQ^c -comodule action ρ_r on kQ is given by:

$$\rho_r(v_1) = v_1 \otimes v_1 + \alpha \otimes \alpha + \beta\alpha \otimes \beta\alpha;$$
$$\rho_r(v_2) = v_2 \otimes v_2 + \beta \otimes \beta;$$
$$\rho_r(v_3) = v_3 \otimes v_3;$$
$$\rho_r(\alpha) = \alpha \otimes v_2 + \beta\alpha \otimes \beta;$$
$$\rho_r(\beta) = \beta \otimes v_3;$$
$$\rho_r(\beta\alpha) = \beta\alpha \otimes v_3.$$

Remark 2. Similarly for a finite quiver Q, we can define a nontrivial left kQ^c -comodule action on path space kQ as

$$\rho_l(p) = \sum_{pp'=q} p' \otimes q.$$

It can also be checked that, when Q has no oriented cycles, this nontrivial left comodule structure is isomorphic to the dual comodule structure of the trivial right kQ^a -module structure on kQ.

In the following of this section, we will compare comodule structures (kQ,ρ_r) and (kQ,Δ) in term of subcomodule.

Theorem 4. Let Q be a finite quiver, then

(1) M is a simple subcomodule of (kQ,ρ_r) if and only if $M = k\{p\}$, where p is a sink path in Q;

(2) M is a simple subcomodule of (kQ,Δ) if and only if $M = k\{v_i\}$, where v_i is a trivial path of length 0 corresponding to vertex i in Q.

Proof.

(1) Sufficiency. Let $M = k\{p\}$, where p is a sink path in Q. Since

$$\rho_r(p) = p \otimes v_{t(p)} \in M \otimes kQ^c,$$

So $M = k\{p\}$ is a subcomodule of (kQ,ρ_r). For that $M = k\{p\}$ is a space of dimension 1, therefore it is a simple subcomodule.

Necessity. Let M be a simple subcomodule of (kQ,ρ_r), then for any path q in M, it has $\rho_r(q) = \sum_{q'q=p} p \otimes q' \in M \otimes kQ^c$, which shows that all paths that have q as a starting subpath must be in M.

Similarly as the proof of Theorem 2 (1), it can be proved that in M there is no path being a subpath of some oriented cycles, and that in M there must be a sink path. Then in M choose any one sink path p, from sufficiency, $k\{p\}$ is a simple subcomodule of M. Since M is also a simple comodule, therefore, $M = k\{p\}$.

(2) **Sufficiency.** Suppose $M = k\{v_i\}$, where v_i is a trivial path of length 0 corresponding to vertex i in Q. Since

$$\Delta(v_i) = v_i \otimes v_i \in M \otimes kQ^c,$$

So $M = k\{v_i\}$ is a subcomodule of (kQ,Δ). For that $M = k\{v_i\}$ is a space of dimension 1, therefore it is a simple subcomodule.

Necessity. Let M be a simple subcomodule of (kQ,Δ). For any path p in M, since

$$\Delta(p) = v_{t(p)} \otimes p + \cdots \in M \otimes kQ^c,$$

then in M there must be a path of length 0. Choose any one path v_i of length 0 in M, from sufficiency, $k\{v_i\}$ is a simple subcomodule of M. Since M is also a simple subcomodule, so $M = k\{v_i\}$. In fact, when Q is a finite quiver without oriented cycles, Theorem 4 can be deduced by Theorem 1-3, since kQ is a finite dimensional space. From Theorem 4, we can get some common properties held by the two comodule structures on path space.

Corollary 2. Let Q be a finite quiver, then

(1) the simple subcomodule of (kQ,ρ_r) is of dimension 1;

(1') the simple subcomodule of (kQ,Δ) is of dimension 1;

(2) (kQ,ρ_r) is a simple comodule if and only if Q contains only one vertex without loops;

(2') (kQ,Δ) is a simple module if and only if Q contains only one vertex without loops.

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