On Four Models of a Three Server Queue with Optional Server Vacations

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***Abstract*—**We study four models of a three server queueing system with Bernoulli schedule optional server vacations. Customers arriving at the system one by one in a Poisson process are provided identical exponential service by three parallel servers according to a first-come, first served queue discipline. In model A, all three servers may be allowed a vacation at one time, in Model B at the most two of the three servers may be allowed a vacation at one time, in model C at the most one server is allowed a vacation, and in model D no server is allowed a vacation. We study steady the state behavior of the four models and obtain steady state probability generating functions for the queue size at a random point of time for all states of the system. In model D, a known result for a three server queueing system without server vacations is derived.

***Keywords*—**A three server queue, Bernoulli schedule server vacations, queue size distribution at a random epoch, steady state.

I. Introduction

LITERATURE on queues is full of papers on vacation queues with many different vacation policies. A majority of these papers are on a single server queueing systems compared to two-server or multi-server systems. To mention a few single server queueing systems with vacations, we refer the reader to [1], [9] [10] and [6], [7]. Some papers on two server queueing systems and multi-servers queuing systems with vacations have been studied by [2]-[5], and [8].

In this paper, we study four models of a queueing system with three parallel servers which are entitled to take an optional Bernoulli schedule vacation on completion of each service. In model A, we assume that all three servers are allowed to take a vacation after completion of service which means that at times the system may be empty with no server working. In model B, we assume that at the most two servers can be on vacation at a time, which means that there is always at least one server available in the system at all times. In model C, at the most one server may be allowed to take a vacation, which means that at least two servers are always available in system. In model D, we assume that no server is allowed to take a vacation, which means that all three servers are always available in the system. We have assumed that servers follow a Bernoulli schedule server vacations, which means that after completion of a service, if a server is eligible to take a vacation then it may take a vacation with probability p or else with probability 1-p may opt to remain in the system.

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Note that the multi-server queueing systems studied by [2] and [8] follow different vacation policies.

II. Model A: All Three Servers Can Be Away on Vacation at a Time

A. States of the System and the Underlying Assumptions

- $A_n(t)$: The probability that at time t, all three servers are available in the system, and there are $n \geq 0$ customers in the system including those in service, if any,
- $T_n(t)$: The probability that at time t, only two of the three servers are available in the system, one is on vacation and there are $n \geq 0$ customers in the system including those in service, if any,
- $O_n(t)$: The probability that at time t, only one of the three servers is available in the system, the other two servers are on vacation and there are $n \geq 0$ customers in the system including those in service, if any.
- $N_n(t)$: The probability that at time t, none of the three servers are available in the system and there are $n \geq 0$ customers waiting in the queue

As soon as a service is complete, the server which completed the service is eligible to take a vacation of random length with probability p or with probability 1-p, it may opt to remain in the system.

Let $\eta\, dt$ be the first order probability that a server's vacation will complete during the interval $(t, t+dt]$.

If two servers are already on vacation, then the third server cannot take a vacation, which implies that there is always at one server available in the system.

The three parallel servers provide one by one service to customers, based on a first-come, first-served basis according to an exponential distribution with mean service time $\frac{1}{\mu}$, $\mu > 0$.

Arrivals are according to Poisson process with mean rate $\lambda (> 0)$.

B. Steady State Equations

Assuming that the steady state exists, we define $A_n = \lim_{t\to\infty} A_n(t)$, $T_n = \lim_{t\to\infty} T_n(t)$, $O_n = \lim_{t\to\infty} O_n(t)$ and $N_n = \lim_{t\to\infty} N_n(t)$ to be the corresponding steady states of the system. Employing the usual probability arguments based on the underlying assumptions, we obtain the following set of forward equations:

$$(\lambda+3\mu)A_n = \lambda A_{n-1} + \eta T_n + 3\mu(1-p)A_{n+1}, n \geq 3, \quad (1)$$

$$(\lambda+2\mu)A_2 = \lambda A_1 + \eta T_2 + 3\mu(1-p)A_3, \quad (2)$$

$$(\lambda+\mu)A_1 = \lambda A_0 + \eta T_1 + 2\mu(1-p)A_2, \quad (3)$$

$$\lambda A_0 = \eta T_0 + \mu(1-p)A_1, \quad (4)$$

$$(\lambda+2\mu+\eta)T_n = \lambda T_{n-1} + 3\mu p A_{n+1} + 2\mu(1-p)T_{n+1} + 2\eta O_n, \; n \geq 2, \quad (5)$$

$$(\lambda+\mu+\eta)T_1 = \lambda T_0 + 2\mu p A_2 + 2\mu(1-p)T_2, +2\eta O_1 \quad (6)$$

$$(\lambda+\eta)T_0 = \mu p A_1 + \mu(1-p)T_1, +2\eta O_0 \quad (7)$$

$$(\lambda+\mu+2\eta)O_n = \lambda O_{n-1} + \mu(1-p)O_{n+1} + 2\mu p T_{n+1} + 3\eta N_n, \; n \geq 1, \quad (8)$$

$$(\lambda+2\eta)O_0 = \mu(1-p)O_1 + 2\mu p T_1 + 3\eta N_0, n \geq 1, \quad (9)$$

$$(\lambda+3\eta)N_n = \lambda N_{n-1} + \mu p O_{n+1}, n \geq 1, \quad (10)$$

$$(\lambda+3\eta)N_0 = \mu p O_1, n \geq 1, \quad (11)$$

C. Steady State Average Queue Size at a Random Point of Time

Define the following probability generating functions:

$$A(z) = \sum_{n=0}^{\infty} A_n z^n, \; T(z) = \sum_{n=0}^{\infty} T_n z^n, \; O(z) = \sum_{n=0}^{\infty} O_n z^n,$$
$$N(z) = \sum_{n=0}^{\infty} N_n z^n \; |z|<1 \quad (12)$$

We perform $\sum_{n=3}^{\infty}(1)z^{n+1} + (2)z^3 + (3)z^2 + (4)z$, use (12) and simply to obtain

$$\begin{aligned}&\left[z\left(\lambda-\lambda z+3\mu\right)-3\mu(1-p)\right]A(z) = \eta z T(z)\\ &+\mu\left(z-(1-p)\right)A_2 z^2 + 2\mu\left(z-(1-p)\right)A_1 z - 3\mu z(z-1)A_0\end{aligned} \quad (13)$$

Similarly, we perform $\sum_{n=2}^{\infty}(5)z^{n+1} + (6)z^2 + (7)z$, use (12) and simply to obtain

$$\begin{aligned}&\left[z\left(\lambda-\lambda z+2\mu+\eta\right)-2\mu(1-p)\right]T(z) =\\ &3\mu p A(z) + \mu\left(z-(1-p)\right)T_1 z + 2\mu T_0 z\\ &-\mu p A_2 z^2 - 2\mu p A_1 z + 2\eta z O(z).\end{aligned} \quad (14)$$

and again, we perform $\sum_{n=1}^{\infty}(8)z^{n+1} + (9)z$, use (12) and simply to obtain

$$\begin{aligned}&\left[z\left(\lambda-\lambda z+\mu+2\eta\right)-\mu(1-p)\right]O(z) =\\ &2\mu p T(z) - 2\mu p T_0 + \mu(z-(1-p))O_0 + 3\eta z N(z)\end{aligned} \quad (15)$$

and yet again, we perform $\sum_{n=1}^{\infty}(10)z^{n+1} + (11)z$, use (12) and simply to obtain

$$\left[z\left(\lambda-\lambda z+3\eta\right)-\mu p\right]N(z) = \mu p O(z) - \mu p O_0$$

We re-write (11), (12) and (13) in matrix form as follows:

$$\begin{bmatrix} g_1(z) & -\eta z & 0 & 0 \\ -3\mu p & g_2(z) & -2\eta z & 0 \\ 0 & -2\mu p & g_3(z) & -3\eta z \\ 0 & 0 & -\mu p & g_4(z) \end{bmatrix}\begin{bmatrix} A(z) \\ T(z) \\ O(z) \\ N(z) \end{bmatrix} = \begin{bmatrix} K_1(z) \\ K_2(z) \\ K_3(z) \\ -\mu p O_0 \end{bmatrix} \quad (16)$$

where;

$$g_1(z) = z\left(\lambda-\lambda z+3\mu\right)-3\mu(1-p),$$

$$g_2(z) = z\left(\lambda-\lambda z+2\mu+\eta\right)-2\mu(1-p)$$

$$g_3(z) = z\left(\lambda-\lambda z+\mu+2\eta\right)-\mu(1-p)$$

$$g_4(z) = \left[z\left(\lambda-\lambda z+3\eta\right)-\mu p\right]$$

$$K_1(z) = \mu\left(z-(1-p)\right)A_2 z^2 + 2\mu\left(z-(1-p)\right)A_1 z - 3\mu z(z-1)A_0$$

$$K_2(z) = \mu\left(z-(1-p)\right)T_1 z + 2\mu T_0 z - \mu p A_2 z^2 - 2\mu p A_1 z$$

$$K_3(z) = -2\mu p T_0 + \mu(z-(1-p))O_0.$$

Now, we solve (16) for $A(z), T(z), O(z)$ and $N(z)$ and obtain

$$A(z) = \frac{N_1(z)}{D(z)}, \; T(z) = \frac{N_2(z)}{D(z)}, \; O(z) = \frac{N_3(z)}{D(z)} \text{ and}$$
$$N(z) = \frac{N_4(z)}{D(z)} \quad (17)$$

where

$$\begin{aligned}&N_1(z) = K_1(z)\left[g_2(z)g_3(z)g_4(z) - 3\mu p\eta z g_2(z) - 4\mu p\eta z g_4(z)\right]\\ &+K_2(z)\left[\eta z g_3(z)g_4(z) - 3\mu p\eta^2 z^2\right] + K_3(z)\left(2\eta^2 z^2 g_4(z)\right)\\ &-6\mu p\eta^3 z^3\end{aligned}$$

$$N_2(z) = K_1(z)\left[3\mu p g_3(z) g_4(z) - 9\mu^2 p^2 \eta z\right] + K_2(z)\left[g_1(z) g_3(z) g_4(z) - 3\mu p\eta z g_1(z)\right] + 2\eta z K_3(z) g_1(z) g_4(z) - 6\mu p\eta^2 z^2 g_1(z) O_0$$

$$N_3(z) = 6\mu^2 p^2 g_4(z) K_1(z) + 2\mu p\, g_1(z) g_4(z) K_2(z) + K_3(z)\left[g_1(z) g_2(z) g_4(z) - 6\mu p\eta g_4\right] + O_0\left[9\mu^2 p^2 \eta^2 z^2 - 3\mu p\eta z g_1(z) g_2(z)\right]$$

$$N_4(z) = 18\mu^3 p^3 K_1(z) - 2\mu p g_1(z) g_3(z) K_2(z) + K_3(z)\left[\mu p g_1(z) g_2(z) - 6\mu^2 p^2 \eta z\right] - O_0\left[\mu p g_1(z) g_2(z) g_3(z) - 4\mu^2 p^2 \eta z g_1(z) - 3\mu^2 p^2 \eta z g_3(z)\right]$$

$$D(z) = g_1(z) g_2(z) g_3(z) g_4(z) - 3\mu p\eta z g_1(z) g_2(z) - 4\mu p\eta z g_1(z) g_4(z) - 3\mu p\eta z g_3(z) g_4(z) + 9\mu^2 p^2 \eta^2 z^2$$

Now, we have to determine six unknown probabilities, namely, A_2, A_1, A_0, T_1, T_0 and O_0 which appear in the right side of terms $K_1(z)$, $K_2(z)$ and $K_3(z)$ defined above. In order to achieve this, we proceed as follows:

First of all, we have to utilize the normalizing condition

$$A(1) + T(1) + O(1) + N(1) = 1 \tag{18}$$

Next, it can be verified that the common denominator $D(z)$ found above is zero at z=1. Since each of the generating functions $A(z)$, $T(z)$ $O(z)$ and $N(z)$ are analytic inside the unit circle $|z|=1$ and have a finite value at z=1, therefore their numerators $N_1(z)$, $N_2(z)$, $N_3(z)$ and $N_4(z)$ must vanish at z=1, thus giving us 4 equations in terms of the unknown probabilities A_2, A_1, A_0, T_1, T_0 and O_0. In addition to the normalizing condition (18) and these four equations, we have equations (3), (4) and (7) in terms of the same six unknown probabilities A_2, A_1, A_0, T_1, T_0 and O_0. Thus a total of 8 equations are sufficient to determine these six unknown probabilities.

III. Model B: Up to Two of the Three Servers Can Be Away on Vacation at a Time

A. States of the System and the Underlying Assumptions

$A_n(t)$: The probability that at time t, all three servers are available in the system and there are $n \geq 0$ customers in the system including those in service, if any,

$T_n(t)$: The probability that at time t, only two of the three servers are available in the system, one is on vacation and there are $n \geq 0$ customers in the system including those in service, if any,

$O_n(t)$: The probability that at time t, only one of the three servers is available in the system, the other two servers are on vacation and there are $n \geq 0$ customers in the system including those in service, if any.

As soon as a service is complete, the server which completed the service is eligible to take a vacation of random length with probability p or else with probability 1-p he may choose to stay in the system. However, if two servers are already on vacation, then the third server cannot take a vacation, which implies that there is always at one server available in the system.

Let $\eta\, dt$ be the first order probability that a server's vacation will complete during the interval $(t, t+dt]$.

The three parallel servers provide one by one service to customers, based on a first-come, first served basis according to an exponential distribution with mean service time $\frac{1}{\mu}$, $\mu > 0$.

Arrivals occur one by one according to Poisson process with mean rate $\lambda (> 0)$.

B. Steady State Equationss

$$(\lambda + 3\mu) A_n = \lambda A_{n-1} + \eta T_n + 3\mu(1-p) A_{n+1}, n \geq 3, \tag{19}$$

$$(\lambda + 2\mu) A_2 = \lambda A_1 + \eta T_2 + 3\mu(1-p) A_3, \tag{20}$$

$$(\lambda + \mu) A_1 = \lambda A_0 + \eta T_1 + 2\mu(1-p) A_2, \tag{21}$$

$$\lambda A_0 = \eta T_0 + \mu(1-p) A_1, \tag{22}$$

$$(\lambda + 2\mu + \eta) T_n = \lambda T_{n-1} + 3\mu p A_{n+1} + 2\mu(1-p) T_{n+1} + 2\eta O_n, \quad n \geq 2, \tag{23}$$

$$(\lambda + \mu + \eta) T_1 = \lambda T_0 + 2\mu p A_2 + 2\mu(1-p) T_2, + 2\eta O_1 \tag{24}$$

$$(\lambda + \eta) T_0 = \mu p A_1 + \mu(1-p) T_1, + 2\eta O_0 \tag{25}$$

$$(\lambda + \mu + 2\eta) O_n = \lambda O_{n-1} + \mu\, O_{n+1} + 2\mu\, p T_{n+1}, n \geq 1 \tag{26}$$

$$(\lambda + 2\eta) O_0 = \mu\, O_1 + 2\mu p T_1, n \geq 1 \tag{27}$$

Define the following probability generating functions:

$$A(z) = \sum_{n=0}^{\infty} A_n z^n, \; T(z) = \sum_{n=0}^{\infty} T_n z^n, \; O(z) = \sum_{n=0}^{\infty} O_n z^n, \quad |z| < 1. \tag{28}$$

C. Steady State Average Queue Size at a Random Point of Time

Proceeding as we did in Model A, we get

$$\left[z\left(\lambda-\lambda z+3\mu\right)-3\mu(1-p)\right]A(z)=\eta zT(z)$$
$$+\mu\left(z-(1-p)\right)A_2z^2+2\mu\left(z-(1-p)\right)A_1z-3\mu z(z-1)A_0 \tag{29}$$

$$\left[z\left(\lambda-\lambda z+2\mu+\eta\right)-2\mu(1-p)\right]T(z)=3\mu pA(z)$$
$$+\mu\left(z-(1-p)\right)T_1z+2\mu T_0z-\mu p\,A_2z^2-2\mu p\,A_1z+2\eta zO(z) \tag{30}$$

$$\left[z\left(\lambda-\lambda z+\mu+2\eta\right)-\mu\right]O(z)=$$
$$2\mu pT(z)-2\mu\,pT_0+\mu(z-1)O_0 \tag{31}$$

We re-write (29), (30) and (31) in matrix form as follows:

$$\begin{bmatrix} g_1(z) & -\eta z & 0 \\ -3\mu p & g_2(z) & -2\eta z \\ 0 & -2\mu p & g_3(z) \end{bmatrix}\begin{bmatrix} A(z) \\ T(z) \\ O(z) \end{bmatrix}=\begin{bmatrix} K_1(z) \\ K_2(z) \\ K_3(z) \end{bmatrix}, \tag{32}$$

where;

$$g_1(z)=z\left(\lambda-\lambda z+3\mu\right)-3\mu(1-p),$$

$$g_2(z)=z\left(\lambda-\lambda z+2\mu+\eta\right)-2\mu(1-p)$$

$$g_3(z)=z\left(\lambda-\lambda z+\mu+2\eta\right)-\mu$$

$$K_1(z)=\mu\left(z-(1-p)\right)A_2z^2+2\mu\left(z-(1-p)\right)A_1z-3\mu z(z-1)A_0$$

$$K_2(z)=\mu\left(z-(1-p)\right)T_1z+2\mu T_0z-\mu p\,A_2z^2-2\mu p\,A_1z$$

$$K_3(z)=-2\mu\,pT_0+\mu(z-1))O_0.$$

Now, we solve (32) for $A(z), T(z)$ and $O(z)$ and obtain

$$A(z)=\frac{N_1(z)}{D(z)},\; T(z)=\frac{N_2(z)}{D(z)} \text{ and } O(z)=\frac{N_3(z)}{D(z)} \tag{33}$$

where;

$$N_1(z)=K_1(z)\left[g_2(z)g_3(z)-4\mu p\eta z\right]+\eta z\left[K_2(z)g_3(z)+2\eta z\,K_3(z)\right]$$

$$N_2(z)=3\mu p\,K_1(z)g_3(z)+K_2(z)g_1(z)g_3(z)+2\eta zK_3(z)g_1(z)$$

$$N_3(z)=g_1(z)g_2(z)g_3(z)+2\mu pg_1(z)K_2(z)$$
$$-3\mu p\eta zK_3(z)+6\mu^2p^2K_1(z)$$

$$D(z)=g_1(z)g_2(z)g_3(z)-4\mu p\eta z\,g_1(z)-3\mu p\eta z\,g_3(z)$$

All six unknown probabilities, namely, A_2, A_1, A_0, T_1, T_0 and O_0 which appear in the right side of terms $K_1(z)$, $K_2(z)$ and $K_3(z)$ can be determined as in Model A.

IV. Model C: Only One of the Three Servers Can Be Away on Vacation at a Time

A. States of the System and the Underlying Assumptions

$A_n(t)$: The probability that at time t, all three servers are available in the system and there are $n \geq 0$ customers in the system including those in service, if any,

$T_n(t)$: The probability that at time t, only two of the three servers are available in the system, one is on vacation and there are $n \geq 0$ customers in the system including those in service, if any,

Let $\eta\,dt$ be the first order probability that a server's vacation will complete during the interval $(t, t+dt]$.

As soon as a service is complete, the server which completed the service is eligible to take a vacation of random length with probability p, or else with probability 1-p, it may choose to stay in the system. However, if one server is already on vacation, then the second server cannot take a vacation, which implies that there are always at least two servers available in the system.

If one server is already on vacation, then another server cannot take a vacation, which implies that there are always at two servers available in the system.

The three parallel servers provide one by one service to customers, based on a first-come, first served basis according to an exponential distribution with mean service time $\frac{1}{\mu}$, $\mu > 0$.

Arrivals occur according to Poisson process with mean rate $\lambda (> 0)$

B. Steady State Equationss

$$(\lambda+3\mu)A_n=\lambda A_{n-1}+\eta T_n+3\mu(1-p)A_{n+1}, n\geq 3, \tag{34}$$

$$(\lambda+2\mu)A_2=\lambda A_1+\eta T_2+3\mu(1-p)A_3, \tag{35}$$

$$(\lambda+\mu)A_1=\lambda A_0+\eta T_1+2\mu(1-p)A_2, \tag{36}$$

$$\lambda A_0=\eta T_0+\mu(1-p)A_1, \tag{37}$$

$$(\lambda+2\mu+\eta)T_n=\lambda T_{n-1}+3\mu pA_{n+1}+2\mu(1-p)T_{n+1}, \; n\geq 2, \tag{38}$$

$$(\lambda+\mu+\eta)T_1=\lambda T_0+2\mu pA_2+2\mu(1-p)T_2, \tag{39}$$

$$(\lambda+\eta)T_0=\mu pA_1+\mu(1-p)T_1, \tag{40}$$

Define the following probability generating functions:

$$A(z)=\sum_{n=0}^{\infty}A_n\,z^n,\; T(z)=\sum_{n=0}^{\infty}T_n\,z^n,\; O(z)=\sum_{n=0}^{\infty}O_nz^n,$$
$$|z|<1. \tag{41}$$

C. Steady State Average Queue Size at a Random Point of Time

Proceeding as we did in Model A, we get

$$\left[z\left(\lambda-\lambda z+3\mu\right)-3\mu(1-p)\right]A(z)=\eta zT(z) +\mu\left(z-(1-p)\right)A_2z^2+2\mu\left(z-(1-p)\right)A_1z-3\mu z(z-1)A_0 \quad (42)$$

$$\left[z\left(\lambda-\lambda z+2\mu+\eta\right)-2\mu(1-p)\right]T(z)= 3\mu pA(z)+\mu\left(z-(1-p)\right)T_1z+2\mu T_0z-\mu p\,A_2z^2-2\mu p\,A_1z \quad (43)$$

We re-write (42) and (43) in matrix form as follows:

$$\begin{bmatrix} g_{1(z)} & -\eta z \\ -3\mu p & g_2(z)\end{bmatrix}\begin{bmatrix} A(z) \\ T(z)\end{bmatrix}=\begin{bmatrix} K_1(z) \\ K_2(z)\end{bmatrix}, \quad (44)$$

where;

$$g_1(z)=z\left(\lambda-\lambda z+3\mu\right)-3\mu(1-p),$$

$$g_2(z)=z\left(\lambda-\lambda z+2\mu+\eta\right)-2\mu(1-p)$$

$$K_1(z)=\mu\left(z-(1-p)\right)A_2z^2+2\mu\left(z-(1-p)\right)A_1z-3\mu z(z-1)A_0$$

$$K_2(z)=\mu\left(z-(1-p)\right)T_1z+2\mu T_0z-\mu p\,A_2z^2-2\mu p\,A_1z$$

Now, we solve (32) for $A(z), T(z)$ and $O(z)$ and obtain

$$A(z)=\frac{N_1(z)}{D(z)} \text{ and } T(z)=\frac{N_2(z)}{D(z)} \quad (45)$$

where;

$$N_1(z)=K_1(z)\,g_2(z)+\eta z\,K_2(z)$$

$$N_2(z)=3\mu p\,K_1(z)+g_1(z)K_2(z)$$

$$D(z)=g_1(z)g_2(z)-3\mu p\,\eta z$$

All five unknown probabilities, namely, A_2, A_1, A_0, T_1 and T_0 which appear in the right side of terms $K_1(z)$ and $K_2(z)$ can be determined as in Model A.

V. Model D: No Server Can Be Away on Vacation (All Three Servers Are Always Available in the System)

In this case, we let p=0, $T_n=0$, $O_n=0$ and $N_n=0$, in the main results of any of the above three models. Consequently, we get $T(z)=0$, $O(z)=0$ and $N(z)=0$ and

$$\left[z\left(\lambda-\lambda z+3\mu\right)-3\mu\right]A(z)=\mu\left(z-1\right)A_2z^2+2\mu\left(z-1\right)A_1z -3\mu z(z-1)A_0z$$

This gives

$$A(z)=\frac{\mu(z-1)A_2z^2+2\mu(z-1)A_1z-3\mu z(z-1)A_0z}{\left[z\left(\lambda-\lambda z+3\mu\right)-3\mu\right]}. \quad (46)$$

Alternatively, the above substitutions in the basic equations (1) to (11) gives

$$(\lambda+3\mu)A_n=\lambda A_{n-1}+3\mu A_{n+1}, n\ge 3, \quad (47)$$

$$(\lambda+2\mu)A_2=\lambda A_1++3\mu A_3, \quad (48)$$

$$(\lambda+\mu)A_1=\lambda A_0+2\mu A_2, \quad (49)$$

$$\lambda A_0=\mu A_1, \quad (50)$$

Solving (47)-(50) recursively, we obtain

$$A_n=\left(\frac{1}{n!}\right)\left(\frac{\lambda}{\mu}\right)^n\left(\frac{1}{\sum_0^\infty\left(\frac{1}{n!}\right)\left(\frac{\lambda}{\mu}\right)^n}\right), \; n=0,1,2,\ \ldots \quad (51)$$

The result in (51) is a known result of the M/M/3 queue.

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