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The New Relative Efficiency Based on the Least Eigenvalue in Generalized Linear Model

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***Abstract*—**A new relative efficiency is defined as LSE and BLUE in the generalized linear model. The relative efficiency is based on the ratio of the least eigenvalues. In this paper, we discuss about its lower bound and the relationship between it and generalized relative coefficient. Finally, this paper proves that the new estimation is better under Stein function and special condition in some degree.

***Keywords*—**Generalized linear model, generalized relative coefficient, least eigenvalue, relative efficiency.

I. Introduction

THE generalized Markov-Gauss model we treat here is linear weighted regression, described by

$$\begin{cases} Y = X\beta + \varepsilon, \\ E(\varepsilon) = 0, \\ Cov(\varepsilon) = \sigma^2\Sigma. \end{cases} \tag{1}$$

where Y is the $n\times 1$ observation vector, X is a $n\times p$ column full rank design matrix which we are known, β is a $p\times 1$ unknown parameter vector, ε is the $n\times 1$ observation vector, and Σ is the $n\times n$ positive definite co variance matrix and the rank is $r(\Sigma)=m\le n$. σ^2 is unknown parameter.

In practical application, it is a very common problem to Σ is unknown or computational complexity. So that people often use LSE instead of BLU. But which will bring some damage to the estimates. In order to measure the size of the loss, the relative efficiency is cited. Common examples:[1]

$$e_1(\hat{\beta})=\frac{\left|Cov(\beta^*)\right|}{\left|Cov(\hat{\beta})\right|},\ e_2(\hat{\beta})=\frac{tr(Cov\beta^*)}{tr(Cov\hat{\beta})},\ e_3(\hat{\beta})=\frac{\left\|tr(Cov\beta^*)\right\|}{\left\|tr(Cov\hat{\beta})\right\|}$$

In this paper, a new relative efficiency is defined based on the minimum positive eigenvalue which is $e_4(\hat{\beta})=\dfrac{\lambda_p(Cov\beta^*)}{\lambda_p(Cov\hat{\beta})}$. The lower bound and the relation of some generalized correlation coefficients are studied.

By the algebraic knowledge, when A is n order real non-negative definite matrix, the order of the eigenvalues is satisfied with

$$\lambda_i(A^2)=\lambda_i^2(A), i=1,2,\cdots,n;\ \lambda_i(A^{-1})=\lambda_{n-i+1}^{-1}(A), i=1,2,\cdots,n;$$

If A is $n\times m$ order real matrix, B is $m\times n$ order real matrix, and

$$n\ge m, r(BA)=r,$$

then

$$\lambda_i(AB)=\lambda_i(BA), i=1,2,\cdots,r$$

II. The Lower Bound of Relative Efficiency

According to the least square unified theory of estimation by the India statistician Rao C. R. established [2], the BLU of β is

$$\beta^*=(X'T^-X)^-X'T^-Y,$$

where $T=\Sigma+XUX'$, U is symmetric matrix and $r(T)=r(\Sigma\vdots X)$, T^- is a generalized inverse of T when

$$r(X)=p,\ \Sigma\ge 0,\ \beta^*=(X'T^-X)^-X'T^-Y$$

and

$$Cov(\beta^*)=\sigma^2(X'T^-X)^{-1}X'T^-\Sigma T^-X(X'T^-X)^{-1}$$ [3], [4]

In this paper

$$U=d^2I, d\ne 0, \mu(X)\subset\mu(T),\ \mu(\Sigma)\subset\mu(T),$$

so it can replace T^+ in T^- that is

$$\beta^*=(X'T^+X)^{-1}X'T^+Y,$$

$$Cov(\beta^*)=\sigma^2(X'T^+X)^{-1}X'T^+\Sigma T^+X(X'T^+X)^{-1}. \tag{2}$$

the LSE of the estimable function β^* in model (2) is

$$\hat{\beta}=(X'X)^{-1}X'Y$$

and

$$Cov(\hat{\beta})=\sigma^2(X'X)^{-1}X'\Sigma X(X'X)^{-1}$$

Lemma 1. Assume that A and B are n order real symmetric matrix, $B\ge 0$, then

$$\lambda_n(B)\lambda_i(A^2)\le\lambda_i(ABA)\le\lambda_1(B)\lambda_i(A^2), i=1,2,\cdots,n$$

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where $\lambda_i(A)$ represents the i of the A matrix.

Inference 1. Assume that A and B are n order real non-negative definite matrix, then

$$\lambda_n(A^2)\lambda_i(B) \le \lambda_i(ABA) \le \lambda_1(A)\lambda_i(B),\ i=1,2,\cdots,n$$

where $\lambda_i(A)$ represents the i of the A matrix.

Inference 2. Assume that A and B are n order real non-negative definite matrix, then

$$\lambda_n(A)\lambda_i(B) \le \lambda_i(AB) \le \lambda_1(A)\lambda_i(B),$$

$$\lambda_i(A)\lambda_n(B) \le \lambda_i(AB) \le \lambda_i(A)\lambda_1(B),$$

$$i=1,2,\cdots,n .$$

Lemma 2. Assume that A is $n\times n$ order *Hermite* matrix, U is $n\times n$ order orthogonal matrix ,$U'U=I_k$,then

$$\lambda_{n-k+i}(A) \le \lambda_i(U'AU) \le \lambda_i(A),$$

$$i=1,2,\cdots,k .$$

The proofs of the above lemmas can be found in [6].

Theorem 1. In model of (1), if there are $Cov(\tilde{\beta}_1) > Cov(\tilde{\beta}_2) \ge 0$ for any two unbiased estimators, when $r(X)=p$, and $\Sigma \ge 0$,then $e_4(\tilde{\beta}_1) < e_4(\tilde{\beta}_2)$.

Proof. According to $Cov(\tilde{\beta}_1) > Cov(\tilde{\beta}_2) \ge 0$ it is obvious that

$$\lambda_p(Cov(\tilde{\beta}_1)) > \lambda_p(Cov(\tilde{\beta}_2)) > 0 ,$$

$$\frac{\lambda_v(Cov\beta^*)}{\lambda_v(Cov\tilde{\beta}_1)} < \frac{\lambda_v(Cov\beta^*)}{\lambda_v(Cov\tilde{\beta}_2)}, e_4(\tilde{\beta}_1) < e_4(\tilde{\beta}_2) .$$

Theorem 2. In model of (1),

$$r(X)=p, \Sigma \ge 0,\ \hat{\beta}=(X'X)^{-1}X'Y,$$
$$\beta^*=(X'T^+X)^{-1}X'T^+Y,$$

then

$$\max(\frac{\mu_m^2\lambda_m(\Sigma^*)}{\mu_1^2\lambda_1(\Sigma^*)}, \frac{\mu_m^2\delta_p\lambda_m(\Sigma^*)}{\mu_1^2\delta_1\lambda_p(\Sigma^*)}) \le e_4(\hat{\beta}) \le 1.\ \delta_1 \ge \delta_2 \ge \cdots \ge \delta_p$$

is the sequential characteristic root of $X'X$. $\lambda_i=\lambda_i(\Sigma)$, $\mu_i=\mu_i(T)$.

Proof. In model of (1),

$$X'X \ge 0, \Sigma \ge 0, T \ge 0$$

Thus,

$$P'TP = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix},$$

where $P_{n\times n}$ is $n\times n$ order orthogonal matrix, $\Delta = diag(\mu_1,\mu_2,\cdots\mu_m)$.

$$\mu(X) \subset \mu(T), X=TC,$$

Thus

$$PX = PTP'PC = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} PC \triangleq \begin{pmatrix} W \\ 0 \end{pmatrix} ,$$

$$W = P_1\Lambda^{\frac{1}{2}}P_2' ,$$

where C is $n\times p$ order matrix, W is $m\times p$ order matrix, P_1 is $m\times p$ order orthogonal matrix, P_2 is $p\times p$ order orthogonal matrix,

$$r(X)=r(W)=p ,$$

$$\Lambda^{\frac{1}{2}} = diag(\delta_1^{\frac{1}{2}}, \delta_2^{\frac{1}{2}}, \cdots\delta_p^{\frac{1}{2}}) ,$$

$$X'T^+X = X'P'PT^+P'PX = (PX)'PT^+PPX$$
$$= (W' \quad 0)\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} W \\ 0 \end{pmatrix} = W'\Delta^{-1}W ,$$

$$PTP' = P(\Sigma + X'd^2X)P' = P\Sigma P' + d^2PX'XP',$$

Thus

$$P\Sigma P' = PTP' - d^2PX'XP' = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} WW' & 0 \\ 0 & 0 \end{pmatrix} \triangleq \begin{pmatrix} \Sigma^* & 0 \\ 0 & 0 \end{pmatrix},$$

$$X'T^+\Sigma T^+X = X'P'PT^+P'P\Sigma P'PT^+P'PX$$
$$= \begin{pmatrix} W' & 0 \end{pmatrix}\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} W \\ 0 \end{pmatrix} = W'\Delta^{-1}\Sigma^*\Delta^{-1}W ,$$

then

$$\frac{1}{\sigma^2}Cov(\beta^*) = (X'T^+X)^{-1}X'T^+\Sigma T^+X(X'T^+X)^{-1}$$
$$= P_2\Lambda^{-\frac{1}{2}}R^{-1}SR^{-1}\Lambda^{-\frac{1}{2}}P_2'$$

where

$$R=P_1'\Delta^{-1}P_1, S=P_1'\Delta^{-1}\Sigma^*\Delta^{-1}P_1.$$

In a similar way:

$$\frac{1}{\sigma^2}Cov(\hat{\beta}) = P_2\Lambda^{-\frac{1}{2}}P_1'\Sigma^*P_1\Lambda^{-\frac{1}{2}}P_2'$$

$$\frac{1}{\sigma^2}\lambda_p(Cov\beta^*) = \lambda_p(P_2\Lambda^{-\frac{1}{2}}R^{-1}SR^{-1}\Lambda^{-\frac{1}{2}}P_2^{'})$$

$$= \lambda_p(R^{-1}SR^{-1}\Lambda^{-1}) \geq \lambda_p(\Lambda^{-1})\lambda_p(R^{-1}SR^{-1})\lambda_1^{-2}(R)\lambda_p(S)\delta_1^{-1}$$

$$\lambda_1(R) = \lambda_1(P_1^{'}\Delta^{-1}P_1) \leq \lambda_1(\Delta^{-1}) = \mu_m^{-1}$$

$$\lambda_p(S) = \lambda_p(P_1^{'}\Delta^{-1}\Sigma\Delta^{-1}P_1) = \lambda_m(\Delta^{-1}\Sigma^*\Delta^{-1})$$

$$\geq \lambda_m(\Sigma^*)\lambda_m^2(\Delta^{-1}) = \frac{\lambda_m(\Sigma^*)}{\mu_1^2}$$

then

$$\frac{1}{\sigma^2}\lambda_p(Cov\beta^*) \geq \frac{\mu_m^2\lambda_m(\Sigma^*)}{\mu_1^2\delta_1} \tag{4}$$

In the similar way:

$$\frac{1}{\sigma^2}\lambda_p(Cov(\hat{\beta})) = \lambda_p(P_2\Lambda^{-\frac{1}{2}}P_1^{'}\Sigma^*P_1\Lambda^{-\frac{1}{2}}P_2^{'})$$

$$\leq \lambda_p(\Lambda^{-1})\lambda_1(P_1^{'}\Sigma^*P_1) \leq \lambda_1^{-1}(\Lambda)\lambda_1(\Sigma^*) = \frac{\lambda_1(\Sigma^*)}{\delta_1} \tag{5}$$

then it is readily verified that

$$e_4(\hat{\beta}) \geq \frac{\mu_m^2\lambda_m(\Sigma^*)}{\mu_1^2\lambda_1(\Sigma^*)}$$

$$\frac{1}{\sigma^2}\lambda_p(Cov(\hat{\beta})) = \lambda_p(P_2\Lambda^{-\frac{1}{2}}P_1^{'}\Sigma^*P_1\Lambda^{-\frac{1}{2}}P_2^{'})$$

$$\leq \lambda_1(\Lambda^{-1})\lambda_p(P_1^{'}\Sigma^*P_1) \leq \lambda_p^{-1}(\Lambda)\lambda_p(\Sigma^*) = \frac{\lambda_p(\Sigma^*)}{\delta_p}$$

$$e_4(\hat{\beta}) \geq \frac{\mu_m^2\lambda_m(\Sigma^*)}{\mu_1^2\lambda_1(\Sigma^*)}$$ [5], [6]

III. The Relationship between the Lower Bound of Relative Efficiency and the Generalized Correlation Coefficient

In the model of (1),

$$r(X) = p, \Sigma \geq 0,$$

$$Cov\begin{pmatrix}\beta^*\\ \hat{\beta}\end{pmatrix} = \sigma^2\begin{pmatrix}(X^{'}T^+X)^{-1}X^{'}T^+\Sigma T^+X(X^{'}T^+X)^{-1} & (X^{'}T^+X)^{-1}X^{'}T^+\Sigma X(X^{'}X)^{-1}\\ (X^{'}X)^{-1}X^{'}\Sigma X(X^{'}T^+X)^{-1} & (X^{'}X)^{-1}X^{'}\Sigma X(X^{'}X)^{-1}\end{pmatrix}$$

$$= \sigma^2\begin{pmatrix}A & C\\ C^{'} & B\end{pmatrix}$$

then

$$\rho_Z^{(3)} = \lambda_1(B^{-1}C^{'}A^{-1}C),$$

$$\rho_Z^{(4)} = \lambda_p(B^{-1}C^{'}A^{-1}C),$$

where $\rho_Z^{(3)} = \max_{1\leq i\leq p}\rho_i^2$, $\rho_Z^{(4)} = \min_{1\leq i\leq p}\rho_i^2$, which are two kinds of generalized correlation coefficient.

Theorem 3. In model of (1),

$$r(X) = p, \Sigma \geq 0,\ \hat{\beta} = (X^{'}X)^{-1}X^{'}Y,$$
$$\beta^* = (X^{'}T^+X)^{-1}X^{'}T^+Y,$$

then

$$\frac{\delta_p^4\lambda_m^2\mu_m^6}{\delta_1\lambda_1^2\mu_1^4}\rho_Z^{(4)} \leq e_4(\hat{\beta}) \leq \frac{\delta_1^5\lambda_1^3\mu_1^6}{\delta_p^3\lambda_p^3\mu_m^6}\rho_Z^{(4)},\ e_4(\hat{\beta}) \leq \frac{\delta_1^4\lambda_1^2\mu_1^6}{\delta_p^2\lambda_p^2\mu_m^6}\rho_Z^{(3)},$$

where $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_p$ is the sequential characteristic root of $X^{'}X$. $\lambda_i = \lambda_i(\Sigma),\ \mu_i = \mu_i(T)$.[7]

Proof.

$$\lambda_p(A) = \lambda_p(A^{-1}A^2) \geq \lambda_p(A^{-1})\lambda_p^2(A),$$

$$\lambda_p(A^{-1}) = \lambda_p((CC^{'})^{-1}A^{-1}(CC^{'})) \geq \lambda_p((C^{'}C)^{-1})\lambda_p(A^{-1}C^{'}C)$$

$$= \lambda_p((C^{'}C)^{-1})\lambda_p(A^{-1}C^{'}CBB^{-1}) \geq \lambda_p((C^{'}C)^{-1})\lambda_p(BC^{'}A^{-1}C)\lambda_p(B).$$

thus

$$e(\beta) = \frac{\lambda_p(Cov\beta^*)}{\lambda_p(Cov\hat{\beta})} = \frac{\lambda_p(A)}{\lambda_p(B)} \geq \lambda_p((C^{'}C)^{-1})\lambda_p^2(A)\rho_Z^{(4)}$$

$$\lambda_1(CC^{'}) \leq \frac{\mu_1^2\lambda_1^2}{\mu_m^2\delta_p^4}$$

$$\lambda_p(A) \geq \frac{\mu_m^2\lambda_m}{\mu_1^2\delta_1}$$

then it is readily verified that

$$e_4(\hat{\beta}) \geq \frac{\delta_p^4\lambda_m^2\mu_m^6}{\delta_1\lambda_1^2\mu_1^4}\rho_Z^{(4)}$$

$$\lambda_p(A^{-1}) = \lambda_p((CC^{'})^{-1}A^{-1}(CC^{'})) \leq \lambda_1((C^{'}C)^{-1})\lambda_p(B^{-1}C^{'}A^{-1}C)\lambda_1(B),$$

$$\lambda_p(A) = \lambda_p(A^{-1}A^2) \leq \lambda_p(A^{-1})\lambda_1^2(A).$$

so that

$$e(\beta) = \frac{\lambda_p(Cov\beta^*)}{\lambda_p(Cov\hat{\beta})} = \frac{\lambda_p(A)}{\lambda_p(B)} \leq \frac{\lambda_1((CC^{'})^{-1})\lambda_1(B)\lambda_1^2(A)}{\lambda_p(B)}\rho_Z^{(4)}$$

$$\lambda_1(A) \leq \frac{\mu_1^2\lambda_1(\Sigma)}{\mu_p^2\delta_p}$$

$$\lambda_1((CC^{'})^{-1}) = \lambda_p^{-1}(CC^{'}) \leq \frac{\mu_1^2\delta_1^4}{\mu_m^2\lambda_p^2}$$

thus

$$e_4(\hat{\beta}) \leq \frac{\delta_1^5\lambda_1^3\mu_1^6}{\delta_p^3\lambda_p^3\mu_m^6}\rho_Z^{(4)}. \tag{6}$$

again

$$\lambda_p(A^{-1}) = \lambda_p((CC^{'})^{-1}A^{-1}(CC^{'})) \le \lambda_p((C^{'}C)^{-1})\lambda_1(A^{-1}C^{'}C)$$
$$\le \lambda_1((C^{'}C)^{-1})\lambda_p(B)\rho_Z^{(3)}$$

thus

$$e(\beta) = \frac{\lambda_p(Cov\beta^*)}{\lambda_p(Cov\hat{\beta})} = \frac{\lambda_p(A)}{\lambda_p(B)} \le \lambda_1((C^{'}C)^{-1})\lambda_1^2(A)\rho_z^{(3)} \quad (7)$$

thus analyzing $\rho_z^{(3)}$ about β in the model of (6) and(7), we get a new estimation of $e_4(\beta)$,

$$e_4(\hat{\beta}) \le \frac{\delta_1^4\lambda_1^2\mu_1^6}{\delta_p^2\lambda_p^2\mu_m^6}\rho_z^{(3)}.$$

IV. Conclusions

Due to the determining theorem of minimum variance unbiased estimator, the estimation of BLUE in linear model is used. But the co variation is not given, people are hardly to get the solution. So that people like to use LSE. And this substitution will take some losses. To quantitative the losses, many scholars have presented many different relative efficiencies in different senses.

For relative efficiency of linear model, we can make a further research from the following several aspects:

1. Discussions are made on the lower bound of relative efficiency, and conduct a further study about which.
2. People can make a further research about the relation with the generalized relative coefficient and relative efficiency
3. Try to find another better relative efficiency.

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