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Existence and Uniqueness of Positive Solution for Nonlinear Fractional Differential Equation with Integral Boundary Conditions

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Abstract—By using fixed point theorems for a class of generalized concave and convex operators, the positive solution of nonlinear fractional differential equation with integral boundary conditions is studied, where $n \geq 3$ is an integer, μ is a parameter and $0 \leq \mu < \alpha$. Its existence and uniqueness is proved, and an iterative scheme is constructed to approximate it. Finally, two examples are given to illustrate our results.

Keywords—Fractional differential equation, positive solution, existence and uniqueness, fixed point theorem, generalized concave and convex operator, integral boundary conditions.

I. Introduction

FRACTIONAL differential equations arise in many engineering and scientific disciplines such as the mathematical modeling of systems and processes in the fields of mechanics, engineering and biological sciences fields, etc., (see [2] and the references therein). Its existence and multiplicity is studied by using of Guo-Krasnoselskiis fixed point theorem etc., (see [3] and the references therein). Boundary value problems for fractional differential equations with integral boundary conditions are very interesting and largely unknown. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers (see [4] and the references therein).

In this paper, we consider the existence and uniqueness of positive solution for nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \quad \alpha \in (n-1,n] \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \quad u(1) = \mu\int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $n \geq 3$ is an integer, μ is a parameter and $0 \leq \mu < \alpha$, and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and monotone with respect to the second argument, and $q : (0,1) \to [0,\infty)$ is continuous. A function u is called a positive solution of the problem (1) if $u(t)$ satisfies (1) and $u(t) > 0$ on $(0,1)$.

Yongping Sun and Yan Sun [6] investigated the positive solutions for the problem (1). Its existence is proved by means of a monotone iterative method. In this study, our work is to extend and improve the main results of the paper [6]. By means of fixed point theorems for a class of generalized concave and convex operators, we get the existence and uniqueness of positive solutions for the problem (1). Meanwhile, an iterative scheme is constructed to approximate this unique solution.

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II. Preliminaries and Previous Results

Definition 1. [1] The integral

$$I_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t, \quad x > 0$$

where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function, is called the Riemann-Liouville fractional integral of order α.

Definition 2. [1] For a function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}x})^n\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

In [5], the author obtained the Greens function associated with the problem (1). More precisely, the author proved the following lemma.

Lemma 1. [5] Let $h \in C[0,1]$ be a given function and $n-1 < \alpha \leq n$, then the boundary-value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + h(t) = 0, \quad 0 < t < 1, \quad \alpha \in (n-1,n] \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \quad u(1) = \mu\int_0^1 u(s)ds, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s, \quad t \in [0,1],$$

where

$$G(t,s) = H(t,s) + \frac{\mu t^{\alpha-1}}{(\alpha-\mu)\Gamma(\alpha)}s(1-s)^{\alpha-1}, \quad t,s \in [0,1],$$

$$H(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, \\ \qquad 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, \\ \qquad 0 \leq t \leq s \leq 1, \end{cases}$$

Obviously,

$$G(t,s) = \frac{1}{(\alpha-\mu)\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s) \\ -(\alpha-\mu)(t-s)^{\alpha-1}, \\ \qquad 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s), \\ \qquad 0 \leq t \leq s \leq 1, \end{cases} \tag{2}$$

and $G(t,s)$ is continuous on the unit square $[0,1]\times[0,1]$.

Lemma 2. The Green function $G(t,s)$ defined by (2) has the following property:

$$\begin{aligned}&\tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}\mu s\\ &\le G(t,s)\\ &\le \tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s)\end{aligned} \tag{3}$$

for $t,s\in(0,1)$.

Proof: Evidently, the right inequality holds. So we only show that the left inequality. When $0\le s\le t\le 1$, we have $0\le t-s\le t-ts=(1-s)t$, and thus $(t-s)^{\alpha-1}\le(1-s)^{\alpha-1}t^{\alpha-1}$. Hence,

$$\begin{aligned}G(t,s)&=\tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s)\\ &\quad-(\alpha-\mu)(t-s)^{\alpha-1}]\\ &\ge\tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s)\\ &\quad-(\alpha-\mu)t^{\alpha-1}(1-s)^{\alpha-1}]\\ &=\tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}\mu s.\end{aligned}$$

When $0\le t\le s\le 1$, by assumption $0\le\mu<\alpha$, we have

$$\begin{aligned}G(t,s)&=\tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s)\\ &\ge\tfrac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}\mu s.\end{aligned}$$

So the left inequality also holds.

A non-empty closed convex set $P\subset E$ is a cone if it meets: (i)$x\in P,\lambda\ge 0\Rightarrow\lambda x\in P$;(ii)$x\in P,-x\in P\Rightarrow x=\theta$. Suppose $(E,||\cdot||)$ is an order Banach space and a cone $P\subset E$, i.e. $x\le y$ if and only if $y-x\in P$. If $x\le y$ and $x\ne y$, then we denote $x<y$. We denote the zero element of E by θ.

Putting $P^0=\{x\in P|x \text{ is an interior point of } P\}$, a cone P is said to be solid if P^0 is non-empty. Moreover, if there is a positive constant $N>0$ such that,for all $x,y\in E,\theta\le x\le y$ implies $||x||\le N||y||$, then P is called normal; N is called the normality constant of P.

If $x\le y$ implies $Ax\le Ay$, we say that an operator $A:E\to E$ is increasing.

For all $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\le y\le\mu x$. Clearly $\sim$ is an equivalence relation. Given $w>\theta$ (i.e. $w\ge\theta$ and $w\ne\theta$), we denote the set $P_w=\{x\in E|x\sim w\}$ by P_w. It is easy to see that $P_w\subset P$ for $w\in P$.

Lemma 3. [7] Let P be a normal cone in a real Banach space E, and $\omega>\theta$. $A:P_\omega\to P_\omega$ is an increasing operator and

$$A(tx)\ge t^{\alpha(t)}Ax,\qquad \forall t\in(0,1),x\in P_w, \tag{4}$$

where $0<\alpha(t)<1,\forall t\in(0,1)$. Then operator A has a unique solution x^* in P_ω. Moreover, constructing successively the sequence $x_n=Ax_{n-1},n=1,2,\cdots$ for any initial value $x_0\in P_\omega$, we have $\|x_n-x^*\|\to 0(n\to\infty)$.

Lemma 4. [7] Let P be a normal cone in a real Banach space E, and $\omega>\theta$. $A:P_\omega\to P_\omega$ is a decreasing operator and

$$A(tx)\le t^{-\alpha(t)}Ax,\qquad \forall t\in(0,1),x\in P_w, \tag{5}$$

where $0<\alpha(t)<1,\forall t\in(0,1)$. Then operator A has a unique solution x^* in P_ω. Moreover, constructing successively the sequence $x_n=Ax_{n-1},n=1,2,\cdots$ for any initial value $x_0\in P_\omega$, we have $\|x_n-x^*\|\to 0(n\to\infty)$.

III. Main Results

In this section, we apply lemma 3 and lemma 4 to investigate the problem (1), new results on the existence and uniqueness of positive solution are obtained.

In this paper, we will work in the Banach space $C[0,1]=\{x:[0,1]\to R \text{ is continuous}\}$ with the standard norm $\|x\|=\sup|x(t)|:t\in[0,1]$. Notice that this space can be endowed with a partial order given by $x,y\in C[0,1],x\le y\Leftrightarrow x(t)\le y(t)$ for $t\in[0,1]$. Let $P=x\in C[0,1]|x(t)\ge 0,t\in[0,1]$ be the standard cone. Evidently, P is a normal cone in $C[0,1]$ and the normality constant is 1.

Theorem 1. Assume that

(H1) $f(t,u):[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and increasing with respect to the second argument,and $f(t,0)\not\equiv 0,\quad t\in[0,1]$;

(H2) $q:(0,1)\to[0,\infty)$ is continuous and

$$0<\int_0^1(1-s)^{\alpha-1}q(s)\mathrm{d}s<\infty,$$

$$0<\int_0^1(1-s)^{\alpha-1}sq(s)\mathrm{d}s<\infty.$$

(H3) for $\forall\lambda\in(0,1)$ and $\forall u\in[0,\infty)$, there exists a function $\varphi(\lambda)\in(\lambda,1]$, such that $f(t,\lambda u)\ge\varphi(\lambda)f(t,u)$. Then the problem (1) has a unique positive solution u^* in P_w, where $w(t)=t^{\alpha-1},t\in[0,1]$. Moreover, for any initial value $u_0\in P_w$, constructing successively the iterative scheme

$$u_n(t)=\int_0^1 G(t,s)q(s)f(s,u_{n-1}(s))\mathrm{d}s,\quad n=1,2,\cdots,$$

we have $u_n(t)\to u^*(t)$ as $n\to\infty$, where $G(t,s)$ is given as (2).

Proof: Solution of the problem (1) can be converted to solution of the operator equation which is equivalent to the problem (1), i.e.:

$$Au(t)=\textstyle\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s$$

where $G(t,s)$ is given as (2).

Note that $G(t,s)\ge 0,t,s\in[0,1]$. By assumption (H1),(H2) and Lemma 2, we know that $Au(t)\ge 0,t\in[0,1]$ and $A:P\to P$ is an increasing operator. By assumption (H3), for $\lambda\in(0,1)$, we know that

$$\begin{aligned}A(\lambda u)(t)&=\textstyle\int_0^1 G(t,s)q(s)f(s,\lambda u(s))\mathrm{d}s\\ &\ge\varphi(\lambda)\textstyle\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s=\varphi(\lambda)Au(t).\end{aligned}$$

That is,$A(\lambda u)\ge\varphi(\lambda)Au,\ \forall u\in P,\ \lambda\in(0,1)$.

Let $\alpha(t)=\frac{\ln\varphi(t)}{\ln t},\ t\in(0,1)$, then $\alpha(t)\in(0,1)$ and

$$A(\lambda u)\ge\lambda^{\alpha(\lambda)}Au,\ \forall u\in P,\lambda\in(0,1).$$

Next we show that $A:P_\omega\to P_\omega$, where $\omega(t)=t^{\alpha-1}$.

Let

$r=\min\{f(t,0):t\in[0,1]\},\ R=\max\{f(t,1):t\in[0,1]\}$, then $0<r\le R$. By assumption (H1), (H2) and Lemma 2, we have

$$\begin{aligned}Aw(t)&=\textstyle\int_0^1 G(t,s)q(s)f(s,w(s))\mathrm{d}s\\ &\ge\textstyle\int_0^1\frac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}\mu sq(s)f(s,0)ds\\ &\ge[\frac{r}{(\alpha-\mu)\Gamma(\alpha)}\textstyle\int_0^1(1-s)^{\alpha-1}\mu sq(s)\mathrm{d}s]t^{\alpha-1},\end{aligned}$$

$$\begin{aligned}&Aw(t)\\&=\int_0^1 G(t,s)q(s)f(s,w(s))\mathrm{d}s\\&\le \int_0^1 \frac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)f(s,1)\mathrm{d}s\\&\le [\frac{R}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)\mathrm{d}s]t^{\alpha-1}\end{aligned}$$

So we have

$$\begin{aligned}&[\frac{r}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mu sq(s)\mathrm{d}s]w(t)\\&\le Aw(t)\\&\le [\frac{R}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)\mathrm{d}s]w(t).\end{aligned}$$

By assumption (H2), we note that

$$\begin{aligned}&\frac{r}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mu sq(s)\mathrm{d}s>0,\\&\frac{R}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)\mathrm{d}s>0.\end{aligned}$$

We easily prove that $A\omega \in P_\omega$, so $A : P_\omega \to P_\omega$.

Finally, by means of lemma 3, the operator equation $Au = u$ has a unique positive solution u^* in P_w. Moreover, constructing successively the iterative scheme $u_n = Au_{n-1}, n = 1, 2, \cdots$ for any initial value $u_0 \in P_w$, we have $u_n \to u^*$ as $n \to \infty$. That is, the problem (1) has a unique positive solution u^* in P_w, where $\omega(t) = t^{\alpha-1}, t \in [0, 1]$. For any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_n(t)=\int_0^1 G(t,s)q(s)f(s,u_{n-1}(s))\mathrm{d}s,\quad n=1,2,\cdots,$$

we have $u_n \to u^*, t \in [0, 1]$ as $n \to \infty$.

Theorem 2. Assume that

(H4) $f(t, u) : [0, 1] \times [0,\infty) \to [0,\infty)$ is continuous and decreasing with respect to the second argument,and $f(t, 1) \not\equiv 0, \quad t \in [0, 1]$;

(H5) $q : (0, 1) \to [0,\infty)$ is continuous and

$$0<\int_0^1 (1-s)^{\alpha-1}q(s)\mathrm{d}s<\infty,$$

$$0<\int_0^1 (1-s)^{\alpha-1}sq(s)\mathrm{d}s<\infty.$$

(H6) for $\forall\lambda \in (0, 1)$ and $\forall u \in [0,\infty)$, there exists a function $\varphi(\lambda) \in (\lambda, 1]$, such that $f(t, \lambda u) \ge \frac{1}{\varphi(\lambda)}f(t, u)$. Then the problem (1) has a unique positive solution u^* in P_w, where $w(t) = t^{\alpha-1}, t \in [0, 1]$. Moreover, for any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_n(t)=\int_0^1 G(t,s)q(s)f(s,u_{n-1}(s))\mathrm{d}s,\quad n=1,2,\cdots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t, s)$ is given as (2).

Proof: Solution of the problem (1) can be converted to solution of the operator equation which is equivalent to the problem (1), i.e.:

$$Au(t)=\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s$$

where $G(t, s)$ is given as (2).

Note that $G(t, s) \ge 0, t, s \in [0, 1]$. By assumption (H4),(H5) and Lemma 2, we know that $Au(t) \ge 0, t \in [0, 1]$ and $A : P \to P$ is a decreasing operator. By assumption (H6), for $\lambda \in (0, 1)$, we know that

$$\begin{aligned}A(\lambda u)(t)&=\int_0^1 G(t,s)q(s)f(s,\lambda u(s))\mathrm{d}s\\&\le \frac{1}{\varphi(\lambda)}\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s=\frac{1}{\varphi(\lambda)}Au(t).\end{aligned}$$

That is,$A(\lambda u) \le \frac{1}{\varphi(\lambda)}Au, \ \forall u \in P, \ \lambda \in (0, 1)$.

Let $\alpha(t) = \frac{\ln\varphi(t)}{\ln t}, \ t \in (0, 1)$, then $\alpha(t) \in (0, 1)$ and

$$A(\lambda u)\le \lambda^{-\alpha(\lambda)}Au,\ \forall u\in P,\lambda\in(0,1).$$

Next we show that $A : P_\omega \to P_\omega$, where $\omega(t) = t^{\alpha-1}$.

Let $r' = \min\{f(t, 1) : t \in [0, 1]\}, \ R' = \max\{f(t, 0) : t \in [0, 1]\}$, then $0 < r' \le R'$. By assumption (H4), (H5) and Lemma 2, we have

$$\begin{aligned}Aw(t)&=\int_0^1 G(t,s)q(s)f(s,w(s))\mathrm{d}s\\&\ge \int_0^1 \frac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}\mu sq(s)f(s,1)ds\\&\ge [\frac{r'}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mu sq(s)\mathrm{d}s]t^{\alpha-1},\end{aligned}$$

$$\begin{aligned}&Aw(t)\\&=\int_0^1 G(t,s)q(s)f(s,w(s))\mathrm{d}s\\&\le \int_0^1 \frac{1}{(\alpha-\mu)\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)f(s,0)\mathrm{d}s\\&\le [\frac{R'}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)\mathrm{d}s]t^{\alpha-1}\end{aligned}$$

So we have

$$\begin{aligned}&[\frac{r'}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mu sq(s)\mathrm{d}s]w(t)\\&\le Aw(t)\\&\le [\frac{R'}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)\mathrm{d}s]w(t).\end{aligned}$$

By assumption (H5), we note that

$$\begin{aligned}&\frac{r'}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mu sq(s)\mathrm{d}s>0,\\&\frac{R'}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)\mathrm{d}s>0.\end{aligned}$$

We easily prove that $A\omega \in P_\omega$, so $A : P_\omega \to P_\omega$.

Finally, by means of lemma 4, the operator equation $Au = u$ has a unique positive solution u^* in P_w. Moreover, constructing successively the iterative scheme $u_n = Au_{n-1}, n = 1, 2, \cdots$ for any initial value $u_0 \in P_w$, we have $u_n \to u^*$ as $n \to \infty$. That is, the problem (1) has a unique positive solution u^* in P_w, where $\omega(t) = t^{\alpha-1}, t \in [0, 1]$. For any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_n(t)=\int_0^1 G(t,s)q(s)f(s,u_{n-1}(s))\mathrm{d}s,\quad n=1,2,\cdots,$$

we have $u_n \to u^*, t \in [0, 1]$ as $n \to \infty$.

IV. Examples

We present two examples to illustrate Theorem 1 and Theorem 2.

Example 1. Consider the following problem:

$$\begin{cases}-D_{0+}^{\frac{5}{2}}u(t)=u^{\frac{1}{2}}+u^{\frac{1}{3}}+t^2+1,\ \ 0\le t\le 1,\\ u^{(j)}(0)=0,\quad 0\le j\le 1,\quad u(1)=2\int_0^1 u(s)ds,\end{cases}\tag{6}$$

In this example, we have $\alpha = \frac{5}{2}$. Let

$$q(t)\equiv 1,\quad f(t,u)=u^{\frac{1}{2}}+u^{\frac{1}{3}}+t^2+1,\ t\in[0,1].$$

Obviously, $f(t,u):[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and increasing with respect to the second argument, and $f(t,0)=t^2+1>0, t\in[0,1]$. And $q(t)$ is continuous and

$$0<\int_0^1(1-s)^{\alpha-1}q(s)\mathrm{d}s=\int_0^1(1-s)^{\frac{3}{2}}\mathrm{d}s<\infty,$$
$$0<\int_0^1(1-s)^{\alpha-1}sq(s)\mathrm{d}s=\int_0^1(1-s)^{\frac{3}{2}}s\mathrm{d}s<\infty.$$

Besides, for $t\in[0,1], \lambda\in(0,1), u\in[0,\infty)$, we have

$$\begin{aligned} f(t,\lambda u) &= (t^3+1)[\frac{1}{\lambda^{\frac{1}{2}}u^{\frac{1}{2}}+2}+\frac{1}{\lambda^{\frac{1}{3}}u^{\frac{1}{3}}+2}+\frac{1}{3}] \\ &\le (t^3+1)[\frac{1}{\lambda^{\frac{1}{2}}(u^{\frac{1}{2}}+2)}+\frac{1}{\lambda^{\frac{1}{2}}(u^{\frac{1}{3}}+2)}+\frac{1}{3\lambda^{\frac{1}{2}}}] \\ &= \frac{1}{\lambda^{\frac{1}{2}}}(t^3+1)[\frac{1}{u^{\frac{1}{2}}+2}+\frac{1}{u^{\frac{1}{3}}+2}+\frac{1}{3}] \\ &= \frac{1}{\lambda^{\frac{1}{2}}}f(t,u) \end{aligned}$$

So, for $\forall\lambda\in(0,1)$ and $\forall u\in[0,\infty)$, there exists a function $\varphi(\lambda)=\lambda^{\frac{1}{2}}\in(\lambda,1]$,such that$f(t,\lambda u)\ge\varphi(\lambda)f(t,u)$.

Hence all the conditions of Theorem 1 are satisfied. An application of Theorem 1 implies that problem (6) has a unique positive solution in P_w, where $w(t)=t^{\frac{3}{2}}$,$t\in[0,1]$.

Example 2. Consider the following problem:

$$\begin{cases} -D_{0+}^{\frac{5}{2}}u(t)=(t^3+1)[\frac{1}{u^{\frac{1}{2}}+2}+\frac{1}{u^{\frac{1}{3}}+2}+\frac{1}{3}], \quad 0\le t\le 1, \\ u^{(j)}(0)=0, \quad 0\le j\le 1, \quad u(1)=2\int_0^1 u(s)ds, \end{cases} \tag{7}$$

In this example, we have $\alpha=\frac{5}{2}$. Let

$$q(t)\equiv 1, \quad f(t,u)=(t^3+1)[\frac{1}{u^{\frac{1}{2}}+2}+\frac{1}{u^{\frac{1}{3}}+2}+\frac{1}{3}],\ t\in[0,1].$$

Obviously, $f(t,u):[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and decreasing with respect to the second argument, and $f(t,0)=t^3+1>0, t\in[0,1]$. And $q(t)$ is continuous and

$$0<\int_0^1(1-s)^{\alpha-1}q(s)\mathrm{d}s=\int_0^1(1-s)^{\frac{3}{2}}\mathrm{d}s<\infty,$$
$$0<\int_0^1(1-s)^{\alpha-1}sq(s)\mathrm{d}s=\int_0^1(1-s)^{\frac{3}{2}}s\mathrm{d}s<\infty.$$

Besides, for $t\in[0,1], \lambda\in(0,1), u\in[0,\infty)$, we have

$$\begin{aligned} f(t,\lambda u) &= \lambda^{\frac{1}{2}}u^{\frac{1}{2}}(t)+\lambda^{\frac{1}{3}}u^{\frac{1}{3}}(t)+t^2+1 \\ &\ge \lambda^{\frac{1}{2}}u^{\frac{1}{2}}(t)+\lambda^{\frac{1}{2}}u^{\frac{1}{2}}(t)+\lambda^{\frac{1}{2}}t^2+\lambda^{\frac{1}{2}} \\ &= \lambda^{\frac{1}{2}}(u^{\frac{1}{2}}(t)+u^{\frac{1}{2}}(t)+t^2+1) \\ &= \lambda^{\frac{1}{2}}f(t,u). \end{aligned}$$

So, for $\forall\lambda\in(0,1)$ and $\forall u\in[0,\infty)$, there exists a function $\varphi(\lambda)=\lambda^{\frac{1}{2}}\in(\lambda,1]$,such that$f(t,\lambda u)\le\frac{1}{\varphi(\lambda)}f(t,u)$. Hence all the conditions of Theorem 2 are satisfied. An application of Theorem 2 implies that problem (7) has a unique positive solution in P_w, where $w(t)=t^{\frac{3}{2}}$,$t\in[0,1]$.

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