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Lyapunov Type Inequalities for Fractional Impulsive Hamiltonian Systems

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Abstract—This paper deals with study about fractional order impulsive Hamiltonian systems and fractional impulsive Sturm-Liouville type problems derived from these systems. The main purpose of this paper devotes to obtain so called Lyapunov type inequalities for mentioned problems. Also, in view point on applicability of obtained inequalities, some qualitative properties such as stability, disconjugacy, nonexistence and oscillatory behaviour of fractional Hamiltonian systems and fractional Sturm-Liouville type problems under impulsive conditions will be derived. At the end, we want to point out that for studying fractional order Hamiltonian systems, we will apply recently introduced fractional Conformable operators.

Keywords—Fractional derivatives and integrals, Hamiltonian system, Lyapunov type inequalities, stability, disconjugacy.

I. Introduction

As we know, the historical perspective of Lyapunov inequalities turns back to the last decade of nineteenth century. A. M Lyapunov, in 1892, in the study of periodic motion, introduced an effective tool for studying the qualitative behavior of second order differential equations with ω-periodic coefficients of the form

$$y^{''} + q(t)y = 0, \quad -\infty < t < \infty. \tag{1}$$

Lyapunov stated the following theorem:

Theorem 1: [14][Chapter III, Theorem II] If function q can only take positive or zero values(without being identically zero), and if further it satisfies in condition

$$\omega \int_0^\omega q(t) \leq 4,$$

the roots of the characteristic equation corresponding to (1) will always be complex, their moduli being equal to 1.

By means of Floquet theory, one can conclude that the result of Theorem 1 is equal to stability of second order ODE (1) in the sense that all solutions of (1) are bounded as $t \to \pm\infty$. The inequality

$$\omega \int_0^\omega q(t)dt > 4. \tag{2}$$

is called Lyapunov inequality.

After introducing the Lyapunov inequality (2) by now, extensive studies about Lyapunov (type) inequalities have been recognized; so that, nowadays Lyapunov type inequalities are known as a prefect theory for studying qualitative behaviour of differential equations such as stability, disconjugacy and oscillatory properties. For verification the above discussion, some of great developments of Lyapunov type inequalities can be found in [2], [3], [5]-[8], [11], [13], [15]-[17], [19]-[21].

Recently, by means of Lyapunov type inequalities corresponding to the fractional boundary value problems, some interesting results about real zeros of Mittag-Leffler functions have been obtained. More details are available in [6], [7], [11]. We notice here this fact that non of above mentioned qualitative gestures for fractional differential equations have been investigated by now. So in this paper, authors are concerned with a study about reconstruction of fractional Lyapunov type inequalities for estimating the mentioned qualitative gestures of fractional Hamiltonian systems under impulse effects. On the other hand, one can observe the boom of developments of theory of impulsive differential equations since 1990 by now. Unerring seeking reason for these developments, leads us to the great applicability of mathematical simulations via impulsive differential equations for many important research fields of science and technology such as optimal control, population dynamics, biotechnologies, industrial robotics and so on. In this way, the monographs [4], [18] contain invaluable applications of impulsive differential equations. Reference [10] considered the impulsive Hamiltonian system with T-periodic coefficients

$$\begin{cases} x^{'}(t) = a(t)x(t) + b(t)u(t), \\ u^{'}(t) = -c(t)x(t) - a(t)u(t), \end{cases} \quad t \neq \tau_i, \tag{3}$$

subjected to the following impulse effects

$$\begin{aligned} &x(\tau_{i+}) = \alpha_i x(\tau_{i-}), \quad u(\tau_{i+}) = \alpha_i u(\tau_{i-}) - \beta_i x(\tau_{i-}), \\ &t, \alpha_i, \beta_i \in \mathbb{R}, i \in \mathbb{Z}, \end{aligned} \tag{4}$$

and under certain classes of assumptions obtained the inequality

$$\begin{aligned} \int_0^T |a(t)|dt + \left(\int_0^T b(t)dt\right)^{1/2} \Bigg\{ \int_0^T c^+(t)dt \\ + \sum_{i=1}^r \left(\frac{\beta_i}{\alpha_i}\right)^+ \Bigg\}^{1/2} > 2, \end{aligned} \tag{5}$$

where $f^+(t) = \max\{f(t), 0\}$.

They used the obtained Lyapunov type inequality for providing a criterion for stability and disconjugacy of impulsive Hamiltonian system (3). Also, they studied in [9], the second order linear impulsive differential equations with

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T-periodic coefficients of the form

$$\begin{cases} \Big(p(t)y^{'}\Big)^{'} + q(t)y = 0, \quad t \neq \tau_i, \\ y(\tau_{i+}) = \alpha_i y(\tau_{i-}), \quad y^{[1]}(\tau_{i+}) = \alpha_i y^{[1]}(\tau_{i-}) - \beta_i y(\tau_{i-}), \end{cases} \tag{6}$$

where $t \in \mathbb{R}$, $i \in \mathbb{Z}$, $x(\tau_{i\pm}) := \lim_{t\to\tau_{i\pm}} x(t)$ and $y^{[1]}(t) = p(t)y^{'}(t)$ denotes the so called quasi derivative of $y(t)$. By means of Floquet theory, under certain conditions they obtained the following Lyapunov type inequality

$$\left[\int_0^T \frac{dt}{p(t)}\right] \cdot \left[\int_0^T q^{+}(t)dt + \sum_{i=1}^{r}\left(\frac{\beta_i}{\alpha_i}\right)^{+}\right] > 4. \tag{7}$$

Authors applied this inequality as (in-)stability criterion for ODE (6).

Indeed, considering an adequate connection between coefficients one can transform the impulsive Hamiltonian system (3), (4) to the second order linear impulsive differential equation (6). So, we can discuss here about "obtaining relevant Lyapunov type inequality for second order impulsive differential equation (6) via imposing new coefficients instead of attempting to find it with usual calculatory manner". So, in this paper, we tried to develop obtained results in the mentioned papers and refined corresponding fractional Lyapunov type inequalities for fractional impulsive Hamiltonian systems and consequently reconstruction of obtained results for mentioned systems. In the next step, from obtained results for fractional Hamiltonian systems, similar results will be concluded for fractional sturm-liouville type problems reduced from fractional Hamiltonian systems. Consider the following fractional order linear Hamiltonian system:

$$\mathfrak{T}^{\nu} y(t) = JH(t)y(t), \quad t \in \mathbb{R},$$

where

$$\mathfrak{T}^{\nu} y(t) = \begin{pmatrix} T^{\nu}u(t) \\ T^{\nu}v(t) \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$, \; H(t) = \begin{pmatrix} c(t) & a(t) \\ a(t) & b(t) \end{pmatrix}.$$

Equivalently, we consider the following system

$$\begin{cases} T^{\nu}u(t) = a(t)u(t) + b(t)v(t), \\ T^{\nu}v(t) = -c(t)u(t) - a(t)v(t), \end{cases} \tag{8}$$

where

$$\begin{cases} t \in \mathbb{R}, \; \nu \in (0,1), \\ a(t), b(t), c(t) : \text{ piece-wise continuous functions on } \mathbb{R}, \\ T^{\nu} : \text{ Conformable fractional derivative of order } \nu \end{cases} \tag{9}$$

Assume that $\{\tau_i\}_{i\in\mathbb{Z}}$ be a increasing real sequence such that is $\tau_i < \tau_{i+1}$ for all $i \in \mathbb{Z}$. Suppose that there exist positive constants $T \in \mathbb{R}$, $r \in \mathbb{Z}$ such that

$$\tau_{i+r} = \tau_i + T, \; i \in \mathbb{Z}, \quad 0 < \tau_1 < \tau_2 < ... < \tau_r < T. \tag{10}$$

Let $a, b, c : \mathbb{R}\backslash\{\tau_i : \; i \in \mathbb{Z}\} \to \mathbb{R}$ and $\alpha_i, \beta_i \in \mathbb{R}$ for $i \in \mathbb{Z}$ satisfy the following conditions

1. $a(t+T) = a(t), \quad b(t+T) = b(t), \quad c(t+T) = c(t), \quad t \in \mathbb{R}\backslash\{\tau_i : \; i \in \mathbb{Z}\}$,
2. $\alpha_i \neq 0, \quad \alpha_{i+r} = \alpha_i, \quad \beta_{i+r} = \beta_i, \quad i \in \mathbb{Z}$,
3. $a, b, c \in C_p([0,T], \mathbb{R})$,

where

$$C_p\left([0,T], \mathbb{R}\right) = \Big\{ f \big| \; f : [0,T]\backslash\{\tau_1, \tau_2, ..., \tau_r\} \to \mathbb{R}, f \in C([0,T]\backslash\{\tau_1, ..., \tau_r\}) \; , \; f(\tau_{i-}), f(\tau_{i+}) \in \mathbb{R}, \; i \in [1,r]_{\mathbf{N}_0} = \{1, 2, ..., r\} \Big\}.$$

In this paper, we are concerned with the special class of fractional Hamiltonian system (11) of the form

$$\begin{cases} T^{\nu}u(t) = \underline{a(t)}u(t) + \underline{b(t)}v(t), \\ T^{\nu}v(t) = -\underline{c(t)}u(t) - \underline{a(t)}v(t), \end{cases} \tag{11}$$

such that $\underline{g(t)} = tg(t)$, with impulsive conditions

$$\begin{cases} u(\tau_{i+}) - \alpha_i u(\tau_{i-}) = 0, \\ v(\tau_{i+}) - \alpha_i v(\tau_{i-}) = -\beta_i u(\tau_{i-}), \end{cases} \tag{12}$$

where $\nu \in (0,1), \; t, \alpha_i, \beta_i \in \mathbb{R}, i \in \mathbb{Z}$.

Remark 1: If we take $a(t) \equiv 0, \; b(t) \not\equiv 0$ for any $t \in \mathbb{R}$ in fractional Hamiltonian system (11), then setting new variables

$$p(t) = \frac{1}{\Big(\underline{b(t)}\Big)}, \quad q(t) = \underline{c(t)}, \tag{13}$$

reduces the fractional impulsive Hamiltonian system (11), (12) to the fractional impulsive differential equation

$$T^{\nu}\big(p(t)T^{\nu}u\big) + q(t)u = 0, \quad t \in \mathbb{R}, \tag{14}$$

$$\begin{cases} u(\tau_{i+}) - \alpha_i u(\tau_{i-}) = 0, \\ \big(pT^{\nu}u\big)(\tau_{i+}) - \alpha_i\big(pT^{\nu}u\big)(\tau_{i-}) = -\beta_i u(\tau_{i-}). \end{cases} \tag{15}$$

As stated before, we are going to generalize the above mentioned obtained results for ordinary Hamiltonian system (3), (4) and second order differential equation (6) and obtain corresponding results for fractional impulsive problems. In this way, several difficulties will appear such as this issue that we cannot obtain corresponding fractional order inequalities unless we use the certain fractional order operators that preserve Leibnitz rule, instead of considering standard fractional order operators such as fractional Riemann-Liouville or Caputo operators. Despite standard process of Leibnitz rule for integer order differential calculus, in the theory of fractional calculus this classic rule is not satisfied generally. In other means, if we take $\nu = 1$, in Hamiltonian system (8), multiplying the first equation by $v(t)$ and second one by $u(t)$ and then summing resulting equations the left hand side gives us standard Leibnitz rule $(uv)^{'} = u^{'}v + uv^{'}$. So, in order to implement mentioned rule in fractional differential calculus, we must apply certain fractional order operators that preserve the fractional Leibnitz rule, namely for favorite functions $u(t), v(t)$

$$\left(T^{\nu}uv\right)(t) = \left(T^{\nu}u\right)(t)[v(t)] + [u(t)]\left(T^{\nu}v\right)(t).$$

More recently $R.\ Khalil$ et al. and $T.\ Abdeljawad$ in [1], [12] introduced new definitions for fractional order operators by means of generalizing the limit approach of integer order differentiation operators, that retain interestingly some of important algebraic properties for fractional order differentiation. So, we represent these so called fractional Conformable operators in the sequel.

Definition 1: [1] The left and right sided fractional Conformable integrals of order $n<\nu\leq n+1,\ n\in\mathbb{N}\cup\{0\}$, for function $f\in L^1(a,b)$ are defined as:

$$I^\nu f(t)=\begin{cases} I_a^\nu f(t)=\dfrac{1}{n!}\displaystyle\int_a^t (t-s)^n(s-a)^{\nu-n-1}f(s)ds,\\ {}_bI^\nu f(t)=\dfrac{1}{n!}\displaystyle\int_t^b (s-t)^n(b-s)^{\nu-n-1}f(s)ds.\end{cases} \tag{16}$$

Consequently, based on [1], [12], we define corresponding fractional operators as:

Definition 2: The left and right sided fractional Conformable derivatives of order $n<\nu\leq n+1$, for $(n+1)-$differentiable function $f(t)$ on $t>a$ is given by:

$$T^\nu f(t)=\begin{cases} T_a^\nu f(t)=\\ \lim\limits_{\epsilon\to 0}\dfrac{f^{([\nu]-1)}(t+\epsilon(t-a)^{[\nu]-\nu})-f^{([\nu]-1)}(t)}{\epsilon},\\ {}_bT^\nu f(t)=(-1)^{n+1}\times\\ \lim\limits_{\epsilon\to 0}\dfrac{\left\{f^{([\nu]-1)}(t+\epsilon(b-t)^{[\nu]-\nu})-f^{[\nu]-1}(t)\right\}}{\epsilon},\end{cases} \tag{17}$$

where $n\in\mathbb{N}\cup\{0\}$ and $[\nu]$ is the smallest integer greater than or equal to ν.

II. Main Results

In this section, main results will be organized as:

(M_1) In first step, we will obtain Lyapunov type inequality for fractional Hamiltonian systems

$$\mathfrak{T}_0^\nu y(t)=\begin{cases} tJH(t)y(t), & \nu\in(0,1),\ t\in\mathbb{R},\\ JH(t)y(t), & \nu=1,\ t\in\mathbb{R},\end{cases} \tag{18}$$

where

$$\mathfrak{T}_0^\nu y(t)=\begin{pmatrix}T_0^\nu u(t)\\ T_0^\nu v(t)\end{pmatrix},\ J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ H(t)=\begin{pmatrix}c(t) & a(t)\\ a(t) & b(t)\end{pmatrix}.$$

Equivalently, we are going to study the fractional Hamiltonian system

$$\begin{cases}\begin{cases} T_0^\nu u(t)=\underline{a}(t)u(t)+\underline{b}(t)v(t),\\ T_0^\nu v(t)=-\underline{c}(t)u(t)-\underline{a}(t)v(t),\end{cases} & \nu\in(0,1),\\ \begin{cases} u'(t)=a(t)u(t)+b(t)v(t),\\ v'(t)=-c(t)u(t)-a(t)v(t),\end{cases} & \nu=1,\end{cases} \tag{19}$$

under impulse effects

$$\begin{cases} u(\tau_{i+})-\alpha_i u(\tau_{i-})=0,\\ v(\tau_{i+})-\alpha_i v(\tau_{i-})=-\beta_i u(\tau_{i-}),\end{cases} \tag{20}$$

such that $t\in[0,T],\ \alpha_i,\beta_i\in\mathbb{R},i\in\mathbb{Z},\ 0<\nu\leq 1,\ t\neq\tau_i$ and $\underline{g}(t)=tg(t)$.

(M_2) In second step, we will find Lyapunov type inequality for fractional Sturm-Liouville type problem

$$\begin{cases} T_0^\nu\big(p(t)T_0^\nu u\big)+q(t)u=0,\\ t\in[0,T]\backslash\{\tau_i|\ i\in[0,r]_{\mathbb{N}_0}\},\ \nu\in(0,1),\\ \big(p(t)u'\big)'+q(t)u=0,\\ t\in[0,T]\backslash\{\tau_i|\ i\in[0,r]_{\mathbb{N}_0}\},\ \nu=1,\end{cases} \tag{21}$$

subjected to the impulsive conditions

$$\begin{cases}\begin{cases} u(\tau_{i+})-\alpha_i u(\tau_{i-})=0,\ \nu\in(0,1),\\ \big(pT_0^\nu u\big)(\tau_{i+})-\alpha_i\big(pT_0^\nu u\big)(\tau_{i-})=-\beta_i u(\tau_{i-}),\end{cases}\\ \begin{cases} u(\tau_{i+})-\alpha_i u(\tau_{i-})=0,\ \nu=1,\\ \big(pu'\big)(\tau_{i+})-\alpha_i\big(pu'\big)(\tau_{i-})=-\beta_i u(\tau_{i-}),\end{cases}\end{cases} \tag{22}$$

Lemma 1: Assume that

(C_1)
$$\prod_{i=1}^r\alpha_i^2=1, \tag{23}$$

(C_2)
$$b(t)>0,\quad \int_0^T t^\nu\left(c(t)-\frac{a^2(t)}{b(t)}\right)dt+\sum_{i=1}^r\frac{\beta_i}{\alpha_i}>0. \tag{24}$$

If $A^2\geq 4$, then fractional Hamiltonian system (19), (20) has a nontrivial solution $y(t)=(u(t),v(t))$ with the following property:

There exist two points $t_1,t_2\in\mathbb{R}$ with $0\leq t_1\leq T,\ t_2>t_1,\ t_2-t_1\leq T$ such that $u(t)$ has zeros at t_1 and t_2, and $u(t)\neq 0$ for all $t_1<t<t_2$.

Theorem 2: Assume that the conditions of Lemma 1 hold. Then the inequality

$$\int_0^T|a(t)|dt+\left(\int_0^T b(t)dt\right)^{1/2}\left(\int_0^T c^+(t)dt+\sum_{i=1}^r\left(\frac{\beta_i}{\alpha_i}\right)^+\cdot\frac{1}{T^\nu}\right)^{1/2}>\frac{2}{T^\nu}. \tag{25}$$

is satisfied.

III. Applications

Theorem 3: [Stability Criterion] Assume that

(E_1)
$$\prod_{i=1}^r\alpha_i^2=1, \tag{26}$$

(E_2)
$$b(t)>0,\quad \int_0^T t^\nu\left(c(t)-\frac{a^2(t)}{b(t)}\right)dt+\sum_{i=1}^r\frac{\beta_i}{\alpha_i}>0. \tag{27}$$

(E_3)

$$\int_0^T |a(t)|dt + \left(\int_0^T b(t)dt\right)^{1/2} \times \left(\int_0^T c^+(t)dt + \sum_{i=1}^{r} \left(\frac{\beta_i}{\alpha_i}\right)^+ . \frac{1}{T^\nu}\right)^{1/2} \leq \frac{2}{T^\nu}. \tag{28}$$

Then, the Hamiltonian system (19), (20) is stable.

We introduce here new refinement of the fractional Lyapunov-type inequality (25) as below.

Theorem 4: Assume that $a, b, c \in C_p[t_1, t_2](t_1 < t_2)$, $b(t) > 0$ and $\alpha_j \neq 0$ for all $j \in \mathbb{Z}$. Let the Hamiltonian system

$$\begin{cases} T_{t_1}^\nu u(t) = \underline{a}(t)u(t) + \underline{b}(t)v(t), & \nu \in (0,1), \\ T_{t_1}^\nu v(t) = -\underline{c}(t)u(t) - \underline{a}(t)v(t), \end{cases} \tag{29}$$

$$\begin{cases} u(\tau_{i+}) - \alpha_i u(\tau_{i-}) = 0, \\ v(\tau_{i+}) - \alpha_i v(\tau_{i-}) = -\beta_i u(\tau_{i-}), \end{cases} \tag{30}$$

has a real solution $(u(t), v(t))$ such that $u(t_{1+}) = 0 = u(t_{2-})$ and $u(t) \neq 0$ on (t_1, t_2). Then the Lyapunov-type inequality

$$\int_{t_1}^{t_2} |a(t)|dt + \left(\int_{t_1}^{t_2} b(t)dt\right)^{1/2} \left(\int_{t_1}^{t_2} c^+(t)dt + \sum_{\tau_i \in (t_1,t_2)} \left(\frac{\beta_i}{\alpha_i}\right)^+ . \frac{1}{(t_2-t_1)^\nu}\right)^{1/2} > \frac{2}{(t_2-t_1)^\nu} \tag{31}$$

holds.

Theorem 5: [Disconjugacy Criterion] Let $a, b, c \in C_p[t_1, t_2]$, $b(t) > 0$ and $\alpha_j \neq 0$ for all $j \in \mathbb{Z}$. If

$$\int_{t_1}^{t_2} |a(t)|dt + \left(\int_{t_1}^{t_2} b(t)dt\right)^{1/2} \left(\int_{t_1}^{t_2} c^+(t)dt + \sum_{\tau_i \in (t_1,t_2)} \left(\frac{\beta_i}{\alpha_i}\right)^+ . \frac{1}{(t_2-t_1)^\nu}\right)^{1/2} \leq \frac{2}{(t_2-t_1)^\nu}, \tag{32}$$

then the Hamiltonian system

$$\begin{cases} T_{t_1}^\nu u(t) = \underline{a}(t)u(t) + \underline{b}(t)v(t), & \nu \in (0,1), \\ T_{t_1}^\nu v(t) = -\underline{c}(t)u(t) - \underline{a}(t)v(t), \end{cases} \tag{33}$$

$$\begin{cases} u(\tau_{i+}) - \alpha_i u(\tau_{i-}) = 0, \\ v(\tau_{i+}) - \alpha_i v(\tau_{i-}) = -\beta_i u(\tau_{i-}), \end{cases} \tag{34}$$

is disconjugate on $[t_1, t_2]$.

Theorem 6: [Zero Count] Assume that $\{s_k\}_{k=1}^{2N+1}$, $N \geq 1$ be an increasing sequence of zeros of $u(t)$ such that every consecutive pair of its elements satisfy in zeros conditions in Theorem 4. Suppose that $\{s_k\}_{k=1}^{2N+1}$ lies in compact interval I with length l. Then

$$\frac{2N}{l^\nu} < \sum_{k=1}^{N} \left\{ \int_{s_{2k-1}}^{s_{2k+1}} |a(t)|dt + \left(\int_{s_{2k-1}}^{s_{2k+1}} b(t)dt\right)^{\frac{1}{2}} \left(\int_{s_{2k-1}}^{s_{2k+1}} c^+(t)dt + \sum_{\tau_i \in (s_{2k-1}, s_{2k+1})} \left(\frac{\beta_i}{\alpha_i}\right)^+ \frac{1}{(s_{2k+1} - s_{2k-1})^\nu}\right)^{\frac{1}{2}} \right\}. \tag{35}$$

Theorem 7: [Nonexistence Criterion] Assume that $a, b, c \in C_p[t_1, t_2]$, $b(t) > 0$ and $\alpha_j \neq 0$ for all $j \in \mathbb{Z}$. Suppose that

$$\int_{t_1}^{t_2} |a(t)|dt + \left(\int_{t_1}^{t_2} b(t)dt\right)^{1/2} \left(\int_{t_1}^{t_2} c^+(t)dt + \sum_{\tau_i \in (t_1,t_2)} \left(\frac{\beta_i}{\alpha_i}\right)^+ . \frac{1}{(t_2-t_1)^\nu}\right)^{1/2} \leq \frac{2}{(t_2-t_1)^\nu}, \tag{36}$$

Then, the Hamiltonian system (33), (34) has no nontrivial solution.

Theorem 8: [Distance Between Consecutive Zeros] Assume that $a \in L^\delta[0, \infty)$, $1 \leq \delta < \infty$ and $y(t) = (u(t), v(t))$ be an oscillatory solution of the Hamiltonian system (33),(34). Let $\{t_n\}_{n=1}^\infty$ be an increasing sequence of zeros of $u(t)$ in $[0, \infty)$ and for any arbitrary $M > 0$, we have

$$\int_t^{t+M} b^\sigma(t)dt \to 0, \quad t \to \infty, \ \sigma \geq 1. \tag{37}$$

Then $t_{n+1} - t_n \to \infty$ as $n \to \infty$.

In the sequel, we introduce the Lyapunov-type inequality corresponding to the fractional S-L type problem (21) and (22) as:

Theorem 9: Assume that

(S_1)

$$\prod_{i=1}^{r} \alpha_i^2 = 1, \tag{38}$$

(S_2)

$$A^2 \geq 4, \tag{39}$$

(S_3)

$$p(t) > 0, \ \int_0^T t^\nu q(t)dt + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} > 0. \tag{40}$$

Then, the Lyapunov type inequality

$$\left(\int_0^T \frac{dt}{p(t)}\right) \left(\int_0^T q^+(t)dt + \sum_{i=1}^{r} \left(\frac{\beta_i}{\alpha_i}\right)^+ \frac{1}{T^\nu}\right) > \frac{4}{T^{2\nu}}, \tag{41}$$

is satisfied.

In the end, we point out this fact that all the qualitative behaviour established above for the fractional Hamiltonian system (18)-(20) can be concluded for fractional Sturm-Liouville type problem (21) and (22).

IV. Conclusion

In this paper, the periodic and non-periodic Lyapunov-type inequalities corresponding to the fractional impulsive Hamiltonian systems of the form (19), (20) studied. As applications to these inequalities, some qualitative gesture such as stability, disconjugacy, nonexistence of non-trivial solutions, upper bound estimation for number of zeros of the non-trivial solutions and distance between consecutive zeros of the oscillatory solutions of the under consideration fractional Hamiltonian systems concluded. We point out once again this fact that as a result of retaining the Leibnitz rule, we apply fractional conformable derivatives that are needed in extracting Lyapunov-type inequalities from the fractional Hamiltonian systems (19), (20). Beside these applications, the same ones can be concluded for fractional Sturm-Liouville type problems (21), (22).

References

[1] T. Abdeljawad; On conformable fractional calculus, *J. Comput. Appl. Math.*, 279, (2015), 57-66.

[2] Mustafa Fahri Aktaş; On the multivariate Lyapunov inequalities, *Appl. Math. Comput.* 232 (2014) 784-786.

[3] Mustafa Fahri Aktaş, Devrim Çakmak, Aydin Tiryaki; On Lyapunov type inequalities of a three-point boundary value problem for third order linear differential equations, Appl. Math. Lett., (2015), In Press.

[4] Drumi Bainov, Valery Covachev; Impulsive Differential Equations With a Small Parameter, World Scientific, (1994).

[5] Sougata Dhar, Qingkai Kong; Liapunov-type inequalities for third-order half-linear equations and applications to boundary value problems, Nonlinear Anal. Theory, Methods and Applications, 110 (2014), 170-181.

[6] Rui A.C. Ferreira; A Lyapunov-type inequality for a fractional boundary value problem, Fract. Calc. Appl. Anal., Vol. 16, No 4 (2013), pp. 978-984; DOI: 10.2478/s13540-013-0060-5.

[7] Rui A. C. Ferreira; On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function, J. Math. Anal. Appl., 412, 2 (2014), 1058-1063.

[8] G. Sh. Guseinov, B. Kaymakçalan; Lyapunov inequalities for discrete linear Hamiltonian systems, Comput. Math. Appl., 45, (2003), 1399-1416.

[9] G. Sh. Guseinov, A. Zafer; Stability criterion for second order linear impulsive differential equations with periodic coefficients, Math. Nachr., 281, No. 9, (2008), 1273-1282.

[10] G. Sh. Guseinov, A. Zafer; Stability criteria for linear periodic impulsive Hamiltonian systems, J. Math. Anal. Appl., 335, (2007), pp. 1195-1206.

[11] Mohamad Jleli, Bessem Samet; Lyapunov type inequalities for a fractional differential equation with mixed boundary conditions, Math. Inequal. Appl., Vol. 18, No. 2 (2015), 443-451.

[12] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh; A new definition of fractional derivative, J. Comput. Appl. Math., 264, (2014), 65-70.

[13] Xin-Ge Liu, Mei-Lan Tang; Lyapunov-type inequality for higher order difference equations, Appl. Math. Comput. 232 (2014) 666-669.

[14] A. M. Lyapunov; The general problem of the stability of motion, Int. J. Control, Vol. 55, No. 3, 1992, pp. 521-790. http://www.tandfonline.com/toc/tcon20/55/3.

[15] B .G. Pachpatte; Lyapunov tye integral inequalities for certain differential equations, Georgian Math. J., 4, No. 2, (1997), 139-148.

[16] B. G. Pachpatte; Inequalities related to zeros of solutions of certain second order differential equations, Ser. Math. Inform., 16 (2001), 35-44.

[17] B. G. Pachpatte; On Lyapunov type inequalities for certain higher order differential equations, J. Math. Anal. Appl., 195 (1995), 527-536.

[18] Gani T. Stamov; Almost Periodic Solutions of Impulsive Differential Equations, Springer, (2012).

[19] X. Yang; On Lyapunov type inequalities for certain higher order differential equations, Appl. Math. Comput., 134 (2003), 307-317.

[20] Xiaojing Yang, Yong-In Kim, Kueiming Lo; Lyapunov-type inequality for a class of even-order linear differential equations, Appl. Math. Comput., 245 (2014), 145-151.

[21] Xiaojing Yang, Yong-In Kim, Kueiming Lo; Lyapunov-type inequalities for a class of higher-order linear differential equations, Appl. Math. Lett., 34 (2014) 86-89.