Stochastic Control of Decentralized Singularly Perturbed Systems

Walid S. Alfuhaid, Saud A. Alghamdi, John M. Watkins, M. Edwin Sawan

***Abstract*—**Designing a controller for stochastic decentralized interconnected large scale systems usually involves a high degree of complexity and computation ability. Noise, observability, and controllability of all system states, connectivity, and channel bandwidth are other constraints to design procedures for distributed large scale systems. The quasi-steady state model investigated in this paper is a reduced order model of the original system using singular perturbation techniques. This paper results in an optimal control synthesis to design an observer based feedback controller by standard stochastic control theory techniques using Linear Quadratic Gaussian (LQG) approach and Kalman filter design with less complexity and computation requirements. Numerical example is given at the end to demonstrate the efficiency of the proposed method.

***Keywords*—**Decentralized, optimal control, output, singular perturb.

I. INTRODUCTION

SOME systems' setups involve large-scale and high-dimension interconnected systems in such a way as to require huge channel bandwidth for controller communication, complex controller design, and expensive implementation. To overcome such limitation of interconnected large-scale systems, decentralized control setup can simplify controller design and reduce the order of the system, communication channel bandwidth, and implementation cost. The order of interconnected large-scale systems can be reduced using singular perturbation techniques by separating the slow and fast dynamics of the overall system behavior. This reduction can minimize the complexity of control algorithms and implementation cost [1]. Considerable research effort has been concentrated toward singular perturbation for long time and interested many researchers as a mathematical technique to simplify and reduce the order of large-scale control systems [2]–[11].

Interconnected subsystems are controlled separately and independently using decentralized control strategies. Each subsystem can be controlled locally using its own states for state feedback control. This makes controllers communicate locally within each subsystem individually. This technique assumes the availability of all states of each subsystem to the local controller, which is not always applicable in real world systems. Usually, state feedback control techniques are used to control and stabilize an unstable system. In the case when system states are not available, control can be implemented using output feedback by observing and estimating the states or by static output feedback. The unavailability of the states is not the only problem involved in using output feedback techniques; the noise in output measurement and disturbance to controller input can have a considerable effect on system performance and even lead to instability. Stochastic control techniques may be used to describe the system with uncertainties [12]–[14].

Decentralized control is widely used in the control of interconnected power systems, distribution networks, traffic systems, computer and communication networks, civil structure $systems, and$ aerospace systems [7], [8], [15], [16]. Considerable research effort was made for decentralized control for large scale systems, [17] studied the stability of decentralized control of interconnected systems using adaptive control scheme. Stability analysis of networked systems was studied by [18] using observer based decentralized control scheme. Reference [16] discussed optimal control design for decentralized large scale systems using modified LQR control. Optimal stabilization for decentralized linear systems using output feedback control strategy was studied by [13]. Stability of decentralized singularly perturbed systems was discussed by [19] and a robust controller was designed using a unified approach. An output feedback control scheme for discrete time decentralized singularly perturbed systems was developed by [20] using an iteration method.

The investigated model in this research of a large scale stochastic decentralized interconnected system was a reduced order quasi-steady state model of the original system using singular perturbation techniques. The minimization of conditions and constraints reaches a solution which applies Lyapunov equations coupled with constraints equations to optimize the performance index of the reduced-order model. A Kalman filter was designed to optimally estimate the states of the fast subsystems to be used in the observer based feedback controller design. Those observed states to be used in the reduced order model to generate several estimates of the main system's states in controlling the overall system performance.

II. PROBLEM STATEMENT

Consider the linear time-invariant decentralized singularly-

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perturbed model in (1) representing the system shown in Fig. 1:

$$\begin{aligned} \dot{x} &= A_{00}x + A_{01}z_1 + A_{02}z_2 \\ \varepsilon\dot{z}_1 &= A_{10}x + A_{11}z_1 + B_1u_1 + G_1w_1 \\ \varepsilon\dot{z}_2 &= A_{20}x + A_{22}z_2 + B_2u_2 + G_2w_2 \\ y_1 &= C_1z_1 + H_1v_1 \\ y_2 &= C_2z_2 + H_2v_2 \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n, z_1 \in \mathcal{R}^{m_1}$, and $z_2 \in \mathcal{R}^{m_2}$ are the state variables of the slow main system and fast subsystems z_1 and z_2 respectively. Inputs $u_i \in \mathcal{R}^{r_i}$ are the control input vectors, and outputs $y_i \in \mathcal{R}^{p_i}$ are the control output vectors of subsystems i respectively for i=1,2. w_i and v_i are the input disturbance and output measurement noise , G_i and H_i are constant scaling matrices corresponds to noise for subsystem i for i=1,2. $A_{00}, A_{01}, A_{02}, A_{10}, A_{11}, A_{02}, A_{20}, A_{01}, A_{22}, B_1, B_2, C_1$ and C_2 are constant matrices with appropriate matching dimensions and it is assumed that the singular perturbation parameter $\varepsilon \ll 1$ is a small positive number.

The objective of this research is to develop a stabilizing, robust and optimal control of reduced-order output feedback controller. This controller should be able to stabilize a linear time-invariant stochastic singularly-perturbed system via decentralized control. Besides states unavailability, the lack of direct control to the main system is a major issue in the problem formulated in (1). The criteria to be analyzed is a standard LQG performance index. The proposed technique will use singular perturbation approach to reduce the order of the original model to be applicable to large-scale systems.

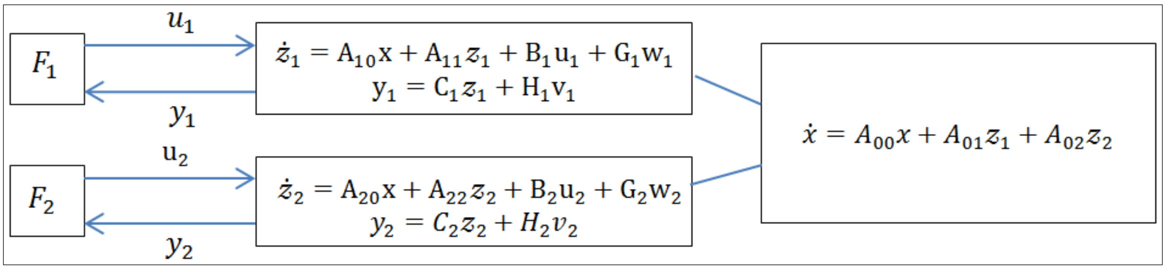

Fig. 1 Decentralized system setup with two subsystems

III. Main Results

In singular perturbation, when ε approaches zero, we assume that the fast subsystems state variables, z_1 and z_2 have reached quasi-steady state. Hence, the system order is reduced to the order of the main system, which is equal to the dimension of the slow state variable x. *When* $\varepsilon \to 0$, *then* $\varepsilon\dot{z} = 0$, which means that only the slow part dynamics affect the system. Now, let $\bar{x}, \bar{z}$ and $\bar{u}$ to represent the quasi-steady state variables of the system. The slow variables of the system can be set as: $x^s = \bar{x}, u^s = \bar{u}$ and $y^s = \bar{y}$. Then (1) reduces to:

$$\begin{aligned} \dot{\bar{x}} &= A_{00}\bar{x} + \textstyle\sum_i A_{0i}\bar{z}_i \\ 0 &= A_{i0}\bar{x} + A_{ii}\bar{z}_i + B_i\bar{u}_i + G_iw_i\ ; \quad i = 1,2 \\ \bar{y}_i &= C_i\bar{z}_i + H_iv_i \end{aligned} \tag{2}$$

By solving the second *equation of* the system (2) for $\bar{z}_i$ we get:

$$\bar{z}_i = -A_{ii}^{-1}(A_{i0}\bar{x} + B_i\bar{u}_i + G_iw_i); \quad i = 1,2 \tag{3}$$

where A_{ii} are non$-$*singular*. By plugging $\bar{z}_i$ from (3) in the first and third equations of (2) we get the reduced order system:

$$\begin{aligned} \dot{\bar{x}} &= A_0\bar{x} + \textstyle\sum[\tilde{B}_i\bar{u}_i + \tilde{G}_iw_i] \\ \bar{y}_i &= \tilde{C}_i\bar{x} + \tilde{D}_i\bar{u}_i + \tilde{\tilde{G}}_iw_i + H_iv_i\ ; \quad i = 1,2 \end{aligned} \tag{4}$$

where,

$$\begin{aligned} A_0 &= A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20} \\ \tilde{B}_i &= -A_{0i}A_{ii}^{-1}B_i \qquad \tilde{C}_i = -C_iA_{ii}^{-1}A_{i0} \\ \tilde{D}_i &= -C_iA_{ii}^{-1}B_i \\ \tilde{G}_i &= -A_{01}A_{11}^{-1}G_1 \quad \text{and} \quad \tilde{\tilde{G}}_i = -C_iA_{ii}^{-1}G_i \end{aligned} \tag{5}$$

The performance index of subsystem i is given by:

$$J_i = \frac{1}{2}\int_0^\infty (\bar{x}^TQ_x\bar{x} + \bar{z}_i^TQ_{zi}\bar{z}_i + \bar{u}_i^TR_i\bar{u}_i)dt \tag{6}$$

where: Q_x ,Q_{z_i} are positive semi definite symmetric matrices and R_i is a positive definite symmetric matrix. By substituting the values of $\bar{z}_i$ (3) we get the reduced performance index as:

$$J_i = \frac{1}{2}\int_0^\infty \{\bar{x}^T\bar{Q}\bar{x} + 2\bar{x}^T\bar{M}_i\bar{u}_i + \bar{u}_i^T\bar{R}_i\bar{u}_i\}dt + J_w \tag{7}$$

where J_w, is a term contains the integral of the variance of the noise which is not affected by control [21], and

$$\begin{aligned} \bar{Q} &= Q_x + A_{i0}^TA_{ii}^{-T}Q_{zi}A_{ii}^{-1}A_{i0} \\ \bar{M}_i &= B_i^TA_{ii}^{-T}Q_{zi}A_{ii}^{-1}A_{i0} \\ \bar{R}_i &= R_i + B_i^TA_{ii}^{-T}Q_{zi}A_{ii}^{-1}B_i \end{aligned} \tag{8}$$

The feedback input for subsystem i is chosen to be:

$$\bar{u}_i = \bar{F}_i\bar{x} = -\bar{R}_i^{-1}\bar{M}_i^T\bar{x} \tag{9}$$

To get rid of the cross product term in (7) we set $\bar{u}_i = \bar{u}_{0i} + \bar{F}_{0i}\bar{x}$ and the performance index (7) will reduce to:

$$J_i = \frac{1}{2}\int_0^\infty \{\bar{x}^T\bar{Q}_{0i}\bar{x} + \bar{u}_{0i}^T\bar{R}_i\bar{u}_{0i}\}dt \tag{10}$$

where,

$$\bar{Q}_{0i} = \bar{Q} - \bar{M}_{0i}\bar{R}_i^{-1}\bar{M}_{0i}^T, \quad \text{and} \quad A_{0c} = A_0 - \tilde{B}_i R_i^{-1}\bar{M}_i^T \tag{11}$$

and the optimal control input of the system will take the form:

$$u_{0i}^* = -R_i^{-1}\left(\bar{M}_i^T + \tilde{B}_i^T K\right)\bar{x} = -F_i^*\bar{x} \tag{12}$$

and K is the solution of the Riccati equation

$$0 = KA_{0c} + A_{0c}^T K - K\tilde{B}_i\bar{R}_i^{-1}\tilde{B}_i^T K + \bar{Q}_{0i} \tag{13}$$

The optimal cost for subsystem *i* will be:

$$J_i = \tfrac{1}{2}tr\{K\Sigma\} + \tfrac{1}{2}tr\{PFW_iF^T\} \tag{14}$$

where W_i is the intensity of w_i, $\Sigma = E\{x_0x_0^T\}$ and P is the solution of the Lyapunov equation

$$0 = PA_{0c} + A_{0c}^T P + \bar{Q}_{0i} \tag{15}$$

When the system reaches quasi-steady state, the reduced system state $\bar{x}$ will be presented as the slow dynamics state x^s. For the system (4), where there is no access to the main state x^s, instead we can use an estimate of the state variable using Kalman filter. Because of the decentralized nature of the system setup, we will not be able to use the same estimate of the state variable x^s for both subsystems. Hence, there will be two estimates for the main system states x^s using two observers (Kalman filters) connected to each subsystem. The observed system:

$$\dot{\hat{x}}_i^s = A_0\hat{x}_i^s + B_{01}u_1^s + B_{02}u_2^s + L(y_i^s - \hat{y}_i^s)\hat{y}_i^s = \tilde{C}_i\hat{x}_i^s + \tilde{D}_i u_i^s \tag{16}$$

where $\hat{x}_i^s$ is the estimate of the quasi-steady state variable of the main system X through the Kalman filter connected to subsystem i ; $i = 1,2$. The error between $\hat{x}_i^s$ and the actual x^s is then calculated:

$$\dot{e}_i = \dot{x}^s - \dot{\hat{x}}_i^s = A_c e_i + w_0 - Lv_i \tag{17}$$

where; $\dot{e}_i$ is the $error\ of\ the\ estimated$ state x using Kalman filter connected to subsystem *I*, and

$$w_0 = G_{01}w_1 + G_{02}w_2 \tag{18}$$

The error $variance$ is defined as:

$$P = E\{e_ie_i^T\} \tag{19}$$

then

$$\dot{P} = A_cP + PA_c^T + W_0 - LV_iL^T \tag{20}$$

where, W_0 and V_i are intensities of w_0 and v_i respectively. For this well-known Lyapunov Equation, minimization is achieved by choosing:

$$L = KC^TV_i^{-1} \tag{21}$$

where K is the solution of the Riccati Equation:

$$\dot{K} = A_cK + KA_c^T + W_0 - KC^TV_i^{-1}CK \tag{22}$$

with $K(0) = \Sigma$

IV. Numerical Example

Consider the $following$ decentralized singularly perturbed system:

$$\begin{gathered}
\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} z_1 + \begin{bmatrix} -3 & 2 \\ -4 & -1 \end{bmatrix} z_2 \\
\varepsilon\dot{z}_1 = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} x + \begin{bmatrix} -5 & 4 \\ -3 & -1 \end{bmatrix} z_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_1 + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w_1 \\
\varepsilon\dot{z}_2 = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} x + \begin{bmatrix} -2 & -1 \\ -2 & -3 \end{bmatrix} z_2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_2 + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w_2 \\
y_2 = [1 \quad 0] z_2 + v_2 \\
y_1 = [0 \quad 1] z_1 + v_1
\end{gathered}$$

with $\varepsilon = 0.1$ and only z_1 the optimal full order model feedback gain found to be:

$$F^* = [-0.16 \quad -0.27 \quad 0.04 \quad 0.46 \quad -0.04 \quad 0.05],$$

with

$$K(0) = \begin{bmatrix} 0.41 & -0.12 & 0.02 & -0.02 & -0.05 & 0.02 \\ -0.12 & 0.46 & 0 & -0.01 & 0.05 & -0.05 \\ 0.02 & 0 & 0.01 & 0 & 0 & 0 \\ -0.02 & -0.01 & 0 & 0.03 & 0 & 0 \\ -0.05 & 0.05 & 0 & 0 & 0.04 & -0.02 \\ 0.02 & -0.05 & 0 & 0 & 0.02 & 0.03 \end{bmatrix}$$

and the close loop poles are:

$$\begin{bmatrix} -33.68 \\ -33.68 \\ -34.99 \\ -11.08 \\ -5.6 \\ -5.6 \end{bmatrix}$$

with $W_1 = W_2 = 0.1$, full order system optimal cost is: 0.3150. The reduced order Model:

$$\begin{gathered}
\dot{x}_s = \begin{bmatrix} -4 & -12.46 \\ 4 & -16.38 \end{bmatrix} x_s + \begin{bmatrix} 0.12 \\ -1.35 \end{bmatrix} u_{s1} + \begin{bmatrix} 0.02 \\ -0.05 \end{bmatrix} w_1 \\
y_s = [-1 \quad 0.53] x_{s1} + 0.41\, u_{s1} + 0.01 w_1 + v_1
\end{gathered}$$

The optimal feedback gain for this reduced order model is: $F^* = [0.5 \quad -0.29]$. By applying this optimal feedback gain to the full order model, the closed loop system optimal cost is: 0.3204. with the following matrix:

$$K(0) = \begin{bmatrix} -0.12 & 0.48 & 0 & -0.02 & 0.05 & -0.05 \\ 0.02 & 0 & 0.01 & -0.01 & 0 & 0 \\ -0.02 & -0.02 & -0.01 & 0.03 & 0 & 0 \\ -0.05 & 0.05 & 0 & 0 & 0.04 & -0.02 \\ 0.02 & -0.05 & 0 & 0 & -0.02 & 0.03 \end{bmatrix}$$

The response of the output and states of the system are shown in Figs. 2 and 3. For subsystem 2, the optimal cost found to be: 0.341 and reduced optimal cost: 0.328.

The optimal cost of the reduced order model is very close to that of the full order system within the order of (ε^2).

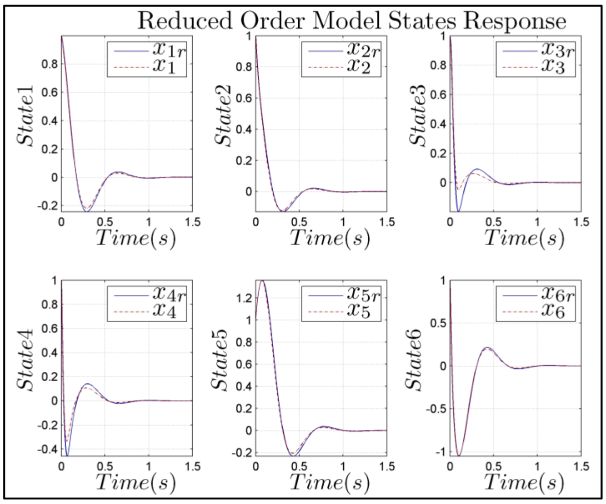

Fig. 2 State response for full and reduced order Models

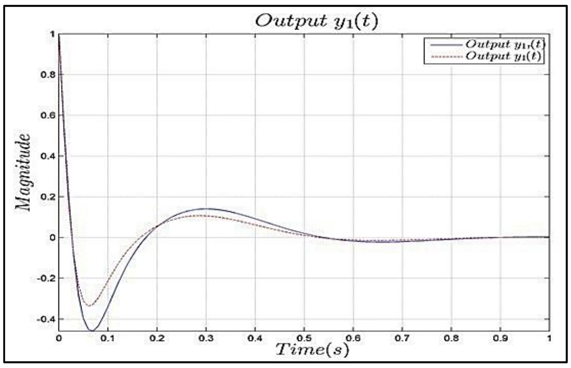

Fig. 3 Output response for full and reduced order Models

V. Conclusion

In this paper, the singularly perturbed decentralized system was found to have an optimal control with optimum feedback gain satisfying the Riccati equation in the quasi-steady state model. When the system is exposed to either a measurement noise, input disturbance or both, a solution exists using LQG approach and Kalman filter with certain assumptions in addition to basic knowledge of the white noise distribution. The numerical example has shown excellent results that the proposed method has succeeded in decreasing the computational workspace and the quadratic convergence has been attained. The new technique in this research will help to simplify system analysis and controller design and can be expanded for multi-subsystems.

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