International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:10, No:2, 2016

Dynamic Analysis of Viscoelastic Plates with Variable Thickness

Gülçin Tekin, Fethi Kadıoğlu

***Abstract*—**In this study, the dynamic analysis of viscoelastic plates with variable thickness is examined. The solutions of dynamic response of viscoelastic thin plates with variable thickness have been obtained by using the functional analysis method in the conjunction with the Gâteaux differential. The four-node serendipity element with four degrees of freedom such as deflection, bending, and twisting moments at each node is used. Additionally, boundary condition terms are included in the functional by using a systematic way. In viscoelastic modeling, Three-parameter Kelvin solid model is employed. The solutions obtained in the Laplace-Carson domain are transformed to the real time domain by using MDOP, Dubner & Abate, and Durbin inverse transform techniques. To test the performance of the proposed mixed finite element formulation, numerical examples are treated.

***Keywords*—**Dynamic analysis, inverse Laplace transform techniques, mixed finite element formulation, viscoelastic plate with variable thickness

I. Introduction

ADOPTING elastic theory for the solution of load bearing structural elements proves to be inconsistent with reality due to the fact that real materials exhibit a mixture of elastic and viscous properties. Therefore, viscoelastic theory appears to be more suitable for describing and analyzing the behavior of structural elements. There are many works in the literature on the theory of viscoelasticity [1], [2].

Due to the mathematical complexity in the viscoelastic constitutive relations, closed-form solutions are often not possible and numerical solution techniques should be performed. Among the computational methods used for viscoelastic problems, the Finite Element Method is the most common and versatile and it has been applied to static and dynamic problems in structural mechanics. The application of the Finite Element Method to viscoelastic problems has been presented by a number of authors [3]-[5]. However, to the best of our knowledge, there are very few published studies on the analysis of viscoelastic plates [6]-[9].

In this study, the dynamic response of viscoelastic Kirchhoff plates with variable thickness is examined by using the mixed finite element method in the transformed Laplace-Carson space. In order to construct a functional for viscoelastic Kirchhoff plates of variable thickness, an efficient systematic procedure based on the Gâteaux differential method is employed. In this functional, there exists four independent variables such as deflection (w), internal forces (M_x, M_y and M_{xy}) in addition to the dynamic and geometric boundary condition terms. In the solution of viscoelastic plate, three-parameter solid model is considered. For transformation of the solutions obtained in the Laplace-Carson domain to the time domain, different numerical inverse transform techniques are employed.

G. Tekin is with the Civil Engineering Department, Istanbul Technical University, Istanbul, Turkey (corresponding author: phone: +90-212-2853701; fax: +902122856587; e-mail: gulcintekin@ itu.edu.tr).
F. Kadıoğlu is with the Department of Civil Engineering Istanbul Technical University, Istanbul, Turkey (e-mail: fkadioglu@itu.edu.tr).

II. Mixed Finite Element Formulation of Viscoelastic Plates with Variable Thickness

Considering a plate with the thickness h under distributed lateral load q as shown in Fig. 1, the equilibrium equation of the plate according to the linear theory of Kirchhoff-Love takes the following form [10-11]:

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x \partial y}+q=0 \tag{1}$$

where M_x, M_y and M_{xy} are the moments depicted in the positive sense in Fig. 1.

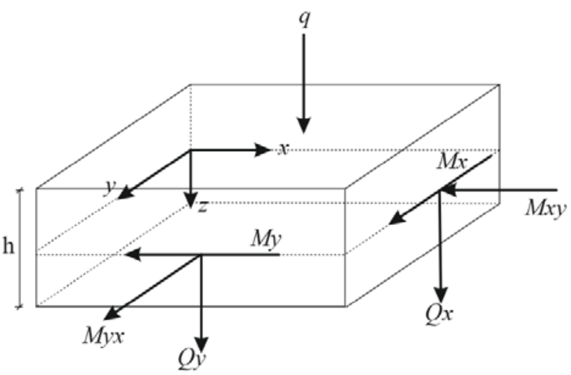

Fig. 1 Positive directions for internal forces

The flexural rigidity of the plate D is given by:

$$D=\frac{E h^3}{12(1-\nu^2)} \tag{2}$$

with E being the Young's modulus and ν being the Poisson's ratio of the plate. The bending moment and twisting moment resultants related to the transverse displacement (w) of the plate's middle surface are given by:

$$
\begin{aligned}
M_x &= -D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] \\
M_y &= -D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \\
M_{xy} &= -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y}
\end{aligned} \tag{3}
$$

After the generalized stress-strain relationships for a viscoelastic Kirchhoff plate is written using two operators E_1*and E_2*which are in the hereditary integral form [8], [12], the field equations of viscoelastic thin plates become:

$$
\begin{aligned}
&\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + q = 0 \\
&M_x = -\frac{h^3}{12}\left[E_1^* \frac{\partial^2 w}{\partial x^2} + E_2^* \frac{\partial^2 w}{\partial y^2}\right] \\
&M_y = -\frac{h^3}{12}\left[E_1^* \frac{\partial^2 w}{\partial y^2} + E_2^* \frac{\partial^2 w}{\partial x^2}\right] \\
&M_{xy} = -\frac{h^3}{12}\left[E_1^* - E_2^*\right] \frac{\partial^2 w}{\partial x\, \partial y}
\end{aligned} \tag{4}
$$

In order to remove the time derivatives from the field equations and boundary conditions, the Laplace-Carson transform method is employed. After taking the Laplace-Carson transform with respect to the time, field equations become:

$$
\begin{aligned}
&\frac{\partial^2 \bar{M}_x}{\partial x^2} + \frac{\partial^2 \bar{M}_y}{\partial y^2} + 2\frac{\partial^2 \bar{M}_{xy}}{\partial x \partial y} + \bar{q} = 0 \\
&\bar{M}_x = -\frac{h^3}{12}\left[\bar{E}_1^* \frac{\partial^2 \bar{w}}{\partial x^2} + \bar{E}_2^* \frac{\partial^2 \bar{w}}{\partial y^2}\right] \\
&\bar{M}_y = -\frac{h^3}{12}\left[\bar{E}_1^* \frac{\partial^2 \bar{w}}{\partial y^2} + \bar{E}_2^* \frac{\partial^2 \bar{w}}{\partial x^2}\right] \\
&\bar{M}_{xy} = -\frac{h^3}{12}\left[\bar{E}_1^* - \bar{E}_2^*\right] \frac{\partial^2 \bar{w}}{\partial x\, \partial y}
\end{aligned} \tag{5}
$$

and boundary conditions in symbolic form become:

$$
\begin{aligned}
&\bar{T} - \hat{\bar{T}} = 0 \\
&-\bar{M} + \hat{\bar{M}} = 0 \\
&\bar{w}' - \hat{\bar{w}}' = 0 \\
&-\bar{w} + \hat{\bar{w}} = 0
\end{aligned} \tag{6}
$$

Field equations in the Laplace-Carson space can be written in operator form as:

$$
\bar{Q} = \bar{L}\,\bar{y} - \bar{f} \tag{7}
$$

where $\bar{Q}$ will be a potential operator in the Laplace-Carson space, if the equality [13]:

$$
< d\bar{Q}\,(\bar{y} - \bar{y}'), \bar{y}^* > = < d\bar{Q}\,(\bar{y} - \bar{y}^*), \bar{y}' > \tag{8}
$$

is satisfied. The < , > symbol indicates the inner product. After satisfying (8), the functional corresponding to the field equations is given by [13]:

$$
I(\bar{y}) = \int_0^1 \left[\bar{Q}\;(s\,\bar{y}), \bar{y}\right] ds \tag{9}
$$

where s is a scalar quantity. Using (9), the explicit form of the functional corresponding to the field equations of viscoelastic Kirchhoff plates with variable thickness in the Laplace-Carson space is given as:

$$
\begin{aligned}
I(\bar{y}) &= \left[\bar{w}_{,x}, \bar{M}_{x_{,x}}\right] + \left[\bar{w}_{,y}, \bar{M}_{y_{,y}}\right] + \left[\bar{w}_{,x}, \bar{M}_{xy_{,y}}\right] + \left[\bar{w}_{,y}, \bar{M}_{xy_{,x}}\right] \\
&-[\bar{q},\bar{w}] - \frac{1}{2\bar{D}\,(1-\nu^2)}\left\{\left[\bar{M}_x,\bar{M}_x\right] + \left[\bar{M}_y,\bar{M}_y\right]\right\} \\
&+ \frac{\nu}{\bar{D}\,(1-\nu^2)}\left[\bar{M}_x,\bar{M}_y\right] - \frac{1}{\bar{D}\,(1-\nu)}\left[\bar{M}_{xy},\bar{M}_{xy}\right] \\
&-\left[\hat{\bar{T}},\bar{w}\right]_\sigma - \left[\left(\bar{M} - \hat{\bar{M}}\right),\bar{w}'\right]_\sigma - \left[\hat{\bar{w}}',\bar{M}\right]_\varepsilon - \left[\left(\bar{w} - \hat{\bar{w}}\right),\bar{T}\right]_\varepsilon
\end{aligned} \tag{10}
$$

In which the brackets with the subscripts σ and ε represents the dynamic and geometric boundary conditions, respectively. The shape functions for the rectangular master element in Fig. 2 are:

$$
\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \quad (i = 1,2,3,4) \tag{11}
$$

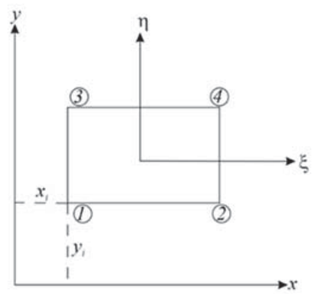

Fig. 2 Rectangular master element

The four variables of the functional given in (10) and the thickness of the plate are expressed by the shape functions ψ_i in the element as:

$$
\begin{aligned}
\bar{w} &= \sum_{i=1}^{4} \bar{w}_i \Psi_i(\xi,\eta) \\
\bar{M}_x &= \sum_{i=1}^{4} \bar{M}_{x_i} \Psi_i(\xi,\eta) \\
\bar{M}_y &= \sum_{i=1}^{4} \bar{M}_{y_i} \Psi_i(\xi,\eta) \\
\bar{M}_{xy} &= \sum_{i=1}^{4} \bar{M}_{xy_i} \Psi_i(\xi,\eta) \\
h &= \sum_{i=1}^{4} h_i \Psi_i(\xi,\eta)
\end{aligned} \tag{12}
$$

and then inserting these approximations into the functional and simplifying with respect to nodal variables, the element matrix can be obtained explicitly.

The numerical solutions obtained in the Laplace-Carson space are transformed to the real time space using different numerical inverse Laplace transform techniques as MDOP, Dubner & Abate, and Durbin. For more information of Laplace inversion process, the reader is referred to the literature [14]-[17].

III. Numerical Examples

In this section, three numerical examples are treated in order to test the performance of the proposed mixed finite element formulation for the dynamic analysis of viscoelastic Kirchhoff plates with variable thickness. Three-parameter solid model as illustrated in Fig. 3 is employed in the solutions.

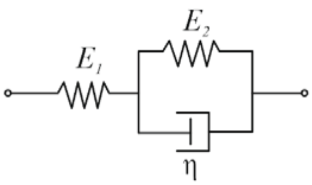

Fig. 3 Three-parameter Kelvin solid model

The material properties are assumed to be:

- E_1= 98 MPa,
- η= 0.245 MPa.s,
- E_2= 24. 5 MPa,
- ν=0.3

Hence, the relaxation modulus of the model is given as:

$$J_{(t)} = \frac{1}{E_1} + \frac{1}{E_2}\left(1 - e^{-\frac{E_2}{\eta}t}\right) \quad (13)$$

Due to the symmetry, the computations are carried out for a quarter of the simply supported plate (as in Fig. 4) by using 4x4 mesh size. Geometrical properties of the plate with variable thickness are a=b=4m.

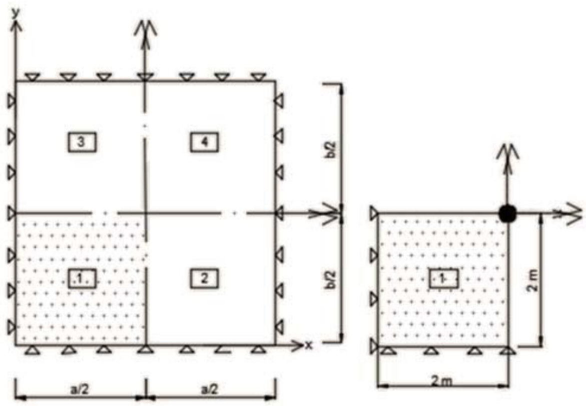

Fig. 4 Symmetry property of simply supported plate

The dynamic behavior of viscoelastic plate is obtained for two different uniformly varying thickness problems.

A. Thickness of the plate is constant along y axis and uniformly varying along x axis as illustrated in Fig. 5 (a).
B. Thickness of the plate is uniformly increasing from supports to the center of the plate as illustrated in Fig. 5 (b).

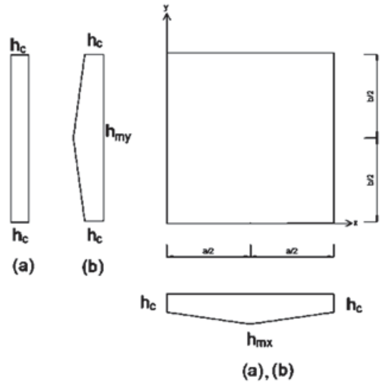

Fig. 5 Plate with variable thickness

For the numerical inversion, the results are obtained when the time increment dt =T/N =0.2 and aT= 5. The time variations of loads considered in numerical examples are illustrated in Fig. 6.

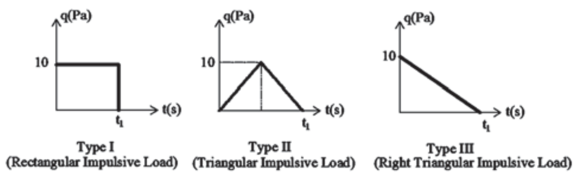

Fig. 6 Time histories of loads

Example 1:

A simply supported viscoelastic plate with uniformly increasing thickness from supports to center of the plate along x and y axes subjected to rectangular impulsive load (Type I) for t_1=10 s is considered. At the center of the plate, the thickness $h_{mx}=h_{my}=2h_c$ and at the supports h_c=0.1 m.

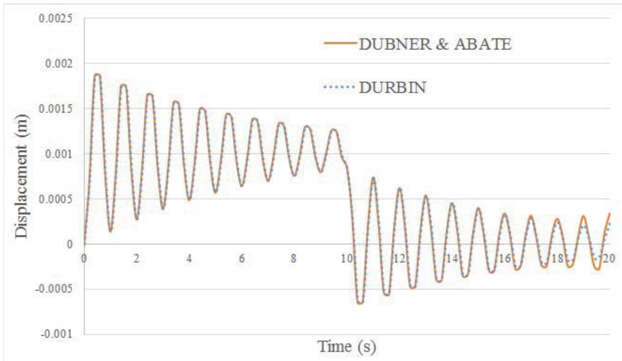

Fig. 7 Dynamic behavior of the displacement at the center of the plate with varying thickness

The central displacement variation with time is illustrated in Fig. 7 by using Dubner & Abate's and Durbin's inverse transform techniques. The material density ρ is assumed as 200 kg/m^3. The dynamic behavior of viscoelastic plate will eventually disappear with time.

Example 2:

In this example, a simply supported viscoelastic plate with uniformly increasing thickness from supports to center of the plate along x and y axes under the triangular impulsive load (Type II) for t_1=10 s is considered. At the center of the plate, the thickness $h_{mx}=h_{my}=2h_c$ and at the supports h_c=0.1 m. The material density ρ is assumed as 200 kg/m^3. For the numerical inversion, MDOP, Dubner & Abate's, and Durbin's inverse transform methods are employed.

In dynamic problems, Durbin's and Dubner & Abate's inverse transform methods give better results when compared to the MDOP. Therefore, the time-dependent central displacement and bending moment values are presented for Dubner & Abate and Durbin inverse transform methods as illustrated in Fig. 8.

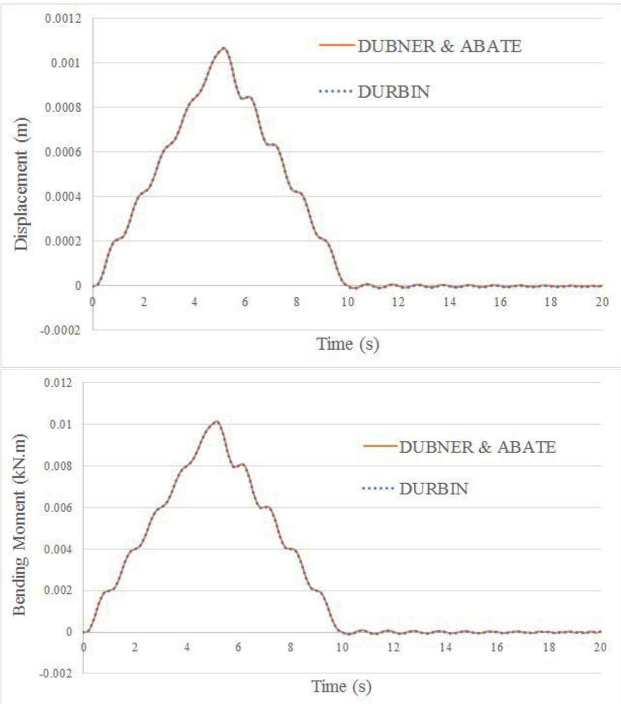

Fig. 8 Dynamic behavior of the displacement and bending moment at the center of the plate with varying thickness

Example 3:

In this example, the time-dependent central displacement of simply supported viscoelastic Kirchhoff plates with constant thickness along y-axis and uniformly varying thickness along x-axis are considered. For the analysis, right triangular impulsive load (Type III) for t_1=10 sec is employed. The effect of the central thickness of the plate on the amplitude of the displacement values and frequency are considered for the center thickness of the plate along x axis, $h_{mx}=2h_{my}=2h_c$ and $h_{mx}=3h_{my}=3h_c$ where h_c=0.1 m. For the numerical inversion, Durbin's inverse transform method is employed.

The material density ρ is assumed as 200 kg/m^3 for dynamic analysis. The effect of the central thickness of the plate on the amplitude of the displacement values and frequency are shown in Fig. 9. As expected, frequency increases with increasing thickness.

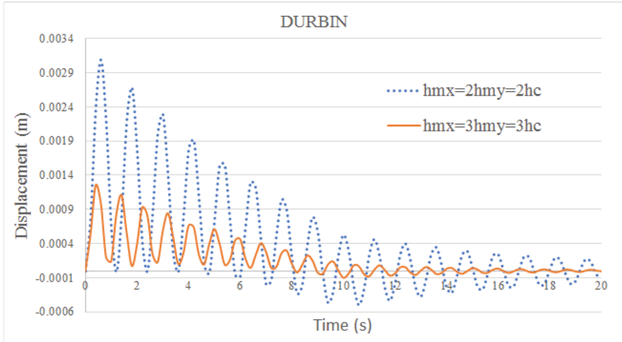

Fig. 9 Effect of thickness on the frequency and displacement

IV. Conclusion

In this study, the dynamic response of viscoelastic thin plates with variable thickness is investigated. For the analysis, mixed finite element formulation based on the Gâteaux differential is used. Derived functional has four independent variables in addition to the dynamic and geometric boundary conditions. In order to remove the time derivatives from governing equations and boundary conditions, the Laplace-Carson transformation is used. For numerical inversion from the Laplace-Carson domain to the time domain, transform methods such as MDOP, Dubner & Abate, and Durbin are employed. The performance of the proposed mixed finite element formulation is tested through various dynamic example problems. It is observed that results are quite reasonable.

Acknowledgment

This research is supported by the Scientific and Technological Research Council of Turkey under the grant number 213M332 and by the Research Foundation of Istanbul Technical University (ITU) under grant number 37961. The authors gratefully acknowledge this support.

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Gülçin Tekin received her BSc degree in Civil Engineering from Yıldız Technical University, Istanbul, Turkey. She completed MSc degree at the Boğaziçi University, Istanbul, Turkey. She is currently a third-year PhD student in the department of Civil Engineering at the Istanbul Technical University, Istanbul, Turkey. Presently she is working as a research assistant at Istanbul Technical University. Her areas of interest are Structural Analysis, Mixed Finite Element Methods.

Dr. Kadıoğlu received his BSc, MSc and PhD from Istanbul Technical University, Istanbul, Turkey. He is currently working as an Associate Prof. in the Department of Civil Engineering at Istanbul Technical University. He published over 35 papers in international journals and conferences, 5 books and book chapters in the fields of Structural Mechanics. Dr. Kadıoğlu's current research interests include: Composite Structures, Stability of Structures and Vibration, Elastic, Viscoelastic Analysis, Structural Analysis and Finite Element Methods.