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Exploring Counting Methods for the Vertices of Certain Polyhedra with Uncertainties

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Abstract**—**Vertex Enumeration Algorithms explore the methods and procedures of generating the vertices of general polyhedra formed by system of equations or inequalities. These problems of enumerating the extreme points (vertices) of general polyhedra are shown to be NP-Hard. This lead to exploring how to count the vertices of general polyhedra without listing them. This is also shown to be #P-Complete. Some fully polynomial randomized approximation schemes (*fpras*) of counting the vertices of some special classes of polyhedra associated with Down-Sets, Independent Sets, 2-Knapsack problems and *2 x n* transportation problems are presented together with some discovered open problems.

Keywords**—**Approximation, counting with uncertainties, mathematical programming, optimization, vertex enumeration.

I. Introduction

VERTEX ENUMERATION (VE) algorithms were of special interest in the 60s, 70s, 80s and 90s [1]-[4]. This is partly due to general real life applications of the methods. Dyer [2] was able to show that for general polyhedra there is no known polynomial methods for enumerating the vertices. This lead to the exploration of counting the extreme-points (vertices) of general polyhedra without listing them. This is also shown to be #P-Complete [5]. Researchers [1], [5] and [6] in recent time concentrated on approximately counting the vertices of some selected classes of polyhedra using known methods of *fully polynomial randomized approximation Schemes* (*fpras*). In this paper, some *fpras* for counting the vertices of polyhedra associated with Down-Set, Independent Set and 2-knapsack problems and $2 \times n$ transportation problems are presented and in the process some newly discovered open problems are raised.

One of the main problems encountered of vertex enumeration and/or counting is that of degeneracy. By definition, a vertex of a polyhedron P is said to be *degenerate* if there are more number of associated inequalities that are binding. Perturbation technique could be employed to deal with polyhedera that are not highly degenerate. This is fully explained in [4]. But, for highly degenerate polyhedera, perturbation technique does not completely solve the problem. This is for the simple reason that many-to-one correspondence might emerged between vertices of the perturbed polytope and the vertices of the original polyhedra. Details of these techniques are available in [1] and [4].

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This article consists of sections that discussed some real life applications of vertex enumeration methods and counting. Some relevant definitions and notations are given. Some introductory methods of approximately counting the vertices of polyhedra associated with polynomially many inequalities, such as, polyhedron associated with: partial order; independent sets; binary knapsack problems; and 2 by n transportation problems are presented.

An important question to ask is: what are the real life applications of vertex enumeration and counting?

II. Applications

Vertex enumeration and counting problems have many real life applications. Dyer [4] was able to present different types of applications of vertex enumeration procedures which can be summarized as: "*dual representation*" which described the set of solutions to the defined inequalities of polytopes represented as convex hull. Another application is that of "*near-optimal*" solutions to a linear program which is used in "*sensitivity analysis*". Multi-parametric linear program for a "*multi-criterion*" linear programming is used for complex decision making.

A more complex application is in the "*game-theory*" such as described in two-person *bi-matrix* games. Non-degenerate cases of vertex enumeration and counting are used in "*fixed-charging*" problems which is applicable for generalized linear-programming in which each variable has a non-negative fixed-charge which is incurred if it is non-zero in the solution. Details of these application methods are described in [3] and [4].

III. Definitions / Notations

Polyhedron: A polyhedron P is defined to be a set satisfying the following:

$$P = \{x \in R^n : \sum_{j=1}^{n} a_{ij} x_j \le b_i, j = 1,2,........, m+n\} \quad (1)$$

Vertex: A vertex v of the Polyhedron P is defined to be the unique point of intersection of at least n of the binding hyperplanes H, where:

$$H = \{x \in R^n : \sum_{j=1}^{n} a_{ij} x_j = b_i\}, \quad i = 1,2,......, m+n \quad (2)$$

#*P*-Counting Problem: A counting problem $f:\Sigma^* \rightarrow N$ is said to belong to the complexity class *#P* if there is a polynomial time predicate $\chi:\Sigma^* \times \Sigma^* \rightarrow \{0,1\}$ and a polynomial p such that $\forall x \in \sum^*$ we have:

$$f(x) = \left|\{w \in \sum^* : \chi(x,w) \wedge |x| \le p(|x|)\}\right| \text{ [5], [6].}$$

Markov Chain (MC): A sequence $(X_t \in \Omega)_{t=0}^{\infty}$ of random variables is a Markov Chain (MC) with state space Ω, if:

$$\Pr[X_{t+1} = y | X_t = x_t, X_{t-1} = x_{t-1}, \ldots X_o = x_o] = \Pr[X_{t+1} = y | X_t = x_t] \quad (3)$$

$$\forall t \in N ,$$

and,

$$\forall x_t, x_{t-1}, \ldots, x_o \in \Omega.$$

Some approximation methods for counting the extreme points of certain classes of polyhedra are presented in detail in [1]. These methods are classified as approximately counting the vertices of polyhedron with polynomially many inequalities.

IV. Polyhedra with Polynomially Many Inequalities

What it meant by 0-1-Permanent?

0-1-PERMANENT: The permanent of $n \times n$ non-negative matrix $A = (a(i,j))$ is defined as:

$$per(A) = \sum_{\pi} \prod_{i} a(i, \pi(i)) \quad (4)$$

where, the sum is over all permutations π of $\{1,2,\ldots,n\}$. If A is $0,1$ matrix, then the permanent is called $0-1-Permanent$.

Claim: There is a $1-1$ correspondence between the number of perfect matching and the $0-1-$ vertices of the polyhedra formed from the following:

$$\sum_{i=1}^{n} a_{ij} x_{ij} = 1, \; (j = 1,2,\ldots,n) \quad (5)$$

$$\sum_{j=1}^{n} a_{ij} x_{ij} = 1, \; (i = 1,2,\ldots, n) \quad (6)$$

Based on the claim we have:

Proposition: There exists a fully polynomial randomized approximation scheme (*fpras*) for counting the $0-1-$ vertices of a polyhedron formed by the permanent of (5) and (6).

V. Partial Order (PO)

A relation R on a set A is called a partial order if it is *reflexive*, *antisymmetric* and *transitive*.

A. Down Set in a Partial Order (PO)

A **Down-Set** S is a subset, $S \subset U$ such that if $j \in S$ and $i \le j$ then $i \in S$ (where $\le$ is the given partial order)

Definition: An $m \times n$ integral matrix A is **totally unimodular** (TU), if the determinant of each square *submatrix* of A is equal to 0, 1 or -1.

Definition: For a given partial order, $\le$ consider the polyhedron in R^n :

$$P^{(0,1)} = \{x : x_i \le x_j \; if \; i \le j, where \; 0 \le x_k \le 1, \forall k\} \quad (7)$$

The coefficient matrix, $A^{(0,1)}$ is a two per row matrix, each with one +1 and one -1.

A directed bipartite graph G is a graph whose nodes are partitioned into two sets such that all the arcs in the graph are directed from a node in the first set – known as *origin* to a node in the second set –known as *destination*. Fig. 1 shows a bipartite graph G with a partial order (PO) and the relationship between an Independent Set and the corresponding Down-Set (S). The *partial order* (PO) on the graph, S, is obtained such that: a < e because there is an *arc* between *node a* and *node e*. Similarly, *b < e*, *b < f*, *c < f*, *c < g*, *d < g* and *d < h*. For example, the Independent Set can be chosen to be {*a, f, g*} which is complemented on the bottom to give Down-Set: {*b, c, d, f, and g*}.

Proposition: For a given partial order, the polyhedron $P^{(0,1)}$ given in (7) is integral with $0,1-vertices$.

Theorem: There is a $1-1$ correspondence between Down-Sets and the vertices of $P^{(0,1)}$ defined in (7).

Proof: If S is a Down-Set, let the vector x^s be defined by, $x_i^s = 1$ if $i \in S$, $x_i^s = 0$ otherwise. Since S is a Down-Set, x^s lies in $P^{(0,1)}$. Also, it exactly satisfies n of the inequalities $0 \le x_i \le 1$, and hence is a vertex. Clearly, this argument can be reversed, since all vertices of $P^{(0,1)}$ have coordinates 0 and 1. QED [1].

VI. Independent Set

Definition: An Independent Set of a graph $G = (V, E)$ is a subset $V^* \subseteq V$ of vertices such that each edge in E is incident on at most one vertex in V^*.

The problem of computing a maximum *independent set* in a graph is $NP-hard$ problem [7].

Dyer and Greenhill [8] have shown that the problem of counting Independent Set in graphs with maximum degree 3 is #P-Complete. An example of Independent Set with complementary properties to Down-Sets (S) is given in Fig. 1.

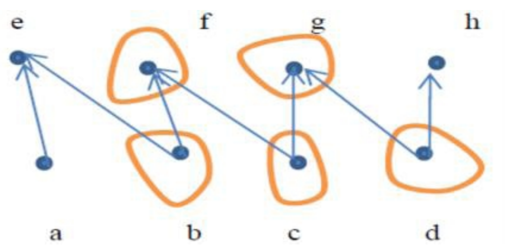

Fig. 1 Complementary relation between elements of IS and S

Claim: Any Independent Set (IS) in a bipartite graph G can be complemented to give a Down-Set S in the associated partial order.

Proof: This follows directly from the fact that $\forall x \in IS$ there exactly no $y \in IS$ such that, $x < y$ i.e. there is no edge between x and y. Also, we have, $\forall y \in S \quad x \in S \ iff \ x < y$ so there is no edge between y and any $x \notin S$. QED [1], [9].

Definition: The polyhedron associated with Independent Sets (IS) in a bipartite graph $G = (V, E)$ is given as:

$$P^{IS} = \{x : x_i + y_j \leq 1, \forall (i,j) \in E, i = 1,2,..,m, j = 1,2,...,n\} \tag{8}$$

Corollary: If A^{IS} is a matrix associated with Independent Set, then it is totally unimodular (TU).

Proof: For j on the bottom put $x_j = 1 - y_j \ (j = 1,2,...,m)$ then we have the polyhedron for the corresponding partial order. QED [1].

From the corollary and proposition it can be concluded that the Down-Sets and Independent Sets are set of integral vertices. The matrices associated with the polyhedra whose sets of solutions (vertices) are Down-Sets or independent sets are totally unimodular (TU), because they have at most two non-zero per row or column that are +1 and -1. It is not yet known whether there is an *fpras* for Down-Set or Independent set, hence the following open problem.

Open Problem: Is there an *fpras* for counting Down-Sets in a partial order or Independent Sets in a bipartite graph? [9].

VII. Binary-Knapsack Problem

The $0{,}1-Knapsack$ problem also called *binary*-Knapsack problem is a one-constraint integer programming problem which can be represented as:

Maximize:

$$z = \sum_{j=1}^{n} c_j x_j \tag{9}$$

Subject to:

$$\sum_{j=1}^{n} a_j x_j \leq b$$

where, $0 \leq x_j \leq 1$ and $x_j, b, c_j, a_j \in Z \ for \ (j = 1,2,.....,n)$ [4, p. 265].

Definition: Let $a = (a)_{i=1}^{n}$ and b be real numbers, then the set of solutions to (9) can be denoted by:

$$\Omega = \{x : a'x \leq b\}$$

where: $x = (x)_{i=1}^{n}$ is a $0-1-vector$ and;

$$a'x \equiv \sum_{i=1}^{n} a_i x_i \tag{10}$$

The problem of computing $|\Omega|$ is #*P*-complete, hence the need to find a good approximation algorithm.

Proposition: Let $P^{Kp} = \{x \in R^n : \sum_{i=1}^{n} a_i x_i \leq b\}$ be a polyhedron where $0 \leq x_i \leq 1$. The number of non-integer vertices of P^{Kp} is less than or equal to n times the number of integer vertices.

Proof: This is true as every non-integer vertex is adjacent on $P^{(KP)}$ to an integer vertex and every integer vertex has at most n neighbors [1].

Theorem: There exist an *fpras* for counting all vertices of non-degenerate $0-1-Knapsack$ polyhedra.

Proof: This follows from general results of [1] and [6].

VIII. $2 \times n$ Transportation Polyhedra

It has also been established that the problem of counting all vertices of transportation polyhedra is #*P*-complete [2]. Is it possible to present approximate counting procedure for the vertices of a transportation polytope also known as $2 \times n$ transportation polyhedra?

Definition: Consider m origin points, where i has a supply, s_i of units of a particular item (commodity). In addition, there are n destination points, where destination j requires d_j units of commodity. It is assumed that $s_i, d_j > 0$. Associated with each link (i, j) from origin i to destination j, there is a unit cost c_{ij} for transportation. The problem is to determine a feasible "shipping pattern" from origins to destinations that minimizes the total transportation cost. This problem is known as Hitchcock or the transportation problem. It is assumed that the total supply equals total demand, that is, the problem is balanced. The problem is that of counting the extreme points of:

$$P^{(transp)} = \{x : A^{(transp)} x = b\} \tag{11}$$

where, $A^{(transp)}$ is $(m+n) \times (m \times n)$ node-arc incidence matrix, it is a 2 nonzero per column matrix, each nonzero is either +1 or -1.

Open Problem: Give *fpras* for enumerating extreme points of the transportation polyhedra $P^{(transp)}$.

IX. Conclusion

Some literature reviews are given for approximately counting the extreme points (vertices) of certain polyhedra associated with 0-1 Permanent, Down-Sets (S) in non-bipartite graphs, Independent-Sets, 0-1 Knapsack Problems and $2 \times n$ transportation problems. Some open problems are suggested and it might be a good line of future research to explore proofs of these open problems posed in this paper.

Reference

[1] Abdullahi, S. D. (2002) Vertex Enumeration and Counting for Certain Classes of Polyhedra. School of Computing. The University of Leeds.
[2] Dyer, M. E. (1983) The Complexity of Vertex enumeration methods. *Mathematics of Operation Research*. 8 381 - 402.
[3] Hadley, G. (1962) Linear Programming Addison-Wesley.
[4] Dyer, M. E. (1979) Vertex Enumeration in Mathematical Programming: Methods and Applications. Department of Computer Studies. The University of Leeds.
[5] Jerrum, M. (1995) The computational complexity of counting. International Congress of Mathematicians. Birkhauser, Basel 1995.
[6] Jerrum, M. and Sinclair, A. (1996) The Markov Chain Monte Carlo method: an approach to approximate counting and integration. PWS Publishing, Boston.
[7] Garey, A. M. and Johnson, D. S. (1979) Computers and Intractability Freeman.
[8] Dyer, M. and Greenhil, C. (2000) On Markov Chains for Independent Sets. *Journal of Algorithms*. 35 17 - 49.
[9] Dyer, M. E., Goldberg, L. A., Greenhill, C. S. and Jerrum, M. R. (2000) On the Relative Complexity of Approximate Counting Problems Springer-Verlag.