

A Survey on Positive Real and Strictly Positive Real Scalar Transfer Functions

Mojtaba Hakimi-Moghaddam

Abstract—Positive real and strictly positive real transfer functions are important concepts in the control theory. In this paper, the results of researches in these areas are summarized. Definitions together with their graphical interpretations are mentioned. The equivalent conditions in the frequency domain and state space representations are reviewed. Their equivalent electrical networks are explained. Also, a comprehensive discussion about a difference between behavior of real part of positive real and strictly positive real transfer functions in high frequencies is presented. Furthermore, several illustrative examples are given.

Keywords—Real rational transfer functions, positive realness property, strictly positive realness property, equivalent conditions.

I. INTRODUCTION

THE concept of Positive Real (PR) transfer functions as a substantial property of driving point impedance of passive electrical networks is developed by Otto Brune in 1930 [1], [2]. A generalization of the necessary and sufficient conditions for PR transfer functions are extracted in [3]. Also, in [3] it is shown that impedance, admittance and hybrid matrices of multi-port passive electrical networks are PR. In addition, in [4] the equivalent conditions for PR transfer function matrices in the state space representation are deduced which nowadays it is called positive real Lemma.

In 1963, Popov introduced the notion of hyperstability in the control theory and consequently the concept of positive realness is applied in the control literature. In fact, he showed that a linear time-invariant system is hyperstable system if and only if its transfer function is PR. Also he developed the concept of strictly positive real (SPR) transfer functions and showed that a linear time-invariant system is asymptotically hyperstable system if and only if its transfer function is SPR [5]. In addition, lossy electrical networks are introduced which contain SPR driving point impedance [6]. Thus PR and SPR transfer functions have been extensively used in the various field of control such as adaptive control [7]-[9], optimal control [10]-[11], nonlinear control [12]-[14], robust control [15]-[20] and even intelligent control [21]. Furthermore, in [12] the equivalent conditions for SPR transfer function matrices in the state space representation are presented which is called Kalman-Yakubovich-Popov (KYP) Lemma.

The paper organized as follows: Section II devoted to PR

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transfer functions, Section III to SPR transfer functions, Section IV to passive electrical networks, Section V to high frequency behavior of $\text{Re}(G(j\omega))$, Section VI to PR Lemma and SPR Lemma, and finally Section VII to conclusion.

II. PR TRANSFER FUNCTIONS

We know if a linear time-invariant (LTI) system is lumped, then its transfer function will be rational function of complex variable s . An arbitrary rational transfer function can be expressed as

$$G(s) = k \frac{s^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}, k \neq 0 \quad (1)$$

Thus, $G(s)$ can be classified as [30]:

- 1) $G(s)$ is *improper* $\Leftrightarrow n < m \Leftrightarrow |G(\infty)| = \infty$
- 2) $G(s)$ is *biproper* $\Leftrightarrow n = m \Leftrightarrow G(\infty) = \text{nonzero constant}$
- 3) $G(s)$ is *proper* $\Leftrightarrow n \geq m \Leftrightarrow G(\infty) = \text{constant}$
- 4) $G(s)$ is *strictly proper* $\Leftrightarrow n > m \Leftrightarrow G(\infty) = 0$

Assumption 1. Hereafter, we consider $G(s)$ in (1) is a rational transfer functions with real coefficients in numerator and denominator and simply we say that $G(s)$ is a *real rational transfer function*.

Definition 1. [4] A real rational transfer function $G(s)$ is called PR if $\text{Re}(G(s)) \geq 0$ for $\text{Re}(s) \geq 0$.

In fact, each real rational transfer function $G(s)$ which maps closed right half plane (CRHP) of s -plane to a subset of CRHP of $G(s)$ -plane is a PR transfer function. Fig. 1 shows the geometric interpretation of above definition.

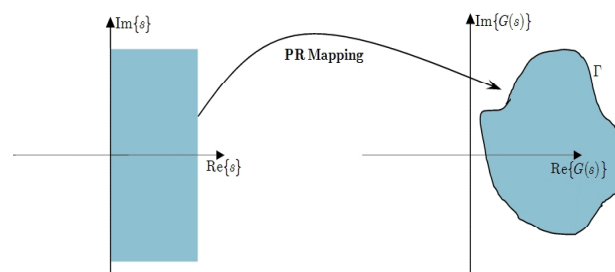


Fig. 1 Geometric Interpretation of Positive Realness

As a result, if $G(s)$ and $H(s)$ are PR then $H(G(s))$ is PR. Since $1/s$ is PR, so $1/G(s)$ is PR. Also, if $G(s)$ and $H(s)$ are PR then $G(s)+H(s)$ is PR. Subsequently, if $G(s)$ and $H(s)$ are PR, then $(1/G(s)+H(s))^{-1}=G(s)/(1+G(s)H(s))$ is PR, and hence the following corollary is concluded.

Corollary 1. [22] The closed loop transfer function resulted from negative feedback interconnection of two PR transfer functions is PR.

Three important properties of a PR real rational transfer function $G(s)$ mentioned in (1) are as follows [31]:

Property 1. All poles are in the closed left half plane (CLHP) and each of poles on the imaginary axis is a simple pole and the associated residue is positive.

Property 2. All zeros are in the closed left half plane (CLHP) and each of zeros on the imaginary axis is a simple zero.

Property 3. The relative degree $(n-m)$ is restricted to -1, 0, and 1.

Definition 2. $G(s)$ is marginally stable if each of whose poles is in the CLHP and each of whose poles on the imaginary axis is a simple pole.

Theorem 1. [12], [31] A real rational transfer function $G(s)$ is PR if and only if

- 1) $G(s)$ is marginally stable,
- 2) Any pole of $G(s)$ on the imaginary axis is a simple pole and the associated residue is positive, and
- 3) $\text{Re}(G(j\omega)) \geq 0, \forall \omega \in \mathbb{R}$ for which $j\omega$ is not a pole of $G(s)$.

Note that the notation $\text{Re}(\cdot)$ means real part.

Definition 3. $G(s)$ is stable if each of whose poles is in the open left half plane (OLHP).

Lemma 1. Let $G(s)$ is stable, then it is PR if and only if its nyquist curve is in CRHP.

Example 1. Let $G(s) = \frac{s^2+1}{s(s^2+4)}$. It is clear that $G(s)$ is marginally stable with three poles on the imaginary axis where their associated residues are positive. On the other hand, $\forall \omega \in \mathbb{R} - \{0, \pm j2\}$ we have $\text{Re}(G(j\omega)) = 0$; thus $G(s)$ is PR.

Example 2. Consider the following stable transfer functions. Their nyquist curves are depicted in Fig. 2. Based on Lemma 1, $G(s)$ is PR and $H(s)$ is not PR.

$$G(s) = \frac{s^2+s+3}{s^2+2s+1}, \quad H(s) = \frac{s^2+s+1}{s^2+s+5}$$

Remark 1. Note that the stability condition in Lemma 1 is a necessary condition. For example, nyquist curve of $G(-s)$ is the same as nyquist curve of $G(s)$ whereas it is not stable and hence it is not PR.

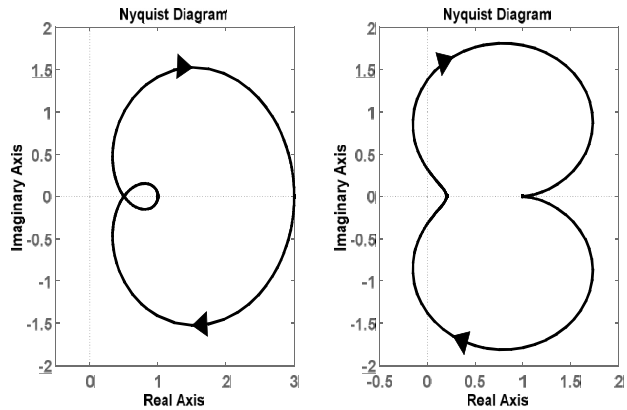


Fig. 2 Nyquist diagram of $G(s)$ (left) and $H(s)$ (right)

Theorem 2. [23] Let $G(s)=d+c(sI-A)^{-1}b$ with $d>0$. Then $G(s)$ is PR if and only if:

- 1) All residues of $G(s)$ at poles on the imaginary axis are positive; and
- 2) The matrix $(A-(1/d)bc)A$ has no eigenvalue of odd algebraic multiplicity on the open negative real axis.

Theorem 3. [23] Let $G(s)=c(sI-A)^{-1}b$ and p be the smallest odd integer such that $cA^p b \neq 0$. Then $G(s)$ is PR if and only if:

- 1) $(-1)^{(p+1)/2} cA^p b > 0$;
- 2) All residues of $G(s)$ at poles on the imaginary axis are positive; and
- 3) The matrix $A(I-(1/(cA^p b))A^p bc)A$ has no eigenvalue of odd algebraic multiplicity on the open negative real axis.

III. SPR TRANSFER FUNCTIONS

In this section, we review definitions and results in the field of SPR transfer functions. Hereafter, wherever we use SPR it means “SPR in the KYP sense” and wherever we use WSPR (Weak SPR) it means “SPR in the Circuit Theory sense”.

Definition 4. [12] $G(s)$ is SPR, if $G(s-\varepsilon)$ is PR for some $\varepsilon>0$.

A geometric interpretation of SPR transfer functions is depicted in Fig. 3 [25].

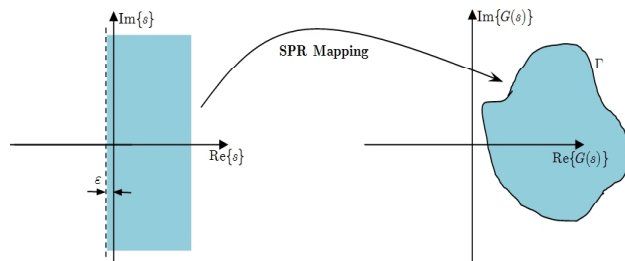


Fig. 3 Geometric Interpretation of SPR in the KYP sense

Considering the geometric interpretation depicted in Fig. 3, it is clear that if $G(s-\varepsilon_1)$ is not PR, then $G(s-\varepsilon)$ will not PR for any $\varepsilon > \varepsilon_1$, so Definition 4 can be modified and restated as follows.

Definition 5. [25], [27] $G(s)$ is SPR, if $G(s-\varepsilon)$ is PR for sufficiently small $\varepsilon > 0$.

Moreover, we have the following corollaries.

Corollary 2. [27] If $G(s)$ is SPR, then there exists an $\varepsilon^* > 0$ such that $G(s-\varepsilon)$ is PR for each $\varepsilon \in (0, \varepsilon^*]$ and there is not any $\varepsilon > \varepsilon^*$ such that $G(s-\varepsilon)$ is PR.

Corollary 3. [27] Consider $G(s)$ in (1) is SPR where p_i and z_j are whose poles and zeros respectively, then we have

$$\varepsilon^* \leq \varepsilon_u = \min \left\{ \min_i |\operatorname{Re}(p_i)|, \min_j |\operatorname{Re}(z_j)| \right\} \quad (2)$$

Example 3. Let $G(s) = \frac{s+1}{s+2}$. It is easy to verify that $\varepsilon^* = \varepsilon_u = 1$.

Corollary 4. [35] $G(s) = \frac{s+a}{s^2+bs+c}$ is PR (SPR) if and only if a, b, c , and $(b-a)$ are nonnegative (positive). Also, we have $\varepsilon^* = \min \{ \varepsilon_u, (b-a) \}$.

Example 4. Let $G(s) = \frac{s+4}{(s+2)(s+3)}$. It is clear that $G(s)$ is SPR and $\varepsilon^* = 1$ whereas $\varepsilon_u = 2$.

Definition 6. [26] The proper and real rational transfer function $G(s)$ is WSPR if

- 1) $G(s)$ is stable, and
- 2) $\operatorname{Re}(G(j\omega)) > 0, \forall \omega \in \mathbb{R}$

Example 5. Let $G(s) = \frac{s^2+s+1}{s^2+s+4}$. It is easy to verify that

$$\operatorname{Re}(G(j\omega)) = \frac{(\omega^2-2)^2}{(\omega^2-4)^2 + \omega^2} \geq 0, \text{ thus } G(s) \text{ is not WSPR because}$$

$$\operatorname{Re}(G(j\sqrt{2})) = 0, \text{ however, } G(s) \text{ is PR.}$$

Remark 2. Note that if $G(s)$ is improper then it is WSPR if in addition, the associated residue of whose simple pole at infinity be positive,

Corollary 5. [27], [28] If $G(s)$ is biproper, then it is SPR if and only if it is WSPR.

Corollary 6. [27], [28] If $G(s)$ is strictly proper, then it is SPR if and only if it is WSPR and in addition, the summation of whose zeros be not equal to the summation of whose poles.

Example 6. Consider the transfer function $G(s)$ in Example 2. It is stable and whose nyquist curve where depicted in Fig. 2 is in ORHP; thus, it is WSPR. Also, from Corollary 5 implies that this transfer function is SPR.

Theorem 4. [27] Let $G(s) = d + c(sI - A)^{-1}b$ be a real rational transfer function with minimal realization; then $G(s)$ is SPR if and only if it is WSPR and one of the following conditions is satisfied

- 1) $G(\infty) = d > 0$; or
- 2) $G(\infty) = d = 0$, and $-cAb > 0$.

We know if the proper transfer function $G(s)$ is PR, WSPR or SPR, then $G(\infty) \geq 0$. Thus we have the following theorem

Theorem 5. [23], [24], [29] The real rational transfer function $G(s) = d + c(sI - A)^{-1}b$ is SPR if and only if one of below cases is satisfied

Case 1 ($d > 0$): A is stable, and the matrix $(A - (1/d)bc)A$ has no eigenvalues on the closed negative real axis.

Case 2 ($d = 0$): $cb > 0, -cAb > 0, A_1$ is stable, and the matrix $(A_1 - (1/d_1)b_1c_1)A_1$ has no eigenvalues on the closed negative real axis where (A_1, b_1, c_1, d_1) is a realization of $cb/G(s) - s$.

Theorem 6. [33] The transfer function $G(s) = c(sI - A)^{-1}b$ is SPR if and only if $G(s) = c(sI - A^{-1})^{-1}b$ is SPR.

IV. PASSIVE ELECTRICAL NETWORKS

We know a passive electrical network is an electrical network which is made up of positive elements including resistance, inductance, capacitance, and coupled inductance.

Lemma 2. [31] The driving point impedance $Z(s)$ or admittance $Y(s)$ of a passive electrical network is PR.

Definition 7. [6] Lossy inductor having impedance $L(s + \varepsilon)$ is inductor L in series with resistor $L\varepsilon$.

Definition 8. [6] Lossy capacitor having impedance $1/C(s + \varepsilon)$ is capacitor C in parallel with resistor $(C\varepsilon)^{-1}$.

A lossy passive electrical network is a passive network which is made up of positive elements including resistance, lossy inductance, and lossy capacitance.

Lemma 3. [6]: The driving point impedance $Z(s)$ or admittance $Y(s)$ of a lossy passive electrical network is SPR.

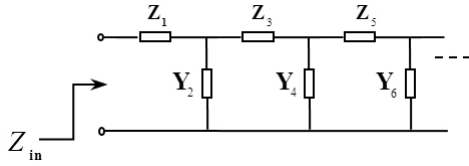


Fig. 4 Ladder Electrical Network Structure

Fig. 4 shows the structure of the ladder network that has the following impedance

$$Z_{in}(s) = Z_1 + \frac{1}{Y_2 + \frac{1}{Z_3 + \frac{1}{Y_4 + \dots}}} \quad (3)$$

In the following Lemma a sufficient condition for PR and SPR transfer function is presented. This Lemma can be very useful in some cases.

Lemma 4. [34] If the transfer function $G(s)$ can be realized as a passive (lossy passive) ladder electrical network then it is PR (SPR).

Example 7. [34] Let $G(s) = \frac{s^2+2s+2}{s^2+3s+3}$. It is easy to verify that

$$G(s) = \frac{s^2+2s+2}{s^2+3s+3} = \frac{1}{1 + \frac{1}{(s+1) + \left(\frac{1}{s+1}\right)}}$$

Hence $G(s)$ is SPR, because it is realized as a lossy passive electrical network.

Example 8. Let $G(s) = \frac{s+1}{s^2+s+1}$. It is easy to verify that

$$G(s) = \frac{s+1}{s^2+s+1} = \frac{1}{s + \left(\frac{1}{s+1}\right)}$$

We cannot obtain any results about SPRness of $G(s)$ based on Lemma 4, however, it can be concluded that $G(s)$ is PR.

V. HIGH FREQUENCY BEHAVIOR OF $\text{Re}(G(j\omega))$

In this section, we review high frequency behavior of $\text{Re}(G(j\omega))$ for PR, WSPR, and SPR real rational transfer functions. A discussion on this topic is presented in [24], [28], [32]. In the next, a review and completion of these references is presented. Consider

$$G(s) = d + c(sI - A)^{-1}b = d + k \frac{s^{n-1} + b_1s^{n-2} + \dots + b_{n-2}s + b_{n-1}}{s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n} \quad (4)$$

is analytic in $|s| > \rho$, then we have

$$\text{Re}(G(j\omega)) = r_0 + \frac{r_2}{\omega^2} + \frac{r_4}{\omega^4} + \dots, \quad |\omega| > \rho \quad (5)$$

where $r_0 = d$, and $r_2 = -cAb = k(a_1 - b_1)$.

Based on (5) the behavior of $\text{Re}(G(j\omega))$ for biproper PR, WSPR, and SPR transfer functions is similar, because for sufficiently high frequencies we have

$$\text{If } d > 0 \Rightarrow \text{Re}(G(j\omega)) \cong r_0 = d, \text{ for } |\omega| \gg \rho \quad (6)$$

In spite of biproper transfer functions, we show that the behavior of $\text{Re}(G(j\omega))$ for strictly proper SPR transfer functions is different from the others. If $G(s)$ is strictly proper, then Definition 5 implies that: $G(s)$ is SPR if and only if it is WSPR, and in addition the following inequality for sufficiently small $\varepsilon > 0$ is satisfied [27].

$$\lim_{\omega \rightarrow \infty} \text{Re}(G(j\omega - \varepsilon)) \geq 0 \quad (7)$$

In the past half century, many researches have been devoted to extract a simple equivalent relation for inequality presented in (7). The first valuable work to simplify this inequality is presented in [6] where argues that: If $G(s)$ is strictly proper SPR transfer function, then $\text{Re}(G(j\omega))$ cannot decay more rapidly than ω^{-2} as $\omega \rightarrow \infty$; i.e.,

$$\lim_{\omega \rightarrow \infty} \omega^2 \text{Re}(G(j\omega)) > 0 \quad (8)$$

On the other hand, (5) for strictly proper SPR transfer function $G(s)$ presented in (4) implies that

$$\lim_{\omega \rightarrow \infty} \omega^2 \text{Re}(G(j\omega)) = -cAb = k(a_1 - b_1) \quad (9)$$

Lemma 5. [24], [28] If $G(s)$ in (4) is strictly proper PR or WSPR, then $-cAb \geq 0$ or $(a_1 - b_1) \geq 0$.

Lemma 6. [24], [28] If $G(s)$ in (4) is strictly proper SPR, then $-cAb > 0$ or $(a_1 - b_1) > 0$.

Lemma 7. There is not any restriction on decay ratio of real part of PR and WSPR transfer functions in high frequencies.

Proof: Let

$$H(s) = \frac{1}{s + G(s)} \quad (10)$$

where $G(s)$ is strictly proper rational transfer function and it is not an odd function of complex variable s . we show that when $\omega \rightarrow \infty$ the following relation is satisfied

$$\text{Re}(H(j\omega)) \approx \omega^{-2} \text{Re}(G(j\omega)) \quad (11)$$

In addition, if $G(s)$ is PR (WSPR), then $H(s)$ in (10) is PR (WSPR). Note that if $G(s)$ is SPR, then $H(s)$ in (10) is not SPR because according to (11) the inequality (8) is contradicted for $H(s)$. It is easy to verify that

$$\text{Re}(H(j\omega)) = \frac{r(\omega)}{(r(\omega))^2 + (i(\omega))^2 + 2\omega i(\omega) + \omega^2} \quad (12)$$

where $r(\omega)$ and $i(\omega)$ are real and imaginary part of $G(j\omega)$ respectively. Since $G(s)$ is strictly proper rational transfer function for sufficiently large ω we have $r(\omega) \rightarrow 0$, $i(\omega) \rightarrow 0$ and $\omega i(\omega) \rightarrow l$, $l \in \mathbb{R}$. Subsequently, (12) reduce to (11). Therefore, we construct a new strictly proper PR (WSPR) transfer function $H(s)$ such that the decay ratio of its real part in high frequencies is equal to ω^{-4} . Furthermore, if $T(s) = (s + H(s))^{-1}$, then the decay ratio of real part of $T(j\omega)$ in high frequencies is equal to ω^{-6} and so on.

Example 9. Let

$$G_1(s) = \frac{2s+1}{s^2+s+1}, \quad G_2(s) = \frac{s+1}{s^2+s+1}$$

$$G_3(s) = \frac{s^2+s+1}{s^3+s^2+2s+1}, \quad G_4(s) = \frac{s^3+s^2+2s+1}{s^4+s^3+3s^2+2s+1}$$

Note that based on Corollary 4 $G_1(s)$ is SPR and $G_2(s)$ is WSPR. Also, note that

$$G_2(s) = \frac{1}{s + \frac{1}{s+1}}, \quad G_3(s) = \frac{1}{s + G_2(s)}, \quad G_4(s) = \frac{1}{s + G_3(s)}$$

Therefore, it can be concluded that $G_3(s)$ and $G_4(s)$ are WSPR and in addition for sufficiently large ω we have

$$\begin{aligned} \text{Re}(G_1(j\omega)) &\approx \omega^{-2} \\ \text{Re}(G_2(j\omega)) &\approx \omega^{-4} \\ \text{Re}(G_3(j\omega)) &\approx \omega^{-6} \\ \text{Re}(G_4(j\omega)) &\approx \omega^{-8} \end{aligned} \quad (13)$$

This fact is depicted in Fig. 5. It is clear that the decay ratio of $\text{Re}(G_k(j\omega))$ is ω^{-2k} .

In the following a simple numerical method for checking high frequency condition of PR (WSPR) and SPR transfer functions is presented. Let

$$\alpha = \frac{\text{Re}(G(j\omega_1))}{\text{Re}(G(j2\omega_1))}, \quad \omega_1 \gg \rho \quad (14)$$

Now, considering (5) we have

$$\text{If } r_0 \neq 0; \Rightarrow \alpha \cong 1$$

$$\text{If } r_0 = 0, \text{ and } r_2 \neq 0; \Rightarrow \alpha \cong 4$$

$$\text{If } r_0 = r_2 = 0, \text{ and } r_4 \neq 0; \Rightarrow \alpha \cong 16$$

⋮

$$\text{If } r_0 = \dots = r_{2(i-1)} = 0, \text{ and } r_{2i} \neq 0; \Rightarrow \alpha \cong 4^i$$

Corollary 7. High frequency behavior of PR, WSPR and SPR biproper transfer functions are the same and for all of them α approaches to 1.

Corollary 8. High frequency behavior of PR, WSPR, and SPR strictly proper transfer functions are different such that for SPR ones α approaches to 4, whereas for PR and WSPR ones α approaches to 4^i , $i = 1, 2, \dots$.

Corollary 9. Consider $G(s)$ in (4) with $d = 0$ is strictly proper WSPR transfer function, then the following statements are equivalent

- 1) $G(s)$ is SPR
- 2) $\text{Re}(G(j\omega))$ cannot decay more rapidly than ω^{-2} as $\omega \rightarrow \infty$ or equivalently $\lim_{\omega \rightarrow \infty} \omega^2 \text{Re}(G(j\omega)) \neq 0$.
- 3) $a_1 \neq b_1$ or $cAb \neq 0$.
- 4) $\sum_{i=1}^{n-1} z_i \neq \sum_{i=1}^n p_i$ where z_i is a zero and p_i is a pole of $G(s)$
- 5) The parameter α defined in (14) approaches to 4.

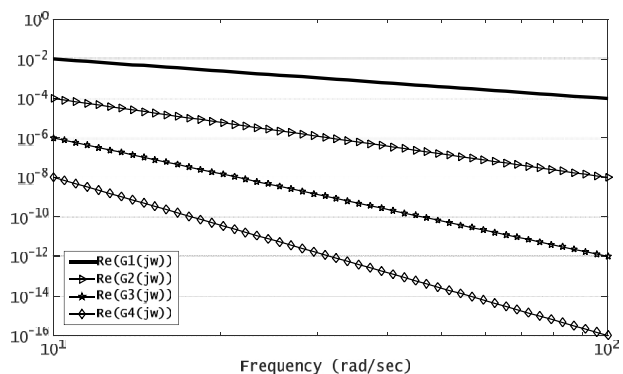


Fig. 5 Real part of transfer functions in Example 9

Considering $\omega_1=50$ in (14) and transfer functions cited in example 9 we have

$$\text{If } G(s) = G_1(s) \Rightarrow \alpha = 4.0024$$

$$\text{If } G(s) = G_2(s) \Rightarrow \alpha = 16.0012$$

$$\text{If } G(s) = G_3(s) \Rightarrow \alpha = 64.0144$$

$$\text{If } G(s) = G_4(s) \Rightarrow \alpha = 255.8058$$

where they confirm the relations presented in (13).

VI. PR LEMMA AND KYP LEMMA

The following Lemmas are called *Positive Real Lemma*, (i.e., *PR Lemma*) and *KYP Lemma* respectively.

Lemma 8. [12] The real rational transfer function $G(s)$ with minimal realization $G(s)=d+c(sI-A)^{-1}b$ is PR if and only if there exists matrix $P=P^T>0$ and vector q such that

$$PA+A^TP=-q^Tq \quad (15)$$

$$Pb=c^T-\sqrt{2d}q^T \quad (16)$$

Lemma 9. [12] The real rational transfer function $G(s)$ with minimal realization $G(s)=d+c(sI-A)^{-1}b$ is SPR if and only if there exists matrix $P=P^T>0$, vector q , and a positive constant ε such that

$$PA+A^TP=-q^Tq-\varepsilon P \quad (17)$$

$$Pb=c^T-\sqrt{2d}q^T \quad (18)$$

Note that (16) and (18) implicitly argue that $d \geq 0$. In addition, for $d=0$ these Lemmas are simplified as follows

$$\text{PR Lemma: } PA + A^T P = -Q \leq 0, \quad Pb = c^T$$

$$\text{KYP Lemma: } PA + A^T P = -Q < 0, \quad Pb = c^T$$

VII. CONCLUSION

In this paper definitions, properties, lemmas and theorems appeared in the literature for PR, WSPR and SPR transfer functions are reviewed. Geometric interpretations are presented for basic definitions; also the equivalent electrical networks of PR and SPR transfer functions are explained. Moreover, an important difference between WSPR and SPR transfer functions which is imposed by high frequencies is illustrated with some virgin examples.

REFERENCES

- [1] E. A. Guillemin, *Synthesis of Passive Networks*. New York: Wiley, 1957.
- [2] M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*, New York: Wiley, 1960.
- [3] B. D. O. Anderson, S. Vongpanitlerd, *Network Analysis and Synthesis*, Prentice-Hall, 1973.
- [4] B. D. O. Anderson, "A System Theory Criterion for Positive Real Matrices," J. SIAM Control, vol. 5, no. 2, 1967, pp. 171-182.
- [5] B. D. O. Anderson, "A Simplified Viewpoint of Hyperstability," *IEEE Transaction of Automatic Control*, 1968, pp. 292-294.
- [6] J. H. Taylor, "Strictly positive-real functions and the Lefschetz-Kalman Yakubovich (LKY) lemma," *IEEE Trans. Circuits and system*, pp. 310-311, 1974.
- [7] P. A. Ioannou, J. Sun, "*Robust Adaptive Control*," Prentice-Hall, Upper Saddle River, New Jersey, 1995.
- [8] K. J. Astrom, B. Wittenmark, *Adaptive Control*, Second Edition, Pearson Education, 2006.
- [9] C.-H. Huang, P. A. Ioannou, J. Margulies, and M. G. Safonov, "Design of Strictly Positive Real Systems Using Constant Output Feedback," *IEEE Trans. on Auto. Cont.*, vol. 44, no. 3, March 1999.
- [10] P. Kokotovic, M. Arcaj, "Constructive nonlinear control: a historical perspective," *Automatica*, 37, 637-662, 2001.
- [11] R. E. Kalman, "When is a Linear Control System Optimal," *trans. ASME, Series D*, 86, 1-10, 1964.
- [12] H. Khalil, *Nonlinear Systems*, third edition, Prentice-Hall, 2002.
- [13] M. Vidyasager, *Nonlinear Systems Analysis*, second edition, 1993.
- [14] C. W. Wu, "Synchronization in Arrays of Coupled Nonlinear Systems: Passivity, Circle Criterion, and Observer Design," *IEEE Trans. Circuits and system*, Vol. 48, No. 10, pp. 1257-1261, Octo. 2001.
- [15] L. Wang, W. Yu, "On Hurwitz stable polynomials and strictly positive real functions," *IEEE Trans. Circuits and system*, Vol. 48, No. 1, pp. 127-128, Janu. 2001.
- [16] D. Henrion, "Linear matrix inequality for robust strictly positive real design," *IEEE Trans. Circuits and system*, Vol. 49, No. 7, pp. 1017-1020, July. 2002.
- [17] A. Betser, E. Zeheb, "Design of robust strictly positive real transfer functions," *IEEE Trans. Circuits and system*, Vol. 40, No. 9, pp. 573-580, Sept. 1993.
- [18] Y. K. Foo, Y. C. Soh, "Strict positive realness of a family of polytopic plants," *IEEE Trans. Auto. Contr.*, Vol. 38, No. 2, pp. 287-289, Feb. 1993.
- [19] H. J. Marquez, C. J. Damaren, "On the design of strictly positive real transfer functions," *IEEE Trans. Circuits and system*, Vol. 42, No. 4, pp. 214-218, April. 1995.
- [20] H. J. Marquez, P. Agathoklin, "On the existence of robust strictly positive real rational functions," *IEEE Trans. Circuits and system*, Vol. 45, No. 9, pp. 962-967, Sept. 1998.
- [21] L.-X. Wang, "*A Course in Fuzzy Systems and Control*," Prentice-Hall International, Inc. 1997.
- [22] J. J. E. Slotine, W. Li, *Applied Nonlinear Control*, Prentice Hall, 1991.
- [23] R. Shorten, C. King, "Spectral Conditions for Positive Realness of Single Input-Single Output Systems," *IEEE Trans. Auto. Control*, vol. 49, no. 10, pp. 1875-1877, 2004.
- [24] M. Hakimi M., H. Khaloozadeh, "Comments on A Note on Spectral Conditions for Positive Realness of Single-Input-Single-Output Systems with Strictly Proper Transfer Functions," *IEEE Trans. Auto. Control*, vol. 58, no. 4, 2013.
- [25] M. Hakimi M., H. Khaloozadeh, "Strict Positive Realness as a Mapping," *SICE-ICASE International Joint Conference*, Korea, pp. 2362-2367, 2006.
- [26] I. Hodaka, N. Sakamoto, M. Suzuki, "New Results for Strict Positive Realness and Feedback Stability," *IEEE Trans. Auto. Contr.*, vol. 45, no. 4, pp. 813-819, 2000.
- [27] M. Hakimi M., H. Khaloozadeh, "Characterization of Strictly Positive Real Multivariable Systems" *IMA Journal of Mathematical Control and Information*, published online: December 3, 2013 (doi:10.1093/imamci/dnt040).
- [28] M. Hakimi M., H. Khaloozadeh, "Revision on the Frequency Domain Conditions for Strict Positive Realness," *International Journal of Control, Automation and System*, vol. 5, no. 1, pp. 1-7, 2007.
- [29] E. Zeheb, R. Shorten, "A Note on Spectral Conditions for Positive Realness of Single-Input-Single-Output Systems with Strictly Proper

- Transfer Functions,” IEEE Trans. Auto. Control, vol. 51, no. 5, pp. 897-900, 2006.
- [30] C-T. Chen, *Linear Systems Theory and Design*. 3rd ed., Oxford University Press, 1999.
- [31] W-K. Chen, *Passive and Active Filters: Theory and Implementations*, wiley, 1986.
- [32] P. Ioannou and G. Tao, “Frequency domain conditions for strictly positive real functions,” IEEE Trans. Automat. Contr., vol. AC-32, pp. 53–54, 1987.
- [33] R. Shorten, M. Corless, K. Wulff, S. Klinge, R. Middleton, “Quadratic Stability and Singular SISO Switching Systems,” IEEE Transactions on Automatic Control, vol. 54, no. 11, pp. 2714–2718, Nov. 2009.
- [34] M. Hakimi M., H. Khaloozadeh, “Strict Positive Realness and Its Equivalent Electrical Networks,” in Proc. 2011 Chinese Control and Decision Conference (CCDC), China, 2011, pp. 139-142.
- [35] M. Hakimi M., Ph.D. Dessertation, “Extension of Concepts and Theorems for SPR Transfer Function Matrices and Its Applications in Control,” (in persian) K.N. Toosi University of Technology, Iran, 2013.