

# A Generalization of Planar Pascal's Triangle to Polynomial Expansion and Connection with Sierpinski Patterns

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**Abstract**—The very well-known stacked sets of numbers referred to as Pascal's triangle present the coefficients of the binomial expansion of the form  $(x+y)^n$ . This paper presents an approach (the Staircase Horizontal Vertical, SHV-method) to the generalization of planar Pascal's triangle for polynomial expansion of the form  $(x+y+z+w+r+\dots)^n$ . The presented generalization of Pascal's triangle is different from other generalizations of Pascal's triangles given in the literature. The coefficients of the generalized Pascal's triangles, presented in this work, are generated by inspection, using embedded Pascal's triangles. The coefficients of I-variables expansion are generated by horizontally laying out the Pascal's elements of (I-1) variables expansion, in a staircase manner, and multiplying them with the relevant columns of vertically laid out classical Pascal's elements, hence avoiding factorial calculations for generating the coefficients of the polynomial expansion. Furthermore, the classical Pascal's triangle has some pattern built into it regarding its odd and even numbers. Such pattern is known as the Sierpinski's triangle. In this study, a presentation of Sierpinski-like patterns of the generalized Pascal's triangles is given. Applications related to those coefficients of the binomial expansion (Pascal's triangle), or polynomial expansion (generalized Pascal's triangles) can be in areas of combinatorics, and probabilities.

**Keywords**—Generalized Pascal's triangle, Pascal's triangle, polynomial expansion, Sierpinski's triangle, staircase horizontal vertical method.

## 1. INTRODUCTION

PASCAL'S triangle which may, at first, looks like a pile of stacked numbers, it carries many secrets and patterns. One of such secrets is its representation of the coefficients of the binomial expansion. In earlier work, the author introduced some polynomial expansion of the form  $(\omega + \lambda_1)(\omega + \lambda_2) \dots (\omega + \lambda_n)$  which, compactly, presented as  $\sum_{k=0}^n \omega^{n-k} \sum_{T_{nk}} \lambda \dots \lambda$ , and is named the Guelph expansion [1]. The term  $\sum_{T_{nk}} \lambda \dots \lambda$  denotes the sum of the products of each and every possible combination of  $k$  elements of the set  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ , and  $T_{nk}$  is the binomial coefficient. Such sum of combinations of roots, also, represents the coefficients,  $a_{n-k}$ , of the expansion:

$$a_n \omega^n + a_{n-1} \omega^{n-1} + a_{n-2} \omega^{n-2} + \dots + a_0 = \sum_{k=0}^n a_{n-k} \omega^{n-k}.$$

Therefore,  $a_{n-k} = \sum_{T_{nk}} \lambda \dots \lambda$  can be used to evaluate the coefficients of a polynomial knowing its roots  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ ,

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or, inversely, one can find the roots knowing the coefficients of a polynomial [2].

In the case of single value  $\lambda$ , the Guelph expansion reduces to the well-known binomial expansion

$$(\omega + \lambda)^n = \sum_{k=0}^n \omega^{n-k} T_{nk} \lambda^k.$$

The binomial coefficient  $T_{nk}$  (for  $n=0, 1, 2, 3, \dots$ , and  $k=0, 1, 2, 3, \dots, n$ ) can generate what is known to the most of the western world as Pascal's triangle [3], in China as Yang Hui's triangle, and in Iran as Khayyam's triangle [4].

Numerous studies are available in the literature about the Pascal's triangles, and binomial expansion [5]–[9], as well as polynomials expansion and their coefficients [10]–[12]. Generalizations of Pascal's triangles both planar and geometrical have also been discussed by many authors [13]–[15]. In geometrical representation, the generalization of Pascal's triangle is made for trinomials expansion (Pascal's pyramid), and hyper pyramids for higher orders of expansions.

The author approached a generalization of binomial expansion to polynomial expansion in the form of:

$$(x + y + z + w + \dots)^n = \sum T_{nk} T_{kk'} T_{k'k''} \dots x^{n-k} y^{k-k'} z^{k'-k''} w^{k''-k'''} \dots, \text{ with}$$

$$k=0, 1, 2, \dots, n; k'=0, 1, 2, \dots, k; k''=0, 1, 2, \dots, k'; \dots \text{etc.}$$

The coefficients of this formalization of the polynomial expansion can be generated by inspection using what is called the Embedded Pascal's Triangles (EPTs) inspection method. In such inspection method, one can generate the coefficients of I-variables expansion by multiplying horizontally laid out classical Pascal elements with the corresponding coefficients of the (I-1) variables expansion [16]. An efficient algorithm is available to generate the coefficients of those polynomials expansion, as well as the monomials accompanying such expansion which are in lexicographic order [17].

The EPTs-based generalization of Pascal's triangles is different from the generalization, given in the literature [18], of Pascal's triangles of  $s^{\text{th}}$  orders ( $s^{\text{th}}$  arithmetic triangle) of polynomials of the form:

$$(1 + x + x^2 + \dots + x^{s-1})^n = \sum_{m=0}^{(s-1)n} \binom{n}{m}_s x^m, \quad s \geq 2$$

where,  $\binom{n}{m}_s$  are the generalized binomial coefficients of order  $s$ . The expansion  $(1 + x + x^2 + \dots + x^{s-1})^n$  is, actually, a

special case of the expansion  $(x + y + z + w + \dots)^n$ , if  $x=1$ ,  $y=x$ ,  $z=x^2$ ,  $w=x^3$ , ..., etc.

The horizontally laid out elements of the classical Pascal's triangle carry a meaning as the coefficients of the binomial expansion raised to the power of  $n$ , where  $n=0, 1, 2, 3, \dots$ . Moreover, they represent the number of non-combined roots, single combined roots, two combined roots, three combined roots, ...,  $k$ -combined roots of a polynomial of degree  $n$  of the form  $\sum_{k=0}^n a_{n-k} \omega^{n-k} = 0$ . Similarly, the vertically laid out elements of Pascal's triangle carry a meaning, simply; each column represents, successively, the number of the coefficients of monomials expansion  $x^n$ , binomial expansion  $(x + y)^n$ , trinomial expansion  $(x + y + z)^n$ , tetranomial expansion  $(x + y + z + w)^n$ , ..., etc., for  $n=0, 1, 2, 3, \dots$ . Those sets of numbers are called Waterloo numbers, and the exact values of these coefficients of the considered expansion (called the attached values of Waterloo numbers) are generated by the EPTs inspection method [19]. Those Waterloo numbers can be generated by a form of a polynomial, called Tripoli's polynomial which have two interesting properties, namely; the roots of a degree  $m$  Tripoli polynomial are at  $-1, -2, -3, \dots, -m$ , and the sum of its coefficients is  $m+1$  [20].

In this paper, a new inspection method to generate planar Pascal's triangles of order  $0, 1, 2, 3, 4, \dots, N$  is introduced. The introduced new inspection method, in this paper, is an improvement of the EPTs method with the advantage of being much straight forward and simple. The 0-order represents the expansion of zero-nomial;  $(0)^n$  expansion which is simply 1, the 1<sup>st</sup> order represents monomials expansion  $(x)^n$ , the 2<sup>nd</sup> order corresponds to the classical Pascal's triangle representing the coefficients of the binomial expansion;  $(x + y)^n$ , whereas order 3 of Pascal's triangle represents the coefficients of the trinomial expansion  $(x + y + z)^n$ , and so forth. In other words, one blows up 1 to many planar Pascal's triangles using the proposed SHV-method to be explained in the Section II of this paper.

The pattern of the odd and even numbers in Pascal's triangle of order 2 (classical Pascal's triangle) is known as the Sierpinski's triangle. In this work, patterns of the odd and even numbers are evolved for the higher orders of the SHV-generated planar Pascal's triangles and are presented at the closing of this paper.

## II. THE STAIRCASE HORIZONTAL VERTICAL (SHV) METHOD

A set of stacked numbers presented in a triangular form was known since the 1st half of the 17th century of the current era (CE) as the Pascal's triangle. Such pattern of stacked numbers were earlier known in India as the staircase of mount Meru with the support of the work of an Indian mathematician (Halayudha) in the late 5th century of (CE). In 2010, Ratemi [16] introduced what is called the Embedded Pascal's triangles (EPTs) method to extend the expansion of binomial expansion to polynomial expansion, and deduced an inspection method to generate the coefficients of I-variables polynomial

expansion in terms of the previous I-1 variables polynomial expansion.

Pascal's triangle carries many secrets and applications, two of them are of the concern of this paper; namely; expansions, and patterns. The Pascal's triangle represents a binomial expansion of the form  $(x + y)^n$ , and the pattern of the odd and even numbers represents what is called The Sierpinski's triangle. In this section, the Staircase Horizontal Vertical (SHV) method which is deduced, solely, from the elements of Pascal's triangle is derived from the EPTs inspection method, and is a much easier straight forward method. Such method starts with 1 and explodes it, successively, to Pascal's triangles of different orders; starting from 0-variables expansion of the form  $(0)^n$ , and proceed to 1-variable expansion of the form  $(x)^n$ , 2-variables expansion of the form  $(x + y)^n$ , 3-variables expansion of the form  $(x + y + z)^n$ , ...until I-variables expansion of the form  $(x + y + z + w + r + \dots)^n$ .

### A. Pascal's Triangle and Waterloo Matrix

The elements of Pascal's triangle are generated by the binomial coefficients

$$T_{nk} = \binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad k = 0, 1, 2, 3, \dots, n \quad (1)$$

are given in Table I.

TABLE I  
PASCAL'S TRIANGLE

n / k	0	1	2	3	4	.....
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
.....						

If one pulls up each column so as to start from the row corresponding to  $n=0$ , then the result will be what is called the Waterloo matrix which presents the Waterloo numbers [7]. The elements of this matrix can be calculated using (2):

$$W_{nk} = \binom{n+k}{k} = \frac{(n+k)!}{n!k!} \quad (2)$$

Table II presents the Waterloo matrix.

TABLE II  
THE WATERLOO MATRIX

n / k	0	1	2	3	4	5	.....
0	1	1	1	1	1	1	
1	1	2	3	4	5	6	
2	1	3	6	10	15	21	
3	1	4	10	20	35	56	
4	1	5	15	35	70	126	
5	1	6	21	56	126	252	
.....							

### B. Generating Pascal's Triangles of Different Orders Using SHV-Method

Starting with the expansion of 0-variables this corresponds to the expansion of  $(0)^n$  which has a value of 1 for  $n=0$ , hence Pascal's triangle of order 0 is 1, (see Fig. 1 (a)). To generate the next monomial expansion of the form  $(x)^n$ , one draws one step and lay out the value of 1 corresponding to  $n=0$ , and to generate the other elements for  $n=1, 2, 3, \dots$  etc., one multiplies the element corresponding to  $n=0$  by the elements of the first column of Waterloo matrix  $W1 = [1 \ 1 \ 1 \ 1 \ 1 \ \dots]^T$ , consecutively. Therefore, Pascal's triangle of order 1 is a column of 1's, (see Fig. 1 (b)). Next, to generate the coefficients of a binomial of the form  $(x + y)^n$ , one lays out, horizontally, all of the 1's of the previous monomial expansion in a staircase manner of step size corresponding to the numbers of the 1<sup>st</sup> column of the Waterloo matrix, then, to complete for the rest of the elements, one multiplies those laid out elements for each step by the columns of the Waterloo matrix in a consecutive manner, i.e.;

$$W1 = [1 \ 1 \ 1 \ 1 \ 1 \ \dots]^T, W2 = [1 \ 2 \ 3 \ 4 \ 5 \ \dots]^T,$$

$$W3 = [1 \ 3 \ 6 \ 10 \ 15 \ \dots]^T, W4 = [1 \ 4 \ 10 \ 20 \ \dots]^T, \dots, \text{etc.}$$

Hence, one gets the classical Pascal's triangle which is of order 2, (see Fig. 1 (c)). Now, to generate the coefficients of trinomials of the form  $(x + y + z)^n$ , one draws a staircase of varying step width, i.e.; one element width, two elements width, three elements width, etc. Such width size of the staircase steps corresponds to the elements of the 2<sup>nd</sup> column of the Waterloo matrix  $W2 = [1 \ 2 \ 3 \ 4 \ 5 \ \dots]^T$ . Next, one lays out the previously generated binomial coefficients under the steps of the new staircase (bolded numbers in Fig. 1 (d)). Finally, to complete the rest of the elements, one multiplies those laid out elements for each step by the elements of the columns of the Waterloo matrix in a consecutive manner, i.e.;  $W1, W2, W3, W4$ , etc. The full coefficients of the trinomial expansion corresponding to planar Pascal's triangle of order 3 are depicted in Fig. 1 (d). To get further expansions, one follows the same algorithm.

### C. Polynomial Expansion

The SHV-method has been presented in Section II to generate the coefficients of different orders of Pascal's triangles. Those coefficients belong to the corresponding polynomial expansions. The monomials of the respected expansion can be generated in a lexicographic order.

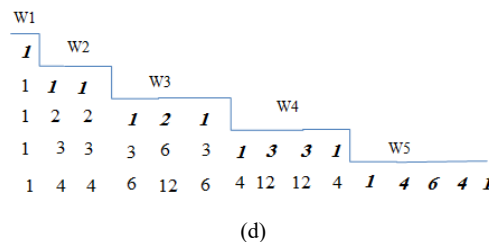
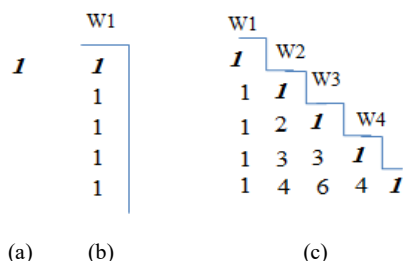


Fig. 1 The Staircase Horizontal Vertical (SHV) method for generating different orders of Pascal's triangles for: (a) zeroomials  $(0)^n$ , (b) monomials  $(x)^n$ , (c) binomials  $(x + y)^n$ , and (d) trinomials  $(x + y + z)^n$

A formula for polynomial expansion [17] is presented in (3) as:

$$(x + y + z + w + \dots)^n = \sum T_{nk} T_{kk'} T_{k'k''} \dots x^{n-k} y^{k-k'} z^{k'-k''} w^{k''-k'''} \dots, \quad (3)$$

with

$$k = 0, 1, 2, \dots, n; \quad k' = 0, 1, 2, \dots, k;$$

$$k'' = 0, 1, 2, \dots, k'; \dots \text{etc.}$$

The monomials shown in (3) are in lexicographic order; hence, they can be used to generate the related expansion. Equation (3) reduces to the well-known formula for binomial expansion given in (4):

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad (4)$$

TABLE III BINOMIAL EXPANSION OF $(x + y)^4$			
$n$	$k = 0, 1, 2, \dots, n$	Coefficients $\binom{n}{k}$ <sup>a</sup>	Monomials <sup>b</sup> $x^{n-k} y^k$
4	0	1	$x^4$
	1	4	$x^3 y$
	2	6	$x^2 y^2$
	3	4	$x y^3$
	4	1	$y^4$

<sup>a</sup> can be read directly from the corresponding row of Pascal's triangle of order 2.

<sup>b</sup> Monomials are in lexicographic order.

Table III presents the required coefficients and monomials for the expansion with  $n=4$ . The full expansion is given by (5):

$$(x + y)^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4 \quad (5)$$

Similarly, (3) reduces to (6) for trinomial expansion:

$$(x + y + z)^n = \sum_{k=0, \dots, n} \sum_{k'=0, \dots, k} \binom{n}{k} \binom{k}{k'} x^{n-k} y^{k-k'} z^{k'} \quad (6)$$

Table IV presents the coefficients, and monomials of the following trinomial expansion:

$$(x + y + z)^3 = x^3 + 3x^2 y + 3x^2 z + 3xy^2 + 6xyz + 3xz^2 + y^3 + 3y^2 z + 3yz^2 + z^3.$$

#### D. Graphical Distributions of the Elements of Pascal's Triangles

A graphical presentation of the distribution of the elements of the different orders of Pascal's triangles as a function of the degree of expansion  $n$ , and the index  $[k] = W_{nk}$  (Waterloo number) representing the number of the coefficients in the corresponding expansion is demonstrated in this section. Referring to Table III which represents the binomial expansion for  $n=4$ , one notices that  $k=0, 1, 2, 3, 4$ , hence there are 5 coefficients for such expansion. This number ( $W_{nk}=W_{41}$ ) can be read directly from the Waterloo matrix for  $n=4$ , and  $k=1$  (2nd column; corresponding to a binomial expansion). On the other hand, for the trinomial expansion raised to power  $n$ ,  $k=0, 1, 2, 3, \dots, n$ , and  $k'=0, 1, 2, 3, \dots, k$ . Referring to Table IV for the trinomial expansion where  $n=3$ , the number of coefficients are 10. This number can be read directly from the Waterloo matrix which corresponds to  $W_{32}$  ( $n=3$ ,  $k=2$ : 3<sup>rd</sup> column representing trinomials).

TABLE IV BINOMIAL EXPANSION OF $(x + y + z)^3$				
Index <sup>a</sup>		Coefficients <sup>b</sup>		Monomials <sup>c</sup>
$n$	$k$	$k'$	$\binom{n}{k} \cdot \binom{k}{k'}$	$x^{n-k} y^{k-k'} z^{k'}$
0	0	0	1	$x^3$
	0	0	3	$x^2 y$
	0	0	3	$x^2 z$
1	1	0	3	$xy^2$
	2	1	6	$xyz$
	2	2	3	$xz^2$
2	0	0	1	$y^3$
	1	1	3	$y^2 z$
	2	2	3	$yz^2$
3	3	1	1	$z^3$

<sup>a</sup>  $k = 0, 1, 2, \dots, n$ , and  $k' = 0, 1, 2, \dots, k$ .

<sup>b</sup>  $\binom{n}{k} \cdot \binom{k}{k'} = \frac{n!}{(n-k)!(k-k')!k'!}$  which can, also, be read directly from the corresponding row of Pascal's triangle of order 3.

<sup>c</sup> Monomials are in lexicographic order.

The distribution of the binomial expansion up to the expansion of  $(x + y)^{19}$  is shown in Fig. 2 (the number of coefficients corresponding to  $n=19$  is  $[k]=W_{nk}=W_{191}=20$ ). The distribution approaches the normal distribution as  $n$  (the degree of the expansion) gets larger. Similarly, the distribution for trinomial expansion up to  $(x + y + z)^5$  is shown in Fig. 3 (the number of coefficients corresponding to  $n=5$  is  $[k]=W_{nk}=W_{52}=21$ ).

#### III. THE SIERPINSKI'S LIKE PATTERNS

It is quite interesting that when shading odd, and even numbers of the classical Pascal's triangle, a pattern is evolved and is known as the Sierpinski's triangle [21]. Fig. 4 (a) presents such pattern for a right angled Pascal's triangle. In a similar manner, one can apply such odd-even shading to generate patterns for higher orders of planar Pascal's triangles. Figs. 4 (b)-(d) present such patterns for Pascal's triangles of orders 3, 4, and 5, respectively.

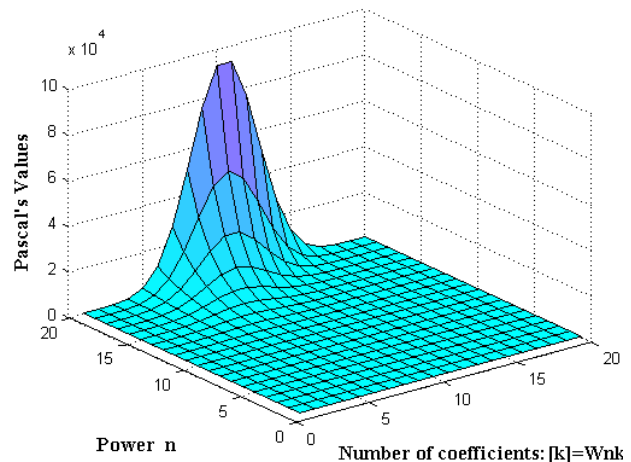


Fig. 2  $(x + y)^n$  distribution function (Pascal's elements of order 2, for  $n=0, 1, 2, 3, \dots, 20$ )

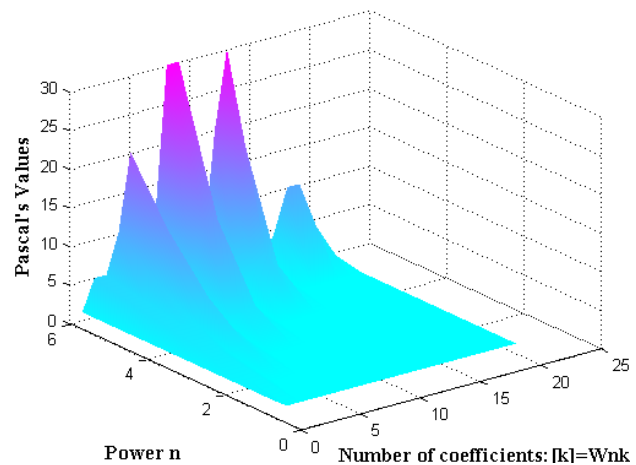


Fig. 3  $(x + y + z)^n$  distribution function (Pascal's elements of order 3, for  $n=0, 1, 2, \dots, 5$ )

#### IV. CONCLUSION

In this work, an inspection simple straight forward method named as SHV- method is introduced to get a generalization of planar Pascal's triangle to extend it from binomial expansion to polynomial expansion.

Furthermore, graphical distributions of the elements of Pascal's triangles are presented. The pattern made by the odd and even numbers of the classical Pascal's triangle which is known as the Sierpinski's triangle is extended for the higher orders of Pascal's triangles. The simple generating method for the generalization of Pascal's triangle can be well used for education. The patterns generated from the odd-even distributions of the elements in Pascal's triangle can be used for the design of carpets, textiles, and other industrial applications.

(a)

(b)

(c)

(d)

## REFERENCES

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