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Multiplicative Functional on Upper Triangular Fuzzy Matrices

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Abstract—In this paper, for an arbitrary multiplicative functional f from the set of all upper triangular fuzzy matrices to the fuzzy algebra, we prove that there exist a multiplicative functional F and a functional G from the fuzzy algebra to the fuzzy algebra such that the image of an upper triangular fuzzy matrix under f can be represented as the product of all the images of its main diagonal elements under F and other elements under G.

Keywords—Multiplicative functional, triangular fuzzy matrix, fuzzy addition operation, fuzzy multiplication operation.

I. Introduction

LET $F[0,\,1]$ denote the fuzzy algebra over the interval $[0,\,1]$ with fuzzy addition $+$ and fuzzy multiplication operation $\times$ defined as

$$a+b=\max\{a,\,b\} \text{ and } a\times b=\min\{a,\,b\} \qquad (1)$$

for all $a,\,b\in[0,\,1]$ and the standard order $\geq$.

If $A=[a_{ij}]$ is a $m\times n$ matrix and every $a_{ij}\in F[0,\,1]$, the matrix A is said to be a fuzzy matrix. It is clear that all fuzzy matrices are matrices but every matrix in general need not be a fuzzy matrix. Hence fuzzy matrices is a subclass of matrices.

The fuzzy addition and fuzzy multiplication of $m\times n$ fuzzy matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ are respectively defined as

$$A+B=[a_{ij}+b_{ij}]=[max\{a_{ij},\,b_{ij}\}] \qquad (2)$$

and

$$A\times B=[max\{min\{a_{ik},\,b_{kj}\},1\leq k\leq m\}]. \qquad (3)$$

Fuzzy sets and fuzzy matrices play an important role in scientific development. Fuzzy sets were presented by L.A. Zadeh [1] in 1965, and fuzzy matrices were introduced by Thomason [2] in 1977. Various properties relative to fuzzy set and fuzzy matrix were subsequently studied by many authors, for example, [3]-[5] presented a number of results on the convergence of the power sequence of fuzzy matrices, respectively. In 1980, K. H. Kim and F. W. Roush [6] studied the canonical form of an idempotent matrix. In 1995, Ragab et al. [7] presented some properties of the min-max composition of fuzzy matrices. In 2007, A. K. Shymal and M. Pal [8] discussed triangular fuzzy matrices. With regard to determinant theory of fuzzy matrix, J. B. Kim [9], [10] discussed the some properties of determinant theory for fuzzy and Boolean matrices in 1988, M.Z. Ragab and E.G. Emam [11] presented the theory of the determinant and adjoint of a square fuzzy matrix in 1994. R. Hemasinha, N.R. Pal and J. C. Bezdek [12] studied the determinant of a fuzzy matrix with respect to t and co-t norms in 1997. More information about fuzzy sets and fuzzy matrices theory, please refer [4] and its references.

It is easy to obtain that the fuzzy matrix addition, the fuzzy matrix multiplication and the determinant of a square fuzzy matrix are different from the usual matrix addition, multiplication and the determinant of a square usual matrix, there are many examples to show that the sum and product of two fuzzy matrices under usual matrix addition and matrix multiplication in general do not give a fuzzy matrix.

Let $A=[a_{ij}]$ be an $n\times n$ fuzzy matrix, then the determinant $|A|$ is defined as

$$|A|=\sum_{\sigma\in S_n}a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

where S_n denotes the symmetric group of all permutations of the indices $(1,2,\ldots,n)$. J. B. Kim [9] proved that if A and B are two fuzzy matrices, then the determinants of $A\times B$, A and B satisfies

$$|A\times B|\geq|A|\,|B|, \qquad (4)$$

such property is obviously different from that of determinant of a real or complex square matrix.

Let $F_{m\,n}$ be the set of all $m\times n$ fuzzy matrices over fuzzy algebra $F[0,\,1]$. If $A=[a_{ij}]\in F_{n\,n}$ and $a_{ij}=0$ for $i>j$, then A is said to be an upper triangular fuzzy matrix, the set of all upper triangular fuzzy matrices is denoted by $TF_{n\,n}$. For convenience, if $a,\,b\in F[0,\,1]$, $A,\,B\in F_{n\,n}$, we replace $a\times b$ and $A\times B$ by ab and AB, respectively.

If $f:\,F_{n\,n}\longrightarrow F[0,\,1]$,

$$f(A\,B)=f(A)\,f(B)$$

for any $A,\,B\in F_{n\,n}$, then f is said to be multiplicative functional, if a functional $f:\,F[0,\,1]\longrightarrow F[0,\,1]$ satisfies

$$f(ab)=f(a)f(b)$$

for any $a,\,b\in F[0,\,1]$, we also call f is multiplicative.

It is well known that determinants of matrices over a field are multiplicative, for example, the determinants of real matrices and complex matrices are multiplicative under usual matrix multiplication, in general, the determinants for fuzzy matrices are not multiplicative according to (4). Note that the difference between the determinants of fuzzy matrices and the determinants of matrices over a field, we will discuss the properties and representations of the multiplicative functionals on fuzzy matrices in this paper.

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II. Preliminaries

Let O, E and $E_{i,j}$ denote the $n \times n$ fuzzy zero matrix, the fuzzy identity matrix and the fuzzy matrix whose (i,j)th entry is 1 and other entries are zero, respectively. If $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ fuzzy matrices and $a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, we write $A = B$.

Let $E^{(1)}_{(i,j)}$, $E^{(2)}_{(i\lambda)}$ and $E^{(3)}_{(i,j\lambda)}$ be the elementary matrices, respectively. That is

- $E^{(1)}_{(i,j)}$ $(i \neq j)$, the matrix obtained form the identity matrix E by interchanging the ith row (column) and the jth row (column).
- $E^{(2)}_{(i\lambda)}$, the matrix obtained form the identity matrix E by multiplying all elements of ith row (column) of E by a nonzero number $\lambda \in F[0, 1]$.
- $E^{(3)}_{(i,j\lambda)}$ $(i \neq j)$, the matrix obtained form the identity matrix E by adding λ times jth row (column) to ith row (column) of E.

In order to discuss the multiplicative functionals on fuzzy matrices, we firstly give the following lemmas. The addition and multiplication operation in the proof of these lemmas are fuzzy addition and fuzzy multiplication operation defined by (1) - (3).

Lemma 1. Let $f : F_{nn} \longrightarrow F[0, 1]$ be the multiplicative functional, then

(1) $f(E^{(3)}_{(i,j1)})f(E^{(2)}_{(i\lambda)}) = f(E^{(2)}_{(i\lambda)})f(E^{(3)}_{(i,j\lambda)})$;

(2) f is not injective;

(3) if f is surjective, then $f(E) = 1$ and $f(O) = 0$.

Proof (1) Since

$$E^{(3)}_{(i,j\lambda)}E^{(2)}_{(i\lambda)} = E^{(2)}_{(i\lambda)}E^{(3)}_{(i,j1)},$$

observe that f is a multiplicative functional, we have

$$f(E^{(3)}_{(i,j1)})f(E^{(2)}_{(i\lambda)}) = f(E^{(2)}_{(i\lambda)})f(E^{(3)}_{(i,j\lambda)}).$$

(2) Note that $E_{ij}\,E_{ij} = O$ for $i \neq j$, hence

$$\begin{aligned} f(E_{ij}\,E_{ij}) &= f(E_{ij})f(E_{ij}) \\ &= f(E_{ij}) = f(O). \end{aligned}$$

If f is injective, then $E_{i,j} = O$, this yields a contradiction. Thus f is not injective.

(3) Since

$$A\,E = A \text{ and } A\,O = O$$

for any fuzzy matrix $A \in F_{nn}$, one has

$$f(A\,E) = f(A)\,f(E) = f(A),$$

$$f(A\,O) = f(A)\,f(O) = f(O).$$

If f is surjective, then $f(E) = 1$ and $f(O) = 0$. The proof is completed.

Lemma 2. Let $f : F_{nn} \longrightarrow F[0, 1]$ be the multiplicative functional and $f(E) = 1$, then $f(E^{(1)}_{(i,j)}) = 1$ and

$$f(\sum_{i=1}^{n} E_{i,\,n-i+1}) = 1.$$

Proof. By simple computation, then

$$E^{(1)}_{(i,j)}\,E^{(1)}_{(i,j)} = E$$

and

$$(\sum_{i=1}^{n} E_{i,\,n-i+1})(\sum_{i=1}^{n} E_{i,\,n-i+1}) = E.$$

Since $f(E) = 1$, one has

$$f(E^{(1)}_{(i,j)})\,f(E^{(1)}_{(i,j)}) = f(E) = 1,$$

$$f(\sum_{i=1}^{n} E_{i,\,n-i+1})f(\sum_{i=1}^{n} E_{i,\,n-i+1}) = f(E) = 1.$$

Hence

$$f(E^{(1)}_{(i,j)}) = 1, \;\; f(\sum_{i=1}^{n} E_{i,\,n-i+1}) = 1.$$

The proof follows.

Remark 1. If $f : F_{nn} \longrightarrow F[0, 1]$ is a multiplicative functional and $f(O) = 0$, since $E_{ij}\,E_{ij} = O$ for $i \neq j$, we have $f(E_{i,j}) = 0$ for $i \neq j$.

Lemma 3. Let $f : F_{nn} \longrightarrow F[0, 1]$ be the multiplicative functional and $f(E) = 1$, then $f(E^{(2)}_{(i,\lambda)}) = f(E^{(2)}_{(1,\lambda)})$ and

$$f(E^{(3)}_{(1,n\lambda)}) = f(E^{(3)}_{(i,j\lambda)})$$

for $i, j = 1, 2 \ldots, n$.

Proof. Since

$$E^{(1)}_{(1,i)}E^{(2)}_{(i,\lambda)}E^{(1)}_{(1,i)} = E^{(2)}_{(1,\lambda)},$$

we have

$$f(E^{(1)}_{(1,i)})f(E^{(2)}_{(i,\lambda)})f(E^{(1)}_{(1,i)}) = f(E^{(2)}_{(1,\lambda)}).$$

By Lemma 2, $f(E^{(1)}_{(i,j)}) = 1$, thus

$$f(E^{(2)}_{(i,\lambda)}) = f(E^{(2)}_{(1,\lambda)})$$

for $i = 1, 2 \ldots, n$.

Since the matrix $E^{(3)}_{(1,n\lambda)}$ can be achieved by pre- and postmultiplication (respectively) of the matrix $E^{(3)}_{(i,j\lambda)}$ by some appropriate matrices $E^{(1)}_{(i,j)}$, again apply Lemma 2, note that f is the multiplicative functional, we obtain that

$$f(E^{(3)}_{(1,n\lambda)}) = f(E^{(3)}_{(i,j\lambda)}).$$

The proof follows.

Lemma 4. Let $f : F_{nn} \longrightarrow F[0, 1]$ be the multiplicative functional and $f(E) = 1$, if $F_0(\lambda) = f(E^{(2)}_{(1,\lambda)})$, then $F_0(1) = 1$ and $F : F[0, 1] \longrightarrow F[0, 1]$ is a multiplicative functional.

Proof. It is clear that $F_0(1) = f(E) = 1$.

Let $\lambda, \mu \in F[0, 1]$, then

$$\begin{aligned} F_0(\lambda\mu) = f(E^{(2)}_{(1,\lambda\mu)}) &= f(E^{(2)}_{(1,\lambda)}E^{(2)}_{(1,\mu)}) \\ &= f(E^{(2)}_{(1,\lambda)})f(E^{(2)}_{(1,\mu)}) \\ &= F_0(\lambda)F_0(\mu). \end{aligned}$$

Lemma 5. Let $f : F_{nn} \longrightarrow F[0, 1]$ be the multiplicative functional and and $f(E) = 1$, if $G_0(\lambda) = f(E^{(3)}_{(1,\, n\lambda)})$, then $G_0(0) = 1$ and $G_0(\lambda + \mu) = G(\lambda)G_0(\mu)$ for $\lambda, \mu \in F[0, 1]$.
Proof. It is clear that $G_0(0) = f(E) = 1$.

Let $\lambda, \mu \in F[0, 1]$, then

$$\begin{aligned} G_0(\lambda + \mu) &= f(E^{(3)}_{(1,\, n(\lambda+\mu))}) = f(E^{(3)}_{(1,\, n\lambda)}E^{(3)}_{(1,\, n\mu)}) \\ &= f(E^{(3)}_{(1,\, n\lambda)})f(E^{(3)}_{(1,\, n\mu)}) \\ &= G_0(\lambda)G_0(\mu). \end{aligned}$$

The proof is complete.

III. Multiplicative Functional on Upper Triangular Fuzzy Matrices

In this section, we firstly show the following Lemma 6, then give the representation of multiplication functional on upper triangular fuzzy matrices.

Lemma 6. Let $A = [a_{ij}] \in F_{nn}$ be a upper triangular matrix. Then A is the product of all fuzzy matrices $E^{(2)}_{(ia_{ii})}$ and $E^{(3)}_{(i,\, ja_{ij})}$, where $i, j = 1, 2 \ldots, n$.

Proof. Let $A_2 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, then

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}. \qquad (5)$$

Let $A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$, write A_3 as partitioned matrix

$$A_3 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where $A_{11} = \begin{pmatrix} a_{11} \end{pmatrix}, A_{12} = \begin{pmatrix} a_{12} & a_{13} \end{pmatrix}$ and

$$A_{22} = \begin{pmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{pmatrix}.$$

By simple computation, we obtain that

$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ 0 & E \end{pmatrix}.$$

Apply (5) to A_{22}, then $\begin{pmatrix} 1 & 0 \\ 0 & A_{22} \end{pmatrix} =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy imply that $\begin{pmatrix} A_{11} & A_{12} \\ 0 & E \end{pmatrix} =$

$$\begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By above discussion, we obtain that $A_3 =$

$$E^{(2)}_{(3\,a_{33})}E^{(3)}_{(2,\,3a_{23})}E^{(2)}_{(2\,a_{22})}E^{(3)}_{(1,\,2a_{12})}E^{(3)}_{(1,\,3a_{13})}E^{(2)}_{(1\,a_{11})}. \qquad (6)$$

Note that $n\times n$ upper triangular fuzzy matrix B_n with matrix representation

$$B_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \end{pmatrix}$$

can be written as

$$B_n = E^{(3)}_{(1,\,2a_{12})}E^{(3)}_{(1,\,3a_{13})}\cdots E^{(3)}_{(1,\,na_{1n})}E^{(2)}_{(1a_{11})},$$

thus apply (6)'s proof method to $A = [a_{ij}] \in F_{nn}$, by induction, we can prove that the fuzzy upper triangular matrix A is the product of $E^{(2)}_{(ia_{ii})}$ and $E^{(3)}_{(i,\,ja_{ij})}$, where $i < j$ and $i, j = 1, 2 \ldots, n$. The proof is complete.

By the above auxiliary results, we can prove the following Theorem 1, the main result of this paper.

Theorem 1. Let $f : TF_{nn} \longrightarrow F[0, 1]$ be multiplicative functional and $f(E) = 1$, then there exist functions $F_f, G_f : F[0, 1] \longrightarrow F[0, 1]$ satisfying

(1) F_f is a multiplicative functional;

(2) $G_f(\lambda+\mu) = G_f(\lambda)\, G_f(\mu)$ for $\lambda,\, \mu \in F[0, 1]$ such that

$$f([a_{ij}]) = (\prod_{i=1}^{n} F_f(a_{ii}))(\prod_{1\le i<j\le n} G_f(a_{ij})).$$

Proof. For $\lambda \in F[0, 1]$, let

$$F_f(\lambda) = F_0(\lambda) = f(E^{(2)}_{(1,\,\lambda)}),$$

$$G_f(\lambda) = G_0(\lambda) = f(E^{(3)}_{(1,\,n\lambda)}).$$

By Lemma 6, the upper triangular fuzzy matrix $[a_{ij}]$ is the product of $E^{(2)}_{(ia_{ii})}$ $(i = 1, \ldots, n)$ and $E^{(3)}_{(i,\,ja_{ij})}$ $(1 \le i < j \le n)$, again apply Lemma 3, we can imply that

$$f([a_{ij}]) = (\prod_{i=1}^{n} F_f(a_{ii}))(\prod_{1\le i<j\le n} G_f(a_{ij})).$$

The proof is complete.

Corollary 1. If $f : TF_{nn} \longrightarrow F[0, 1]$ is multiplicative functional. Let the functions F_1 and $G_1 : F[0, 1] \longrightarrow F[0, 1]$ satisfy $F_1(1) = 1$, $G_1(0) = 1$, and

$$f([a_{ij}]) = (\prod_{i=1}^{n} F_1(a_{ii}))(\prod_{1\le i<j\le n} G_1(a_{ij})).$$

then $F_1 = F_0$ and $G_1 = G_0$, where F_0 and G_0 are the same as that of Lemma 3 and Lemma 4, respectively.

Proof. For $\lambda \in F[0, 1]$, let A =diag $\{\lambda, \lambda, \ldots, \lambda\}$ be a diagonal fuzzy matrix and a fuzzy matrix $B \in TF_{nn}$ with matrix representation

$$B = \begin{pmatrix} 1 & \lambda & \lambda & \ldots & \lambda \\ 0 & 1 & \lambda & \ldots & \lambda \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \lambda \\ 0 & \ddots & \ddots & \ddots & 1 \end{pmatrix}.$$

Note that $G_1(0) = 1$, then

$$F_1(\lambda) = (\prod_{i=1}^{n} F_1(\lambda))(\prod_{1 \le i < j \le n} G_1(0)) = f(A).$$

Since $F_1(1) = 1$, we have

$$G_1(\lambda) = (\prod_{i=1}^{n} F_1(1))(\prod_{1 \le i < j \le n} G_1(\lambda)) = f(B).$$

By Theorem 1, $f(A) = F_0(\lambda)$ and $f(B) = G_0(\lambda)$, hence $F_1 = F_0$ and $G_1 = G_0$. The proof follows.

Remark 2. By Corollary 1, the F_f and G_f with the properties described in Theorem 1 are unique.

Remark 3. By [7. Proposition 3.2], $|A| = a_{11}a_{22}\cdots a_{nn}$ for all $A = [a_{ij}] \in TF_{n\,n}$, thus $f_0(A) = a_{11}a_{22}\cdots a_{nn}$ is multiplicative functional from $TF_{n\,n}$ to $F[0, 1]$, by Theorem 1, then $F_{f_0}(\lambda) = \lambda$, and $G_{f_0}(\lambda) = 1$.

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