International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:9, No:10, 2015

A Comparison of Bias Among Relaxed Divisor Methods Using 3 Bias Measurements

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Abstract—The apportionment method is used by many countries, to calculate the distribution of seats in political bodies. For example, this method is used in the United States (U.S.) to distribute house seats proportionally based on the population of the electoral district. Famous apportionment methods include the divisor methods called the Adams Method, Dean Method, Hill Method, Jefferson Method and Webster Method. Sometimes the results from the implementation of these divisor methods are unfair and include errors. Therefore, it is important to examine the optimization of this method by using a bias measurement to figure out precise and fair results. In this research we investigate the bias of divisor methods in the U.S. Houses of Representatives toward large and small states by applying the Stolarsky Mean Method. We compare the bias of the apportionment method by using two famous bias measurements: the Balinski and Young measurement and the Ernst measurement. Both measurements have a formula for large and small states. The Third measurement however, which was created by the researchers, did not factor in the element of large and small states into the formula. All three measurements are compared and the results show that our measurement produces similar results to the other two famous measurements.

Keywords—Apportionment, Bias, Divisor, Fair, Simulation

I. Introduction

THE apportionment method is used for equal allocation of identical and indivisible objects that may be entitled to unequal shares. A lot of research about the apportionment method such as the determination of the number of members of the U.S. House of Representatives is based on the proportion of the population of each state to the total population of the U.S. Consequently, the results of the election can be implemented unfairly and included errors. For example, in the case of the U.S. House of Representatives, the problems that happened in the United States Congress from the beginning have been solved by apportionment since the early 1790s. Fairness and historical precedents dictate that several properties must be satisfied by any acceptable method. It seems that the present method is not accurate. Therefore, the apportionment method is the unique method for satisfying the essential properties[1]. Seats in the House of Representatives are allocated by the formula known as the Hill method, which is one of six currently approved methods. These current divisor methods consist of five methods of rounding fractions (The Hill method, the Dean method, the Adams method, the Webster method and the Jefferson method.) one ranking fraction.

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The sixth method, called the Hamilton method, used a ranking fraction instead of rounding fraction. In ranking fractions, named the Hamilton method was initiated by Alexander Hamilton and was used by the first congress to enact an apportionment of the House. During 1851 and 1901 the Hamilton method was described as the largest fractional method of remainders ever used. It has been never strictly applied because various external variables, such as increasing number of states, made unstable results of this apportionment method. This is known as the Alabama Paradox, which was found during an increase of the size of the House in 1880 in Alabama State. The state expected to receive 8 seats from the house size of 299, whereas only 7 seats from a house size of 300 were earned. The divisor methods, including the Hill method currently in use, differently allocate seats among the states, but the operational methods only differ where rounding occurs in seat assignments. Three of these methods, the Adams method, the Webster method and the Jefferson method have fixed rounding points. The other two methods, the Dean method and the Hill method, are used for various rounding points that rise as the number of seats assigned to a larger state. The methods can be defined by the rounded point which occurs in a similar way. The Adams method is up for all fractions, the Dean method is at the harmonic mean, the Hill method is at the geometric mean, the Webster method is at the arithmetic mean which is 0.5 for successive numbers, and the Jefferson method is down for all fractions.

Therefore, it is important to examine the bias of the apportionment method in order to figure out precise and fair results. There are two purposes to this research. First the researchers wanted to see if the measurement we created produces similar results to other famous measurements, despite our measurement not including large and small states. Second, the researchers wanted to find out which apportionment method produced the least amount of errors.

II. The apportionment methods

From the general problem of how to find the fair of division problem of the apportionment method in an election system is how to divide the seats in a fair manner.

The first step to find the apportionment problem let "s" denote the number of states, the house size "h" is the total number of seats, "p" is the population of s states, which can be represented by the population of the state vector $=$ $[p_1, \ldots, p_s]$, where p_i is a positive integer, the total Population is $p; p = \sum p_i$ and the quota is "q" where $q_i = \frac{h}{p} \times p_i$ for $i = 1, \ldots, s$. If $a = (a_1, \ldots, a_s) \geq 0$ is a vector of positive integer, then the vector a is called an apportionment of h if $\sum_i a_i = h$. Then, carrying out the constitutional requirement exactly means to achieve the mathematical equality $a = q$ where $q = (q_1, \ldots, q_s)$ is the vector of quotas. Most of common apportionment methods are divisor methods. This research shows that the distribution results of all five divisor methods, the Adam's Method, Dean's Method, Jefferson's Method, Hill's Method, Webster's Method, and the optimal solution of the optimization problem.

Step 2, divisor method of apportionment can be found as N denotes sets of positive integers. A real valued function $d(a)$ for $a \in N$ is defined as a rounding criterion. The function $d(a)$ is assigned as a strictly increasing function in a. It satisfies $a \leq d(a) \leq a+1$ for $a \in N$. Then z is specified be a positive real number and $[z]$ is denoted as an integer. If $z < d(0)$, then $[z] = 0$. If $d(a) < z < d(a+1)$ for some $a \in N$, then $[z] = a + 1$. If $z = d(a)$ for some $a \in N$, then $[z] = a$ or $a + 1$. If an apportionment method is defined with $d(a)$, then the method is called a divisor method. The following divisor methods are especially well known and defined with respective $d(a)$[2], shown in Table I.

Finally Step 3, we used the Stolarsky mean to defined the divisor methods, then denotes R as the set of real numbers. For a real number $\theta \in$ R and a positive integer $a \in N$, are assigned as the rounding criterion $d_\theta(a)$ as follows:

In case $a \neq 0$;

$$d_\theta(a) = \begin{cases} \frac{1}{e}\frac{(a+1)^{a+1}}{a^a} & ,if \quad \theta = 1 \\ \frac{1}{\log\frac{a+1}{a}} & ,if \quad \theta = 0 \\ \left(\frac{(a+1)^\theta - a^\theta}{\theta}\right)^{\frac{1}{\theta-1}} & ,if \quad \theta \neq 0, 1 \end{cases} \tag{1}$$

In case $a = 0$;

$$d_\theta(0) = \begin{cases} 0 & ,if \quad \theta \leq 0 \\ \frac{1}{e} \approx 0.37 & ,if \quad \theta = 1 \\ \left(\frac{1}{\theta}\right)^{\frac{1}{\theta-1}} & ,if \quad \theta > 0, \theta \neq 1 \end{cases} \tag{2}$$

For $d(a)$ that $a < d(a) < a+1$ when θ is finite. The values of $d(a)$ relate intimately with the biases of apportionment methods. It is important to note that the function $d(a)$ is increasing in θ for each fixed positive $a \in N$. In addition, $d(0)$ is also increasing in $\theta > 0$ while $d(0) = 0$ for $\theta \leq 0$[3].

Notice that $-\infty$ corresponds to the Adams method, $\theta = -4$ to the Dean Method, $\theta = -1$ to the Hill Method, $\theta = 0$ to the TS Method[4], $\theta = 1$ to the Theil Method[5], $\theta = 2$ to the Webster Method and ∞ to the Jefferson Method[6], shown in Table II.

III. The Bias Measurements

We measured the famous bias measurement using the Balinski and Young (BY) and the Ernst (ER) measurements. These measurements include large and small state element formula. In addition we used the bias measurement formula which, not including large and small state element in formula were also applied. A method is absolutely unbiased if the bias values become zero[7].

The Balinski and Young[1] formula is,

$$BY(\theta) = \frac{k_S(\theta)}{k_L(\theta)} - 1 \tag{3}$$

where,

k_L is the large state elements,

$$k_L(\theta) = \frac{\sum_{i \in L} a_i(\theta)}{\sum_{i \in L} q_i}$$

k_S is the small state elements,

$$k_S(\theta) = \frac{\sum_{i \in S} a_i(\theta)}{\sum_{i \in S} q_i}$$

The Ernst[8] formula is,

$$ER(\theta) = 1 - \frac{k'_S(\theta)}{k'_L(\theta)} \tag{4}$$

where,

$k'_L$is the large state elements,

$$k'_L(\theta) = \sum_{i \in L} \frac{q_i}{a_i(\theta)}$$

$k'_S$is the small state elements,

$$k'_S(\theta) = \sum_{i \in S} \frac{q_i}{a_i(\theta)}$$

Both BY and ER bias measurement are including large and small state amounts definitions. However sometimes isn't clear because it is difficult to determine what can be considered large or small. Then we used the "B" Bias Measurement to compare those apportionment methods, which no large and small state definition.

The B bias measurement formula is,

$$B(\theta) = \sum_{i=1}^{s} [q_i]_\theta - h \tag{5}$$

where,

$[q_i]_\theta$ is the rounding quota by using the Stolarsky mean in every θ of q_i.

TABLE I
5 Accepted Divisor Methods

Method	:	Adams	Dean	Hill	Webster	Jefferson
$d(a)$	:	a	$\frac{a(a+1)}{\left(a+\frac{1}{2}\right)}$	$\sqrt{a(a+1)}$	$a+\frac{1}{2}$	$a+1$

TABLE II
The values of parameter θ According to Method

θ	:	$-\infty$	-4	-1	0	1	2	$+\infty$
Method	:	Adams	Dean	Hill	TS	Theil	Webster	Jefferson

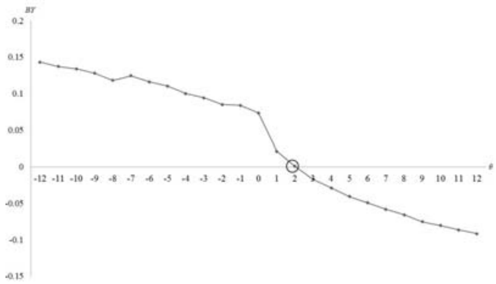

Fig. 1. Bias of Balinski and Young $BY\,(\theta)$ formulas (3)

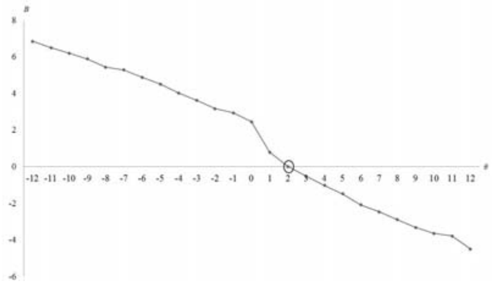

Fig. 3. Bias of B bias measurement $B\,(\theta)$ formulas (5)

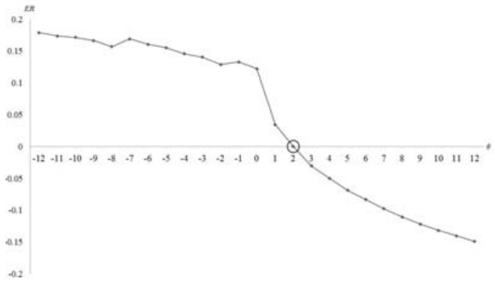

Fig. 2. Bias of Ernst $ER\,(\theta)$ formulas (4)

IV. Results

This research used the random populations with the simulation program to find the bias's results of the apportionment method by using the Stolarsky mean at parameter θ from -12 to 12. The results of mean bias values from the BY, ER and B 's bias measurements are compared in Table III. The trend of the mean biases from the BY ER and B 's bias measurements are shown in Fig.1, Fig.2 and Fig.3. The X axis is the parameter θ values and Y axis is the bias values from every bias measurements. From Table III, the results shown, the lowest bias values are at θ = 2, which correspond to the Webster method.

V. Conclusion

The apportionment method problems of the Representatives base on the proportional of the population of states are often seem easy, but hard to solve because there are many Apportionment methods but In this research shown that the apportionment method by using the Stolarsky Mean can be described in the form of discrete optimization, then the continuous values should have an optimal solution identical to the quota. We compare that value to find the best method by using the famous bias measurement using the Balinski and Young measurement and the Ernst measurement which, including large and small state element in formulas and using the B bias measurement which, not include large and small state element in formula were also applied. The research found that the Webster Method was the lowest bias divisor.

TABLE III
THE BIAS VALUES OF BY, ER AND B AT θ FROM 12 TO -12

θ	BY	ER	B
θ = -12	0.1434	0.1793	6.842
θ = -11	0.1372	0.1739	6.497
θ = -10	0.1342	0.1717	6.1895
θ = -9	0.1281	0.1668	5.878
θ = -8	0.1181	0.1573	5.4284
θ = -7	0.1245	0.1695	5.278
θ = -6	0.1162	0.1607	4.8744
θ = -5	0.1103	0.1556	4.498
θ = -4	0.0999	0.146	4.023
θ = -3	0.0942	0.1406	3.618
θ = -2	0.0849	0.1293	3.1573
θ = -1	0.0838	0.1331	2.9362
θ = 0	0.0734	0.1224	2.452
θ = 1	0.0216	0.0348	0.792
θ = 2	0.0011	0.0005	0.004
θ = 3	-0.0168	-0.0303	-0.534
θ = 4	-0.0287	-0.0495	-1.03
θ = 5	-0.0405	-0.0684	-1.4893
θ = 6	-0.0491	-0.0829	-2.092
θ = 7	-0.058	-0.0976	-2.4708
θ = 8	-0.0657	-0.1104	-2.896
θ = 9	-0.0751	-0.1217	-3.3293
θ = 10	-0.0803	-0.1314	-3.649
θ = 11	-0.0863	-0.1402	-3.794
θ = 12	-0.0916	-0.1491	-4.503

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