

# Computational Simulations on Stability of Model Predictive Control for Linear Discrete-time Stochastic Systems

Tomoaki Hashimoto

*Abstract*—Model predictive control is a kind of optimal feedback control in which control performance over a finite future is optimized with a performance index that has a moving initial time and a moving terminal time. This paper examines the stability of model predictive control for linear discrete-time systems with additive stochastic disturbances. A sufficient condition for the stability of the closed-loop system with model predictive control is derived by means of a linear matrix inequality. The objective of this paper is to show the results of computational simulations in order to verify the effectiveness of the obtained stability condition.

*Keywords*—Computational simulations, optimal control, predictive control, stochastic systems, discrete-time systems.

## I. INTRODUCTION

MODEL predictive control (MPC) is a well-established control method in which the current control input is obtained by solving a finite horizon open-loop optimal control problem using the current state of the system as the initial state. This procedure is repeated at each sampling instant. Thus, MPC is a kind of optimal feedback control in which the control performance over a finite future is optimized and its performance index has a moving initial time and a moving terminal time [1]. MPC is also known as receding horizon control and it is one of the most successful control methodologies because it enables control performance to be optimized while taking into account constraints on state and control variables [2]–[9].

Recently, robust MPC methods against uncertain disturbances attract much attention in this research field. The design methods of robust MPC can be classified into deterministic and stochastic approaches. In the deterministic approach, most studies are based on the min-max approach, where a performance index is minimized over the worst possible disturbance scenario [10]–[14]. However, min-max approaches are often computationally demanding, and the control performance is often too conservative because no statistical properties of the disturbance are taken into account.

The other approach is addressed by stochastic MPC where expected values of performance indices, probabilistic constraints and convergence in probability are considered by exploiting the statistical information on the disturbance [15]–[20]. Although the aforementioned papers [15]–[20] have

achieved tremendous progress in dealing with probabilistic constraints of the stochastic MPC, there are several restrictions on the probability distributions of stochastic disturbances such as the normal (Gaussian) distribution, known distribution, finite-support and time-invariance. On the other hand, the methods proposed in [21], [22] enable us to address arbitrarily unknown probability distributions including non-Gaussian, infinitely-supported and time-variant distributions. The Chebyshev's inequality was applied in [21], [22] to transform probabilistic constraints on the state variables into deterministic constraints on the control inputs. Moreover, a sufficient condition for the stability of the closed-loop system was provided in [22]. However, the validity of the obtained stability condition has not yet been confirmed. Therefore, the objective of this paper is to derive a modified sufficient condition for the stability of the closed-loop system with MPC. Moreover, in this paper, the results of computational simulations are provided in order to verify the effectiveness of the obtained stability condition.

This paper is organized as follows. In section II, we introduce some notations and define the system model. In section III, we provide some preliminary results that are useful to construct the main results. The stochastic MPC problem is formulated in section IV and it is solved in section V. The stability of stochastic MPC is discussed in section VI. In section VII, we provide an illustrative example to verify the effectiveness of the obtained stability condition. Finally, some concluding remarks are given in section VIII.

## II. NOTATION AND SYSTEM MODEL

Let  $\mathbb{R}$  and  $\mathbb{N}$  denote the sets of real numbers and natural numbers, respectively. Let  $\mathbb{R}_+$  denote the set of nonnegative real numbers. For a matrix  $A$ , the transpose and the trace of  $A$  is denoted by  $A'$  and  $\text{tr}A$ , respectively. For matrices  $A = \{a_{i,j}\}$  and  $B = \{b_{i,j}\}$ , let the inequalities between  $A$  and  $B$  such as  $A > B$  and  $A \geq B$  indicate that they are satisfied componentwisely, i.e.,  $a_{i,j} > b_{i,j}$  and  $a_{i,j} \geq b_{i,j}$  hold for all  $i$  and  $j$ , respectively. Likewise, let each notation for the absolute value  $|A|$ , the square root  $\sqrt{A}$  and the multiplication  $A \otimes B$  indicate that it holds componentwisely, i.e.,  $|A| = \{|a_{i,j}|\}$ ,  $\sqrt{A} = \{\sqrt{a_{i,j}}\}$  and  $A \otimes B = \{a_{i,j} \times b_{i,j}\}$  for all  $i$  and  $j$ . Let  $A \succ 0$  indicate that  $A$  is a positive definite matrix, i.e.,  $x'Ax > 0$  for any  $x \neq 0$ . For a vector  $x$ , let the norms  $\|x\|$  and  $\|x\|_A$  be defined by  $\|x\| := x'x$  and  $\|x\|_A := x'Ax$ , respectively, where  $A \succ 0$ .

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T. Hashimoto is with the Department of Mechanical Engineering, Osaka Institute of Technology, Asahi-ku 535-8585, Japan (e-mail: info@thashi.net).

A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathbb{K}$  if it is continuous, strictly increasing and  $\alpha(0) = 0$ . A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathbb{K}_\infty$  if  $\alpha \in \mathbb{K}$  and  $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ .

Let the triple  $(\Omega, \mathcal{F}, \mathcal{P})$  denote a probability space, where  $\Omega \subseteq \mathbb{R}$  is the sampling space,  $\mathcal{F}$  is the  $\sigma$ -algebra and  $\mathcal{P}$  is the probability measure [23].  $\Omega$  is non-empty and is not necessarily finite.  $\mathcal{P}(E)$  denotes the probability that the event  $E$  occurs. If  $\mathcal{P}(E) = 1$  we say that  $E$  occurs almost surely. For a random variable  $z : \Omega \rightarrow \mathbb{R}$  defined on  $(\Omega, \mathcal{F}, \mathcal{P})$ , let the expected value and the variance of  $z$  be denoted by  $\mathcal{E}(z)$  and  $\mathcal{V}(z)$ , respectively. For a random vector  $z = [z_1, \dots, z_n]'$  whose each component is a random variable  $z_i : \Omega \rightarrow \mathbb{R}$  ( $i = 1, \dots, n$ ) defined on the same probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ , we also adopt the same notation  $\mathcal{E}(z)$  and  $\mathcal{V}(z)$  to denote  $\mathcal{E}(z) = [\mathcal{E}(z_1), \dots, \mathcal{E}(z_n)]'$  and  $\mathcal{V}(z) = [\mathcal{V}(z_1), \dots, \mathcal{V}(z_n)]'$  for notational simplicity. Furthermore, a covariance matrix  $C_v(z)$  is defined by  $C_v(z) := \mathcal{E}\{[z - \mathcal{E}(z)][z - \mathcal{E}(z)]'\}$ .

Throughout this paper, we consider the following linear discrete-time system with stochastic disturbances:

$$x(t+1) = Ax(t) + Bu(t) + Cw(t), \quad (1)$$

where  $t \in \mathbb{N}$  is the time step,  $x(t) : \mathbb{N} \rightarrow \mathbb{R}^n$  is the state,  $u(t) : \mathbb{N} \rightarrow \mathbb{R}^m$  is the control input and  $w(t) : \mathbb{N} \rightarrow \mathbb{R}^\ell$  is the unknown stochastic disturbance. More precisely, for each component  $w_i : \mathbb{N} \times \Omega \rightarrow \mathbb{R}$  of  $w$ , the random sequence  $\{w_i(t) : t \in \mathbb{N}\}$  is a collection of random variables on the same probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  equipped with a filtration  $\{\mathcal{F}_t : t \in \mathbb{N}\}$  [23]. The system coefficients  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{n \times \ell}$  are all known constant matrices. The pair  $(A, B)$  is assumed to be controllable. We also assume that the initial state  $x(0)$  is given and all components of the state  $x(t)$  are deterministically observable. Thus, we suppose that  $\mathcal{E}(x(t)) = x(t)$  and  $\mathcal{V}(x(t)) = 0$  at the present time  $t$ .

Next, we introduce some assumptions on the properties of the stochastic disturbances.

*Assumption 1:*  $w_i(t)$  and  $w_j(t)$  are independent each other for all  $i \neq j$  and  $t \in \mathbb{N}$ . Also,  $w_i(t)$  and  $w_j(k)$  are independent each other for all  $t \neq k$  and  $j \in \{1, \dots, \ell\}$ .

Assumption 1 implies that all random variables  $w_i(t)$  for  $i \in \{1, \dots, \ell\}$  and  $t \in \mathbb{N}$  are independent one another.

*Assumption 2:*  $\mathcal{E}(w(t))$  and  $\mathcal{V}(w(t))$  are assumed to be known for every time  $t$ .

Note that the probability distributions of the random variables  $w_i$  are not necessarily assumed to be known. In this study, the assumption on known probability distributions is relaxed as arbitrarily unknown probability distributions.

*Assumption 3:* There exists a positive real constant  $\delta$  such that

$$\|C\mathcal{E}(w(t))\|_A \leq \delta \|\mathcal{E}(x(t))\|_A \quad (2)$$

is satisfied for all  $A \succ 0$  and  $t \in \mathbb{N}$ .

Note that  $\mathcal{E}(w(t))$  is assumed to be bounded, but  $w(t)$  itself may be unbounded. Assumption 3 is introduced to discuss the stability at the origin of the averaged system for (1).

*Definition 1:* System (1) is said to be almost surely asymptotically stable in the mean if the following condition is

satisfied:

$$\mathcal{P}\left(\lim_{t \rightarrow \infty} \mathcal{E}(x(t)) = 0\right) = 1. \quad (3)$$

### III. PRELIMINARIES

In this section, we provide some preliminary results that are useful to derive the main results. The following lemma is well known as Lyapunov stability theory.

*Lemma 1 ([24]):* Consider a system  $x(t+1) = f(x(t))$ , where  $x(t) : \mathbb{N} \rightarrow \mathbb{R}^n$ ,  $f(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $f(0) = 0$ . Suppose that there exist a Lyapunov function  $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , class  $\mathbb{K}_\infty$  functions  $\alpha_1, \alpha_2$  and a positive definite function  $\alpha_3$  satisfying all the following conditions:

$$\begin{aligned} V(x) &\geq \alpha_1(\|x\|) \\ V(x) &\leq \alpha_2(\|x\|) \\ V(f(x)) - V(x) &\leq -\alpha_3(\|x\|) \end{aligned}$$

Then, the origin  $x = 0$  is asymptotically stable.

The equivalence shown below is known as Schur complement.

*Lemma 2:* For given block matrices  $A, B$  and  $C$ , the followings are equivalent.

$$\begin{aligned} \begin{bmatrix} A & B \\ B' & C \end{bmatrix} &\succ 0 \\ \Leftrightarrow C &\succ 0, \quad A - B'C^{-1}B \succ 0 \end{aligned}$$

The following lemmas are fundamental properties of the matrix theory.

*Lemma 3:* For any  $A \succ 0 \in \mathbb{R}^{n \times n}$  and  $b, c \in \mathbb{R}^n$ ,

$$\pm 2b'Ac \leq b'Ab + c'Ac.$$

*Lemma 4:* For any nonsingular matrix  $A$ ,  $(A')^{-1} = (A^{-1})'$  and  $A'A \succ 0$  hold true. For any positive definite matrix  $A$ , it is true that  $A^{-1} \succ 0$  and there exists  $B$  such that  $A = B'B$ .

### IV. PROBLEM STATEMENT

In this section, we formulate the stochastic MPC problem of system (1). The control input at each time  $t$  is determined so as to minimize the performance index given by

$$J := \phi[x(t+N)] + \sum_{k=t}^{t+N-1} L[x(k), u(k)]. \quad (4a)$$

Therein,  $N \in \mathbb{N}$  denotes the length of prediction horizon.  $\phi$  and  $L$  are defined by

$$\phi := \mathcal{E}[x(t+N)'Px(t+N)], \quad (4b)$$

$$L := \mathcal{E}[x(k)'Qx(k)] + u(k)'Ru(k), \quad (4c)$$

where  $P, Q$  and  $R$  are positive definite constant matrices.  $\phi \in \mathbb{R}_+$  is the terminal cost function and  $L \in \mathbb{R}_+$  is the stage cost function over the prediction horizon.

For notational convenience, let  $\mathbf{X} \in \mathbb{R}^{nN}$ ,  $\mathbf{U} \in \mathbb{R}^{mN}$ ,  $\mathbf{W} \in \mathbb{R}^{\ell N}$ ,  $\mathbf{A} \in \mathbb{R}^{nN \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{nN \times mN}$ ,  $\mathbf{C} \in \mathbb{R}^{nN \times \ell N}$ ,  $\mathbf{Q} \in \mathbb{R}^{nN \times nN}$  and  $\mathbf{R} \in \mathbb{R}^{mN \times mN}$  be defined by

$$\mathbf{X}(t) := \begin{bmatrix} x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix}, \quad \mathbf{U}(t) := \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix},$$

$$\mathbf{W}(t) := \begin{bmatrix} w(t) \\ \vdots \\ w(t+N-1) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},$$

$$\mathbf{B} := \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} C & 0 & \cdots & 0 \\ AC & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}C & A^{N-2}C & \cdots & C \end{bmatrix},$$

$$\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix}, \quad \mathbf{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \end{bmatrix}.$$

Using the above notation, the performance index in (4) can be rewritten as follows:

$$J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \mathcal{E}[x(t)'Qx(t)] + \mathcal{E}[\mathbf{X}(t)'Q\mathbf{X}(t)] + \mathbf{U}(t)'R\mathbf{U}(t), \quad (5)$$

Here, note that system (1) over the prediction horizon can be rewritten as

$$\mathbf{X}(t) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathbf{W}(t). \quad (6)$$

Now, the optimal control problem can be formulated as the minimization problem of (5) subject to constraint (6) for given  $x(t)$ ,  $\mathbf{Q}$  and  $\mathbf{R}$ .

## V. STOCHASTIC MPC

In this section, we provide a method for solving the stochastic MPC problem discussed in the previous section.

From (6),  $\mathcal{E}(\mathbf{X}(t))$  and  $\mathcal{V}(\mathbf{X}(t))$  are given by

$$\mathcal{E}(\mathbf{X}(t)) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \quad (7a)$$

$$\mathcal{V}(\mathbf{X}(t)) = \mathbf{C} \otimes \mathbf{C}\mathcal{V}(\mathbf{W}(t)). \quad (7b)$$

In the above, we apply  $\mathcal{E}(x(t)) = x(t)$  because the present state  $x(t)$  is a deterministic vector. Moreover, it follows from (5) that

$$J = x(t)'Qx(t) + \mathbf{U}(t)'R\mathbf{U}(t) + \text{tr}[QC_v(\mathbf{X}(t))] + \mathcal{E}(\mathbf{X}(t))'Q\mathcal{E}(\mathbf{X}(t)). \quad (8)$$

It is worth noting that the covariance matrix  $C_v(\mathbf{X}(t))$  is independent of  $\mathbf{U}(t)$  as shown below.

$$\begin{aligned} C_v(\mathbf{X}(t)) &= \mathcal{E}[\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}'] \\ &= \mathcal{E}[\{\mathbf{C}\mathbf{W}(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t))\}\{\mathbf{C}\mathbf{W}(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t))\}'] \end{aligned}$$

Substituting (7a) into (8) and neglecting the terms that don't contain  $\mathbf{U}(t)$ , we have

$$\begin{aligned} \min_{\mathbf{U}(t)} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] &= \\ \min_{\mathbf{U}(t)} \left\{ \mathbf{U}'(t) (\mathbf{B}'\mathbf{Q}\mathbf{B} + \mathbf{R}) \mathbf{U}(t) \right. & \\ \left. + 2(\mathbf{A}x(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t)))' \mathbf{Q}\mathbf{B}\mathbf{U}(t) \right\}. & \quad (9) \end{aligned}$$

Note that the minimization problem of  $J$  has been reduced to the quadratic programming with respect to  $\mathbf{U}$  that can be solved using a conventional algorithm [25].

## VI. STABILITY OF STOCHASTIC MPC

In this section, we study the stability of the closed-loop system with the stochastic MPC. We employ here the Lyapunov stability theory to derive a sufficient condition for the asymptotic stability in the mean of the stochastic MPC system. It is known that there is a restriction on the choice of a performance index in order to guarantee the stability of the closed-loop system with MPC [24]. More precisely, the terminal cost function should be chosen as a Lyapunov function satisfying (10). Therefore, we need to select a performance index that is appropriate for stability analysis. Hence, in the subsequent discussion, we consider the cost functions  $\phi$  and  $L$  as follows:

$$\begin{aligned} \phi[\mathcal{E}(x(t+N))] &= \mathcal{E}(x(t+N))'P\mathcal{E}(x(t+N)), \\ L[\mathcal{E}(x(k)), u(k)] &= \mathcal{E}(x(k))'Q\mathcal{E}(x(k)) + u(k)'Ru(k) \end{aligned}$$

Note that the minimization problem of the above cost functions can be reduced to the same minimization problem in (9). Therefore, the stability of MPC system with performance index (4) is equivalent to the stability of MPC system with the above performance index.

First, we consider the existence of the control input  $u(t) = K\mathcal{E}(x(t))$  such that the following inequality holds, where  $K \in \mathbb{R}^{m \times n}$  is a constant matrix.

$$\phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] \leq -L[\mathcal{E}(x(t)), u(t)] \quad (10)$$

Recall that  $P$ ,  $Q$  and  $R$  are weighting matrices introduced in (4). Let  $Z$  and  $G$  be matrices such that  $Z = P^{-1}$  and  $G = KZ$ .

The following lemma plays an important role to establish the stability criteria for the closed-loop system with the stochastic MPC.

*Lemma 5:* Inequality (10) is satisfied if there exist  $Z$  and  $G$  such that the following linear matrix inequality (LMI) holds for given  $Q$ ,  $R$ ,  $\delta$ :

$$\begin{bmatrix} (1-2\delta)Z & ZA' + G'B' & ZQ & G'R \\ AZ + BG & \frac{Z}{2} & 0 & 0 \\ QZ & 0 & Q & 0 \\ RG & 0 & 0 & R \end{bmatrix} \succ 0. \quad (11)$$

*Proof:* It is straightforward that

$$\begin{aligned} \phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] &= \mathcal{E}(w(t))' \{C'PC\} \mathcal{E}(w(t)) \\ &+ \mathcal{E}(x(t))' \{(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)) \\ &+ 2\mathcal{E}(x(t))'(A+BK)'PC\mathcal{E}(w(t)). \end{aligned} \quad (12)$$

Applying Lemma 3 to the last term in the right-hand side of (12) yields

$$\begin{aligned} \phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] &\leq 2\mathcal{E}(w(t))' \{C'PC\} \mathcal{E}(w(t)) \\ &+ \mathcal{E}(x(t))' \{2(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)). \end{aligned} \quad (13)$$

Furthermore, applying Assumption 3 to the first term in the right-hand side of (13) yields

$$\begin{aligned} \phi[\mathcal{E}(x(t+1))] - \phi[\mathcal{E}(x(t))] &\leq \\ \mathcal{E}(x(t))' \{2\delta P + 2(A+BK)'P(A+BK) - P\} \mathcal{E}(x(t)). \end{aligned} \quad (14)$$

Noting that

$$L = \mathcal{E}(x(t))'(Q + K'RK)\mathcal{E}(x(t)), \quad (15)$$

we can see that if

$$P - 2(A+BK)'P(A+BK) - 2\delta P - Q - K'RK \succ 0 \quad (16)$$

is satisfied, then inequality (10) holds true.

In the following, it is shown that above inequality (16) is equivalent to inequality (11).

Pre- and post-multiplying (16) by  $Z$  yields

$$(1-2\delta)Z - 2(AZ+BG)'Z^{-1}(AZ+BG) - ZQZ - G'RG \succ 0 \quad (17)$$

Using the following relation

$$ZQZ + G'RG = \begin{bmatrix} QZ \\ RG \end{bmatrix}' \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}^{-1} \begin{bmatrix} QZ \\ RG \end{bmatrix},$$

we can see that (17) is equivalent to the following:

$$\begin{aligned} (1-2\delta)Z \\ - \begin{bmatrix} AZ+BG \\ QZ \\ RG \end{bmatrix}' \begin{bmatrix} \frac{Z}{2} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix}^{-1} \begin{bmatrix} AZ+BG \\ QZ \\ RG \end{bmatrix} \succ 0 \end{aligned} \quad (18)$$

Using Lemma 2, we can see that the Schur compliment of (11) is equivalent to (18). Consequently, the proof has been completed. ■

Let a function  $V[\mathcal{E}(x(t))] : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be defined by

$$V[\mathcal{E}(x(t))] := \min_{\mathbf{U}(t)} J[\mathcal{E}(x(t)), \mathcal{E}(\mathbf{X}(t)), \mathbf{U}(t)]. \quad (19)$$

Let  $\mathbf{U}^*(t)$  denote the sequence of the optimal control input over the prediction horizon defined by

$$\begin{aligned} \mathbf{U}^*(t) &:= \begin{bmatrix} u^*(t) \\ \vdots \\ u^*(t+N-1) \end{bmatrix} \\ &:= \arg \min_{\mathbf{U}(t)} J[\mathcal{E}(x(t)), \mathcal{E}(\mathbf{X}(t)), \mathbf{U}(t)]. \end{aligned} \quad (20)$$

Let  $\mathbf{X}^*(t) = [x^*(t+1), \dots, x^*(t+N)]'$  denote the optimal state sequence of the closed-loop system over the prediction horizon using  $\mathbf{U}^*(t)$ . Let  $\hat{\mathbf{U}}^*(t+1)$  be defined by

$$\hat{\mathbf{U}}^*(t+1) := \begin{bmatrix} u^*(t+1) \\ \vdots \\ u^*(t+N-1) \\ u(t+N) \end{bmatrix}. \quad (21)$$

Therein, the final optimal control input  $u^*(t+N)$  is replaced with any feasible control input  $u(t+N)$ . Accordingly, let  $\hat{\mathbf{X}}^*(t+1)$  be the state sequence of the closed-loop system using  $\hat{\mathbf{U}}^*(t+1)$ .

Here, we introduce the well-known standard assumption for the stability analysis of the MPC system [24].

*Assumption 4:* There exists a function  $\alpha \in \mathbb{K}_\infty$  such that

$$V[\mathcal{E}(x(t))] \leq \alpha(\|\mathcal{E}(x(t))\|) \quad (22)$$

is satisfied for all  $t \in \mathbb{N}$ .

Note that if there exists a positive constant  $\rho$  such that

$$\|u^*(t)\| \leq \rho \|\mathcal{E}(x(t))\|$$

is satisfied for all  $t \in \mathbb{N}$ , then Assumption 4 is satisfied. Thereby, Assumption 4 is called the weak controllability assumption [24].

Here, we provide the stability criteria for the closed-loop system using the stochastic MPC.

*Theorem 1:* Under Assumptions 1–4, the closed-loop system using stochastic MPC input  $\mathbf{U}^*(t)$  is almost surely asymptotically stable in the mean if there exist  $Z$  and  $G$  such that LMI (11) is satisfied.

*Proof:* It follows from (19) that

$$\begin{aligned} V[\mathcal{E}(x(t))] &= L[\mathcal{E}(x(t)), u^*(t)] \\ &+ \sum_{k=t+1}^{t+N-1} L[\mathcal{E}(x^*(k)), u^*(k)] + \phi[\mathcal{E}(x^*(t+N))] \end{aligned} \quad (23)$$

Using the relation

$$\begin{aligned} J[\mathcal{E}(x(t+1)), \mathcal{E}(\mathbf{X}^*(t+1)), \mathbf{U}^*(t+1)] \\ \leq J[\mathcal{E}(x(t+1)), \mathcal{E}(\hat{\mathbf{X}}^*(t+1)), \hat{\mathbf{U}}^*(t+1)], \end{aligned} \quad (24)$$

we have the following:

$$\begin{aligned} V[\mathcal{E}(x(t+1))] &= \sum_{k=t+1}^{t+N} L[\mathcal{E}(x^*(k)), u^*(k)] \\ &+ \phi[\mathcal{E}(x^*(t+N+1))] \\ &\leq \sum_{k=t+1}^{t+N-1} L[\mathcal{E}(x^*(k)), u^*(k)] \\ &+ L[\mathcal{E}(x^*(t+N)), u(t+N)] + \phi[\mathcal{E}(x(t+N+1))] \\ &=: \hat{V}[\mathcal{E}(x(t+1))] \end{aligned} \quad (25)$$

Let  $\hat{V}[\mathcal{E}(x(t+1))]$  be defined as above. Using the above inequality, we have the following:

$$\begin{aligned} V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \\ \leq \hat{V}[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \\ = -L[\mathcal{E}(x(t)), u^*(t)] + L[\mathcal{E}(x^*(t+N)), u(t+N)] \\ + \phi[\mathcal{E}(x(t+N+1))] - \phi[\mathcal{E}(x^*(t+N))] \end{aligned} \quad (26)$$

We can see from Lemma 5 that there exists  $u(t+N)$  such that the following inequality holds.

$$\begin{aligned} \phi[\mathcal{E}(x(t+N+1))] - \phi[\mathcal{E}(x^*(t+N))] \\ \leq -L[\mathcal{E}(x^*(t+N)), u(t+N)] \end{aligned} \quad (27)$$

Applying (27) to (26) yields

$$V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \leq -L[\mathcal{E}(x(t)), u^*(t)]. \quad (28)$$

Here, note that there exists a positive constant  $\nu$  such that the following inequalities hold.

$$\begin{aligned} V[\mathcal{E}(x(t))] &\geq L[\mathcal{E}(x(t)), u^*(t)] \\ &\geq \mathcal{E}(x(t))' Q \mathcal{E}(x(t)) \\ &\geq \nu \|\mathcal{E}(x(t))\| \end{aligned} \quad (29)$$

Therefore, it follows that

$$V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] \leq -\nu \|\mathcal{E}(x(t))\| \quad (30)$$

Consequently, under Assumption 4, we can see that there exist  $\mathbb{K}_\infty$  functions  $\alpha_1$  and  $\alpha_2$  such that the following inequalities are satisfied.

$$\begin{aligned} \alpha_1(\|\mathcal{E}(x(t))\|) &\leq V[\mathcal{E}(x(t))] \leq \alpha_2(\|\mathcal{E}(x(t))\|) \\ V[\mathcal{E}(x(t+1))] - V[\mathcal{E}(x(t))] &\leq -\alpha_1(\|\mathcal{E}(x(t))\|) \end{aligned}$$

Hence, using Lemma 1, we can conclude that  $\mathcal{E}(x(t)) = 0$  is asymptotically stable. This completes the proof. ■

*Remark 1:* From Theorem 1, we can verify the stability of the closed-loop system with the stochastic MPC by checking LMI (11). A brief description of the procedure for solving LMI (11) is provided below.

- (i)  $A$ ,  $B$  and  $\delta$  are given.
- (ii)  $Q$  and  $R$  are arbitrarily chosen.
- (iii) Check LMI (11) using a conventional algorithm [26].
- (iv) If there exist feasible solutions  $Z$  and  $G$ , then go to (v). Otherwise, go back to (ii).
- (v)  $P$  is determined by  $P = Z^{-1}$ . Then, the procedure is terminated.

Following the above procedure, we identify weighting coefficients  $P$ ,  $Q$ ,  $R$  that can guarantee the stability of the closed-loop system with the stochastic MPC.

### VII. ILLUSTRATIVE EXAMPLE

As an illustrative example, we consider here a system whose system coefficients are given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (31)$$

Moreover,  $w(t)$  is set as a uniformly distributed random variable within the interval  $[-5 \ 5]$ . Other parameters employed in the numerical simulations are as follows: The prediction horizon is set as  $N = 3$ .

Here, we consider two cases for the weighting coefficients of the performance index.

In case (I), we consider  $P$ ,  $Q$ ,  $R$  as in (32) that don't satisfy LMI (11), i.e., the asymptotic stability in the mean of the MPC system cannot be guaranteed.

$$P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 70 \quad (32)$$

TABLE I  
CLASSIFICATION OF NUMERICAL SIMULATIONS

	LMI infeasible	LMI feasible
Stochastic MPC	Case (I)	Case (II)

In case (II), we consider  $P$ ,  $Q$ ,  $R$  as in (33) that satisfy LMI (11), i.e., the asymptotic stability in the mean of the MPC system can be guaranteed.

$$P = \begin{bmatrix} 60 & 30 \\ 30 & 100 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 70 \quad (33)$$

Thus, we perform numerical simulations for two cases as shown in Table I. For each case, we perform 100 trials for numerical simulations. Thus, the initial states  $x(0)$  are given by uniformly distributed random variables within the interval  $[5 \ 10]$ .

Figs. 1–3 show that the system in case (I) is not almost surely asymptotically stable in the mean. On the one hand, Figs. 4–6 show that the system in case (II) is almost surely asymptotically stable in the mean. Comparing case (I) with (II), we can see that Theorem 1 is useful to verify the asymptotic stability in the mean for the closed-loop system with MPC. Consequently, we can verify the effectiveness of the proposed condition by numerical simulations.

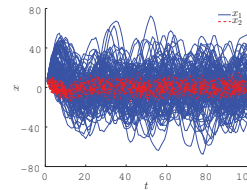


Fig. 1. Time response of  $x(t)$  in case (I)

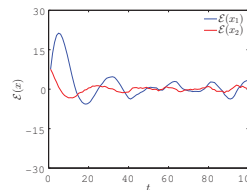


Fig. 2. Time response of  $\mathcal{E}(x(t))$  in case (I)

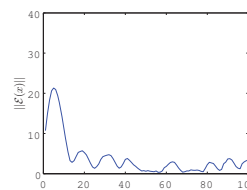
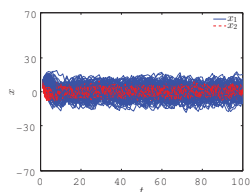
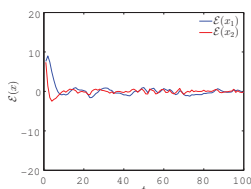
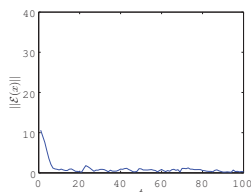


Fig. 3. Time response of  $\|\mathcal{E}(x(t))\|$  in case (I)

Fig. 4. Time response of  $x(t)$  in case (II)Fig. 5. Time response of  $\mathcal{E}(x(t))$  in case (II)Fig. 6. Time response of  $\|\mathcal{E}(x(t))\|$  in case (II)

### VIII. CONCLUSION

In this study, we proposed a design method of model predictive control (MPC) for linear discrete-time systems with additive stochastic disturbances under probabilistic constraints. The stochastic MPC problem was reduced to the quadratic programming that can be solved using a conventional algorithm. Furthermore, we provide a sufficient condition for the stability of the closed-loop system by means of a linear matrix inequality that can be easily verified using a conventional algorithm. The effectiveness of the obtained stability condition was verified by computational simulations.

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