Application of Higher Order Splines for Boundary Value Problems

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Abstract—Bringing forth a survey on recent higher order spline techniques for solving boundary value problems in ordinary differential equations. Here we have discussed the summary of the articles since 2000 till date based on higher order splines like Septic, Octic, Nonic, Tenth, Eleventh, Twelfth and Thirteenth Degree splines. Comparisons of methods with own critical comments as remarks have been included.

Keywords—Septic spline, Octic spline, Nonic spline, Tenth, Eleventh, Twelfth and Thirteenth Degree spline, parametric and non-parametric splines, thermal instability, astrophysics.

I. INTRODUCTION

IN the study of problems arising in astrophysics, problem of heating of infinite horizontal layer of fluid, eigen value problems arising in thermal instability, obstacle, unilateral, moving and free boundary-value problems, problems of the defection of plates and in a number of other scientific applications, we find a system of differential equations of different order with different boundary conditions. In general, it is not possible to obtain the analytical solution of them; we usually resort to some numerical methods for obtaining an approximate solution of these problems. In the present paper, higher order spline techniques for solving boundary value problems in ordinary differential equations are briefly discussed.

We have discussed here several survey papers based on application of Septic, Octic, Nonic, Tenth, Eleventh, Twelfth and Thirteenth Degree spline functions to solve various systems of differential equations. In Section II we have discussed brief definition of spline. In Section III, we consider the papers having septic spline techniques to solve boundary-value problems. In Section IV, we discuss about octic spline techniques to solve these. In Sections V-IX nonic spline, tenth, eleventh, twelfth and thirteenth degree spline technique are discussed respectively. In Section X, the conclusion and further developments are given.

II. SPLINE FUNCTIONS

Usually a spline is a piecewise polynomial function defined in a region, such that there exists a decomposition of the region into sub-regions in each of which the function is a polynomial of some degree d. Also the function, as a rule, is continuous in in the region, together with its derivatives of order up to (d-1).

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We have different types of spline functions such as linear, quadratic, cubic, quartic, quintic, sextic, septic, octic, nonic etc. They are also known as 'polynomial spline' function. To be able to deal effectively with problems under consideration we introduce 'spline functions' containing a parameter. These are 'non-polynomial splines' defined through the solution of a differential equation in each subinterval. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. These 'splines' belong to the class C^∞ and reduce into polynomial splines as parameter [1]-[4], [39]-[42]. The exact form of the spline depends upon the manner in which the parameter is introduced.

III. SEPTIC SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

A septic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most seven.

(ii) The first six derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i) = u(x_i), i = 0(1)N+1$.

In a nonpolynomial septic spline we introduce a parameter k. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. This 'spline' belongs to the class C^2 and reduces into polynomial splines as parameter $k \rightarrow 0$.

A few papers based on Septic spline are as follows:

(1) Considering [5] Having the Sixth-Order Boundary Value Problem of the Type

$$\left.\begin{array}{l} y^{(6)}(x) + f(x)y(x) = g(x), \qquad x \in [a,b], \\ y(a) = \alpha_0, y(b) = \alpha_1, \\ y^{(1)}(a) = \gamma_0, y^{(1)}(b) = \gamma_1, \\ y^{(2)}(a) = \delta_0, y^{(2)}(b) = \delta_1, \end{array}\right\} \quad (1)$$

where α_i, γ_i and $\delta_i, i = 0,1$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$.

In this paper, the authors have applied seventh degree non-polynomial spline functions to derive a numerical method to obtain the solution of such system of sixth-order differential equations. In this paper the consistency relation between the values of spline and its sixth order derivatives at knots is determined using derivative continuities at knots. The error bound of the solution is also discussed. The method discussed here is second order convergent. This paper includes the

comparison of method developed with those discussed by [6] and [7] as well and is observed to be better.

(2) Considering [8] Having the System of Sixth-Order Boundary Value Problem of the Type

$$D^6 y(x) = f(x, y), \qquad a < x < b,\ D \equiv d/dx, \qquad (2)$$

Subject to the boundary conditions

$$\left. \begin{array}{l} y(a) = A_0, D^2 y(a) = A_2, D^4 y(a) = A_4, \\ y(b) = B_0, D^2 y(b) = B_2, D^4 y(b) = B_4 \end{array} \right\} \qquad (3)$$

where $y(x)$ and $f(x, y)$ are continuous functions defined in the interval $[a, b]$. It is considered that $f(x, y) \in C^6[a, b]$ is real and that $A_i, B_i, i = 0, 2, 4,$ are finite real numbers.

In this paper non-polynomial splines, equivalent to seven-degree polynomial splines, based numerical methods are developed for computing approximations to the solution of sixth-order boundary-value problems with two-point boundary conditions. By using standard procedure Second, fourth- and sixth-order convergence is obtained. It is found that the present methods give approximations, which are better than those produced by other spline and domain decomposition methods. A few numerical examples are given to illustrate practical usefulness of the new approach.

The spline functions proposed in this paper can be better explained by:

$$T_7 = \text{span}\left\{1, x, x^2, x^3, x^4, x^5, \sin(kx), \cos(kx)\right\},$$

$$=\text{span}$$

$$\left\{ \begin{array}{l} 1, x, x^2, x^3, x^4, x^5, \dfrac{7!}{k^7}\left(kx - \sin(kx) - \dfrac{(kx)^3}{6} + \dfrac{(kx)^5}{120}\right), \\[2mm] \dfrac{8!}{k^8}\left(\cos(kx) - 1 + \dfrac{(kx)^2}{2} - \dfrac{(kx)^4}{24} + \dfrac{(kx)^6}{720}\right). \end{array} \right\} \qquad (4)$$

In the present paper the special sixth-order, boundary-value problem to be solved is transformed using non-spline polynomial techniques into linear or non-linear algebraic system. The authors have established the convergence of the method. It has been shown that the relative errors in absolute value confirm the theoretical convergence.

(3) Considering [9] Having the System of Sixth-Order Boundary Value Problem of the Type

$$y^{(6)}(x) + f(x)y(x) = g(x), \qquad x \in [a, b], \qquad (5)$$

Subject to the boundary conditions

$$\left. \begin{array}{l} y(a) - A_1 = y^{(2)}(a) - A_2 = y^{(4)}(a) - A_3 = 0 \\ y(b) - B_1 = y^{(2)}(b) - B_2 = y^{(4)}(b) - B_3 = 0 \end{array} \right\} \qquad (6)$$

where $A_i, B_i, i = 1, 2, 3$ are finite real constants. The functions $f(x)$ and $g(x)$ are continuous functions defined in the interval

$[a, b]$. A boundary value problem of this type arises in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars can modeled by sixth-order boundary value problems. Also when an infinite horizontal layer of fluid is heated from below and is subject to rotation, instability occurs. When this instability is like ordinary convection, the differential equation is sixth order [7].

In this paper two new second and fourth order methods based on a septic nonpolynomial spline function for the numerical solution of sixth order two point boundary value problems are discussed. The spline function is used to derive some consistency relations for computing approximations to the solution of this problem. The proposed approach gives better approximations than existing polynomial spline and finite difference methods up to order four and have a lower computational cost. Convergence analysis of these two methods is discussed. To illustrate the practical use of the methods three numerical examples are included. The septic spline used in this paper has a polynomial part and a trigonometric part. A new second-order method is obtained for choices of α, β, γ and ρ to be defined during the development of the method, such that

$$\alpha = 0,\ 1 - 2\beta - \rho = 0, \tfrac{1}{4} - 4\beta - \gamma = 0,$$

In addition a new fourth –order convergent method is obtained for

$$\alpha = 0, \qquad 1 - 2\beta - 2\gamma - \rho = 0, \frac{1}{4} - 4\beta - \gamma = 0$$
$$\frac{105840}{10!} - 16\frac{\beta}{12} - \frac{\gamma}{12} = \frac{0,9516816}{12!} - \frac{64\beta}{360} - \frac{\gamma}{360} = 0$$

These methods are extensions of the solutions of second and fourth-order two-point boundary value problems presented in [10], [11].

Remark: The literature on the numerical solution of sixth-order boundary-value problems is sparse. Such problems are known to arise in astrophysics; the narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth-order boundary-value problems [12]. Also in [13] it is given that dynamo action in some stars may be calculated by such equations. Chandrasekhar [14] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth order decomposition methods to investigate solution of the sixth-order boundary-value problems. Theorems, which list the conditions for the existence and uniqueness of solutions of sixth-order boundary-value problems, are thoroughly discussed in [15]. Non-numerical techniques for solving such problems are contained in [16], [17]. Numerical methods of solution are contained implicitly in [18]. Twizell [19] developed a second-order method for solving special and general sixth-order problems and in later work Twizell and Boutayeb [20] developed finite-difference methods of order two, four, six and eight for solving such problems. Wazwaz [21] used decomposition and modified

domain decomposition methods to investigate solution of the sixth-order boundary-value problems.

IV. OCTIC SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

An octic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most eight.

(ii) The first seven derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i) = u(x_i), i = 0(1)N+1$.

A few papers based on eighth degree spline are as follows

(1) Considering [22] Having the System of Eighth-Order Boundary Value Problem of the Type

$$
\begin{aligned}
&y^{(viii)} + \phi(x)y + \psi(x) \qquad -\infty < a \le x \le b < \infty, \\
&y(a) = A_0, \ y^{(ii)}(a) = A_2, \ y^{(iv)}(a) = A_4, \ y^{(vi)}(a) = A_6, \\
&y(b) = B_0, \ y^{(ii)}(b) = B_2, \ y^{(iv)}(b) = B_4, \ y^{(vi)}(b) = B_6,
\end{aligned}
\tag{7}
$$

where $y = y(x)$ and $\emptyset(x), \varphi(x)$ are continuous functions defined in the interval $[a,b]$ with $A_i, B_i, i = 0, 2, 4,$ are finite real numbers.

In the present paper, octic spline based algorithm is used to solve the system of the type (7). The method discussed in this paper approximates the solutions, and their higher-order derivatives. Numerical illustrations are given to show the usefulness of the algorithm discussed. The algorithm discussed is second-order convergent.

(2) Considering [23] Having the System of Second-Order Boundary Value Problem of the Type

$$y'' = f(x,y,y'), \qquad y(0) = y_0, y'(0) = y'_0$$

where $f(x)$ be a smooth function defined on $[0,1]$ and $f \in C^{n-1}([0,1]x\ R^2), n \ge 2$ and it satisfies the Lipschitz condition.

The authors claimed here that the absolute errors in the solution of the second order initial value problems given by their eighth degree spline based construction are smaller than the errors in the constructions in [24], also we can using this model to find the approximate solution for all order initial value problems with a good result even for small h. Moreover, we found new construction, gives more accurate results in comparison spline used in [24].

V. NONIC SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

A nonic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most nine.

(ii) The first eight derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i) = u(x_i), i = 0(1)N+1$.

Considering [25] having the system of eighth-order boundary value problem of the type

$$
\left.
\begin{aligned}
&y^{(8)}(x) + f(x)y(x) = g(x), \qquad x \in [a,b], \\
&y(a) = \alpha_0, y(b) = \alpha_1, \\
&y^{(1)}(a) = \gamma_0, y^{(1)}(b) = \gamma_1, \\
&y^{(2)}(a) = \delta_0, y^{(2)}(b) = \delta_1, \\
&y^{(3)}(a) = \upsilon_0, y^{(2)}(b) = \upsilon_1,
\end{aligned}
\right\}
\tag{8}
$$

where α_i, y_i, δ_i and $\gamma_i, i = 0,1$ finite real constants and $y(x), g(x)$ are continuous functions defined in the interval $[a,b]$.

In this paper, Nonic spline is used for the numerical solutions of the eighth order linear special case boundary value problem given by (8). The method presented in this paper has been found to be second order convergent. Numerical examples compared with those considered by [22] and [26], and it is found that the algorithm in this paper is more efficient.

Remark: [27] developed finite difference methods for the solution of eighth-order boundary value problems. Twizell et al. [28] developed numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability.

Shahid S. Siddiqi et al. defined end conditions for nonic spline [29] to use the nonic spline not only for interpolation purpose, but for the solutions of eighth order boundary value problems of the type (8), with the end conditions consistent with the boundary value problems. In this paper, nonic spline is defined for interpolation at equally spaced knots along with the end conditions required for the definition of spline. These conditions are in terms of given functional values at the knots and lead to uniform convergence of $O(h^{10})$, throughout the given interval.

To implement the method for the interpolation in this paper, the authors have discussed three examples. As the nonic spline is defined in terms of any four derivatives along with the data to be interpolated, so in these examples the splines are defined in terms of the following forms along with the given data

(i) First, Third, Fifth and Seventh derivatives of the spline

(ii) Second, Fourth, Sixth and Eighth derivatives of the spline

(iv) First, Second Third and Fourth derivatives of the spline

(v) First, Second Third and Fifth derivatives of the spline

VI. TENTH DEGREE SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

A tenth degree spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most ten.

(ii) The first ninth derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i) = u(x_i), i = 0(1)N+1$.

Considering [30] having the system of tenth-order boundary value problem of the type

$$
\left.\begin{array}{l}
y^{(x)}(x)+\phi(x)\,y=\psi(x), \qquad -\infty < a \le x \le b < \infty, \\
y(a)=A_0,\; y^{(ii)}(a)=A_2, \\
y^{(iv)}(a)=A_4,\; y^{(vi)}(a)=A_6, \\
y^{(viii)}(a)=A_8,\; y(b)=B_0,\; y^{(ii)}(b)=B_2, \\
y^{(iv)}(b)=B_4,\; y^{(vi)}(b)=B_6, \\
y^{(viii)}(b)=B_8,
\end{array}\right\} \qquad (9)
$$

where $y=y(x)$ and $\phi(x),\varphi(x)$ are continuous functions defined in the interval $[a,b]$ with $A_i, B_i, i=0,2,4,$ are finite real numbers.

In this paper, linear, tenth-order boundary-value problems are solved, using tenth degree polynomial splines. The algorithm discussed here is found second-order convergent. Numerical illustrations are given to show the practical usefulness of the algorithm developed.

Remark: Numerical results relating to the solution of tenth-order problems are rare in the literature. In this paper problems are discussed to compare the maximum absolute errors with the analytical solutions. Some unexpected results for the higher derivatives were obtained near the boundaries of the given interval. The absolute errors in the function values were, however, very small. The absolute error in the function values and all derivatives were seen to be small at points remote from the boundaries.

VII. ELEVENTH DEGREE SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

An eleventh degree spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1},x_j]$, $S_\Delta(x)$ is a polynomial of degree at most eleven.

(ii) The first tenth derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i)=u(x_i), i=0(1)N+1$.

In a nonpolynomial Eleventh degree spline we introduce a parameter k. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. This 'spline' belongs to the class C^2 and reduces into polynomial splines as parameter $k \to 0$.

A few papers based on eleventh degree spline are as follows

(1) Considering [31] Having the System of Tenth-Order Boundary Value Problem of the Type

$$
\left.\begin{array}{l}
y^{(10)}(x)+f(x)\,y(x)=g(x), \qquad x\in[a,b], \\
y(a)=\alpha_0,\; y(b)=\alpha_1, \\
y^{(1)}(a)=\gamma_0,\; y^{(1)}(b)=\gamma_1, \\
y^{(2)}(a)=\delta_0,\; y^{(2)}(b)=\delta_1, \\
y^{(3)}(a)=\upsilon_0,\; y^{(2)}(b)=\upsilon_1, \\
y^{(4)}(a)=\zeta_0,\; y^{(4)}(b)=\zeta_1,
\end{array}\right\} \qquad (10)
$$

where $\alpha_i, y_i, \delta_i, \gamma_i$ are finite real constants and $y(x), g(x)$ are continuous functions defined in the interval $[a,b]$.

In the present paper eleventh degree spline is used to obtain numerical solutions of the tenth-order linear special case boundary value problems. Comparative study of the errors with those considered by [30] is tabulated and the method is found to be better.

(2) Considering [32] Having the System of Tenth-Order Boundary Value Problem of the Type (10)

In this paper, non-polynomial spline function is used to develop a technique for the solution of 10th-order boundary value problem, extending the method for the solution of eighth-order linear special case boundary value problems, developed by [33]. The method developed is observed to be better than that developed by [30].

In this paper, to develop the spline approximation to the problem (10), the interval $[a,b]$ is divided into k equal subintervals, using the grid points

$$
x_i = a+ih;\, i=0,1,...,k \;\; h=(b-a)/k
$$

Here using the first, third, fifth, seventh and ninth derivative continuities at knots, the consistency relation between the values of spline and its 10th-order derivatives at knots are determined.

Remark: The literature on the numerical solutions of tenth-order boundary value problems and associated eigenvalue problems is seldom. If an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modelled by tenth-order boundary value problem. When instability sets here in as over stability, it is modelled by twelfth-order boundary value problem [14].

(3) Considering [34] Having the System of Tenth-Order Boundary Value Problem of the Type (10)

In this paper, Non-polynomial spline is used for solution of the tenth-order linear boundary value problems. The authors obtained the classes of numerical methods for a specific choice of the parameters involved in non-polynomial spline. The end conditions consistent with the boundary value problems are derived. Truncation errors are given. A new approach convergence analysis of the presented methods is discussed. Two examples are considered for the numerical illustration. However, it is observed that the approach produces better numerical solutions in the sense that max $|e_i|$ is minimum.

VIII. TWELFTH DEGREE SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

A twelfth degree spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1},x_j]$, $S_\Delta(x)$ is a polynomial of degree at most twelve.

(ii) The first eleven derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i) = u(x_i), i = 0(1)N+1$.

Considering the paper [35] having the system of twelfth-order boundary value problem of the type

$$y^{(xii)} + \phi(x)y = \psi(x), \qquad -\infty < a \le x \le b < \infty$$
$$y^{(2k)}(a) = A_{2k}, y^{(2k)}(b) = B_{2k}, \qquad k = 0,1,2,....,5$$

where $y = y(x)$ and $\emptyset(x), \varphi(x)$ are continuous functions defined in the interval $[a,b]$ and $A_i, B_i, i = 0,2,4,6,8,10$ are finite real constants.

In this paper, linear, twelfth-order boundary-value problems are solved, using polynomial splines of degree twelve. Numerical examples are given to show the usefulness of the algorithm developed. The algorithm is found second-order convergent.

Remark: In this paper two problems are discussed to compare the maximum absolute errors with the analytical solutions. Some unexpected results for the higher derivatives were obtained near the boundaries of the given interval. The absolute errors in the function values were, however, very small. The absolute error in the function values and all derivatives were seen to be small at points remote from the boundaries

IX. THIRTEENTH DEGREE SPLINE TECHNIQUE TO SOLVE BOUNDARY VALUE PROBLEMS

A thirteen degree spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most thirteen.

(ii) The first twelve derivatives of $S_\Delta(x)$ are continuous on $[a,b]$

(iii) $S_\Delta(x_i) = u(x_i), i = 0(1)N+1$.

In a nonpolynomial thirteen degree spline we introduce a parameter k. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. This 'spline' belongs to the class C^2 and reduces into polynomial splines as parameter $k \to 0$.

A few papers based on thirteenth degree spline are as follows:

(1) Considering [36] Having the System of Twelfth-Order Boundary Value Problem of the Type

$$\left.\begin{array}{l} y^{(12)}(x) + f(x)y(x) = g(x), \qquad x \in [a,b], \\ y(a) = \alpha_0, y(b) = \alpha_1, \\ y^{(1)}(a) = \gamma_0, y^{(1)}(b) = \gamma_1, \\ y^{(2)}(a) = \delta_0, y^{(2)}(b) = \delta_1, \\ y^{(3)}(a) = \upsilon_0, y^{(2)}(b) = \upsilon_1, \\ y^{(4)}(a) = \zeta_0, y^{(4)}(b) = \zeta_1, \\ y^{(5)}(a) = \omega_0, y^{(5)}(b) = \omega_1, \end{array}\right\} \qquad (11)$$

where

$$\alpha_i, \gamma_i, \delta_i, \upsilon_i, \zeta_i \; \omega_i, i = 0,1$$

are finite real constants and the functions and $y(x), g(x)$ are continuous functions defined in the interval $[a,b]$.

In the present paper thirteen degree spline is used to obtain numerical solutions of the twelfth order linear special case boundary value problems. Numerical examples demonstrate the practical usefulness of the method.

(2) Considering [37] Having the System of Twelfth-Order Boundary Value Problem of the Type (8).

In this paper, non-polynomial spline function is used to develop a technique for the solution of 12th order boundary value problem, extending the method for the solution of 8th order linear special boundary value problems developed by [38]. The spline functions proposed in this paper have the form

$$T_7 = \underset{\text{span}}{} \left\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, \cos(kx), \sin(kx)\right\}$$

where k is the frequency of the trigonometric part of the spline functions.

Scott and Watts [43] developed a numerical method for the solution of linear BVP using a combination of superposition and orthonormalization. In this paper, difficult examples of BVP of orders 2, 4 and 8 are considered. Scott and Watts [44] described several computer codes that were developed using the superposition and orthonormalization technique and invariant imbedding. Watson and Scott [45] proved that Chow-Yorke algorithm was globally convergent for a class of spline collocation approximations to non-linear two point boundary value problems. Several numerical implementations of the algorithm are briefly described, and computational results are presented for a fairly difficult fluid dynamics boundary value problem.

Remark: The literature on the numerical solutions of twelfth order boundary value problems and associated eigenvalue problem is seldom. If an infinite horizontal layer of fluid is heated from below, with the assumption that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modelled by tenth order boundary value problem. When instability sets in as overstability, it is modelled by twelfth order BVP [14].

X. CONCLUSION

In solving problems arising in astrophysics, problem of heating of infinite horizontal layer of fluid, eigenvalue problems arising in thermal instability, obstacle, unilateral, moving and free boundary-value problems, problems of the deflection of plates and in a number of other problems of scientific applications, spline functions are not only more accurate but also we have a variety of choices to use septic, octic, nonic or higher splines to solve them. The survey paper contains a large amount of work done in the area of application of spline functions to solve differential equations and it may provide a better platform to do more work in this

field. Though here we have considered a limited research papers since 2000 till date due to space constraints, yet we see that a standard work is going on in the field concerned and now-a-days several researchers are doing work on spline solution to differential equations.

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