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Closed-Form Solutions for Nanobeams Based On the Nonlocal Euler-Bernoulli Theory

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***Abstract*—**Starting from nonlocal continuum mechanics, a thermodynamically new nonlocal model of Euler-Bernoulli nanobeams is provided. The nonlocal variational formulation is consistently provided and the governing differential equation for transverse displacement is presented. Higher-order boundary conditions are then consistently derived. An example is contributed in order to show the effectiveness of the proposed model.

***Keywords*—**Bernoulli-Euler beams, Nanobeams, nonlocal elasticity.

I. Introduction

NOWADAYS, beam-type structures are more and more used in micro and nanotechnology. They are commonly adopted in various Micro Electro-Mechanical Systems (MEMS) and Nano Electro-Mechanical Systems (NEMS) [1]-[4].

The operating principle of many of such sensors is based on the bending of a nanocantilever or on the three point bending scheme [5]-[10].

It is well-known that the small size of the samples requires high precision in experiments so that tests are problematic to conduct and they often deliver substantial fluctuating measurements [11], [12]. Hence a detailed understanding of analytical models and numerical studies of nanostructures is of paramount importance in projecting nanomechanical structures.

Consequently, in order to effectively implement nanobeams one needs reliable models of beam mechanical behavior. Straightforward utilization of beam models used at macroscale cannot be carried out since small-size effects tend to be important at the nanoscale level [12], [13]. In particular, a local state is influenced by its neighborhood so that nonlocal models have to be used in such beams [14]. Accordingly, several nonlocal beam models have been proposed such as strain gradient theories, modified coupled stress theories and nonlocal elasticity theories [15]-[23]

Many of the studies on micro- and nanobeams are based on the Eringen nonlocal elasticity theory [24]-[26].

In this paper the nonlocal elasticity theory is adopted to propose a new nonlocal model for bending of Euler-Bernoulli nanobeams which provides, as a special case, the Eringen nonlocal model. In the framework of nonlocal thermodynamics [27], the nonlocal model follows by a suitable definition of the Helmolthz free energy.

Such a free energy is given in terms of a combination of the axial strain and of its gradient so that the proposed nonlocal model depends on an internal length scale and on a positive dimensionless participation factor, which can modify the stiffness of the nanobeam.

Nonlocal thermodynamics allows us to build up a reliable methodology to provide nonlocal models. The related variational formulation is obtained. Then the differential equation with the relevant higher-order boundary conditions can be derived in a direct way.

As an example, a nanocantilever is considered in order to investigate the influence of the nonlocal parameters.

II. Bernoulli-Euler Nanobeam Kinematics

In this paper, we pursue the solution due to in-plane bending of nanobeams. The longitudinal axis is denoted by x and the cross section lies in the y-z plane. Bending due to mechanical forces takes place in the x-y plane. The displacement field of the nanobeam is then given by

$$\begin{cases} s_x(x,y,z) = -v^{(1)}(x)y \\ s_y(x,y,z) = v(x) \\ s_z(x,y,z) = 0 \end{cases} \tag{1}$$

and the kinematically axial compatible deformation is

$$\varepsilon(x,y,z) = -v^{(2)}(x)y \tag{2}$$

where v is the transverse displacement along the y-axis and the superscript $\cdot^{(n)}$ denotes the n-derivative along the nanobeam axis x. Note that the nanobeam bending curvature is $\chi = v^{(2)}$.

III. Constitutive Model

To develop a constitutive model, we start from the Helmholtz free energy

$$\psi(\varepsilon, \varepsilon^{(1)}) = \frac{1}{2}E\varepsilon^2 + c^2\left(\frac{1}{2}E\alpha^2\varepsilon^{(1)2} + \frac{q}{A}\chi(\varepsilon)\right) \tag{3}$$

where E is the Young modulus of the material, c is the length-scale parameter, the dimensionless parameter α is a participation factor. The straight nanobeam occupies a domain V of a three-dimensional Euclidean space and A denotes the cross-section area.

The energy balance equation along with the above

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expression of the Helmholtz free energy results in:

$$\int_V \bar{\sigma}\dot{\varepsilon}\, dV = \int_V \left(E\varepsilon + \tfrac{q}{A}c^2\partial_\varepsilon\chi(\varepsilon)\right)\dot{\varepsilon}\, dV + \int_V E\alpha^2c^2\varepsilon^{(1)}\dot{\varepsilon}^{(1)}\, dV \quad (4)$$

where ∂_ε is the derivative with respect to the variable ε and $\bar{\sigma}$ is the nonlocal axial stress.

With the above form of the free energy at hand, the stress and the stress-like variable conjugated to the strain gradient can be calculated as

$$\begin{aligned}\sigma_0 &= E\varepsilon\,,\\ \sigma_1 &= E\varepsilon^{(1)}.\end{aligned} \quad (5)$$

By the application of the Bernoulli-Euler beam kinematic relation reported in (2), the equality (4) can be transformed into

$$\int_V \bar{\sigma}\dot{v}^{(2)}y\, dV = \int_V \left(\sigma_0\dot{v}^{(2)}y + \alpha^2c^2\sigma_1\dot{v}^{(3)}y + \tfrac{q}{A}c^2\dot{v}^{(2)}\right)dV \quad (6)$$

At this point the bending moments are defined as

$$\begin{aligned}M &= -\int_A \bar{\sigma}\, ydA\\ M_0 &= -\int_A \sigma_0\, ydA = -\int_A E\varepsilon\, ydA\\ M_1 &= -\int_A \sigma_1\, ydA = -\int_A E\varepsilon^{(1)}\, ydA\end{aligned} \quad (7)$$

so that the following variational formulation is obtained

$$\int_0^L M\dot{v}^{(2)}dx = \int_0^L M_0\dot{v}^{(2)}dx + \int_0^L c^2q\dot{v}^{(2)}dx + \int_0^L \alpha^2c^2M_1\dot{v}^{(3)}dx \quad (8)$$

where L represents the beam length.

IV. Governing Equations and Boundary Conditions

Integrating by parts the variational formulation above, according to

$$\begin{aligned}\int_0^L(\blacksquare)\dot{v}^{(1)}dx &= [(\blacksquare)\dot{v}]_0^L - \int_0^L(\blacksquare)^{(1)}\dot{v}dx\\ \int_0^L(\blacksquare)\dot{v}^{(2)}dx &= \left[(\blacksquare)\dot{v}^{(1)}\right]_0^L - \left[(\blacksquare)^{(1)}\dot{v}\right]_0^L + \int_0^L(\blacksquare)^{(2)}\dot{v}dx\\ \int_0^L(\blacksquare)\dot{v}^{(3)}dx &= \left[(\blacksquare)\dot{v}^{(2)}\right]_0^L - \left[(\blacksquare)^{(1)}\dot{v}^{(1)}\right]_0^L + \left[(\blacksquare)^{(2)}\dot{v}\right]_0^L - \int_0^L(\blacksquare)^{(3)}\dot{v}dx\end{aligned} \quad (9)$$

the following ordinary differential equation (ODE) for the nonlocal model is provided

$$M^{(2)} - c^2q^{(2)} = M_0^{(2)} - \alpha^2c^2M_1^{(3)}. \quad (10)$$

The boundary conditions at $\{0, L\}$ are reported in Table I.

TABLE I
KINEMATIC AND STATIC BOUNDARY CONDITIONS

Kinematic BCs	Static BCs
v	$-M^{(1)} + c^2q^{(1)} = -M_0^{(1)} + \alpha^2c^2M_1^{(2)}$
$v^{(1)}$	$M - c^2q = M_0 - \alpha^2c^2M_1^{(1)}$
$v^{(2)}$	$0 = \alpha^2c^2M_1$

External forces can be obtained by integrating by parts the left end side of the variational formulation (8) given by $\int_0^L M\dot{v}^{(2)}dx$ to get the classical differential relation $M^{(2)} = q$. The boundary conditions at $\{0, L\}$ are $T = -M^{(1)} = F$ and $M = \mathcal{M}$ where T is the shear force, q is the distributed transverse load and $(F, \mathcal{M})$ are the transverse force and couple respectively.

The bending moments (7) can be expressed in terms of the transverse displacement in

$$\begin{aligned}M_0 &= EIv^{(2)}\\ M_1 &= EIv^{(3)}\end{aligned} \quad (11)$$

so that the final nonlocal differential equation is now obtained from (10) by the application of (11)

$$EI\alpha^2c^2v^{(6)} - EIv^{(4)} = -q + c^2q^{(2)}. \quad (12)$$

The boundary conditions at $\{0, L\}$ for the transverse displacement v are reported in Table II.

TABLE II
KINEMATIC AND STATIC BOUNDARY CONDITIONS IN TERMS OF DISPLACEMENT

Kinematic BCs	Static BCs
v	$-EIv^{(3)} + EI\alpha^2c^2v^{(5)} = T + c^2q^{(1)}$
$v^{(1)}$	$EIv^{(2)} - EI\alpha^2c^2v^{(4)} = M - c^2q$
$v^{(2)}$	$0 = \alpha^2c^2EIv^{(3)}$

If $\alpha = 0$, the proposed coupled model reduce to the Eringen model (EM) [14].

V. Cantilever Beam Loaded by a Linearly Decreasing Distributed Load

In order to show the effectiveness of the proposed model, a nanocantilever with length L is subjected to a linearly decreasing distributed load $q(x) = a(1 - x/L)$ where a is the value of the distributed load at the cantilever cross-section $x = 0$.

A. Closed Form Solution

The differential equation is provided by

$$EI\alpha^2c^2v^{(6)} - EIv^{(4)} = -q \quad (13)$$

and the boundary conditions follow from Table II in

$$\begin{aligned}&v(0) = 0\\ &v^{(1)}(0) = 0\\ &EI\alpha^2c^2v^{(3)}(0) = 0\\ &-EIv^{(3)}(L) + EI\alpha^2c^2v^{(5)}(L) = c^2q^{(1)}(L)\\ &EIv^{(2)}(L) - EI\alpha^2c^2v^{(4)}(L) = -c^2q(L)\\ &EI\alpha^2c^2v^{(3)}(L) = 0.\end{aligned} \quad (14)$$

Solving ODE (13) with the boundary conditions (14), the following displacement is obtained

$$v(x) = v_{\mathcal{L}}(x) - \frac{2c^2 a x^3(-1+\alpha^2)}{12\mathrm{EI}L} - \frac{a c^2 x \alpha^2 (L^2+2c^2(-1+\alpha^2))}{2\mathrm{EI}L} + \frac{6c^2 a x^2(-1+\alpha^2)}{12\mathrm{EI}} + \frac{a c^3 e^{\frac{x}{c\alpha}} \alpha^3 (-L^2+2c^2(-1+e^{\frac{L}{c\alpha}})(-1+\alpha^2))}{2(-1+e^{\frac{2L}{c\alpha}})\mathrm{EI}L} - \frac{a c^3 e^{\frac{L}{c\alpha}-\frac{x}{c\alpha}} \alpha^3 (e^{\frac{L}{c\alpha}} L^2+2c^2(-1+e^{\frac{L}{c\alpha}})(-1+\alpha^2))}{2(-1+e^{\frac{2L}{c\alpha}})\mathrm{EI}L} + \frac{a c^3 \alpha^3 ((1+e^{\frac{2L}{c\alpha}})L^2+2c^2(-1+e^{\frac{L}{c\alpha}})^2(-1+\alpha^2))}{2(-1+e^{\frac{2L}{c\alpha}})\mathrm{EI}L} \quad (15)$$

The local transverse displacement $v_{\mathcal{L}}$ of the classical Euler-Bernoulli beam is

$$v_{\mathcal{L}}(x) = \frac{aL^2x^2}{12\mathrm{EI}} - \frac{aLx^3}{12\mathrm{EI}} + \frac{ax^4}{24\mathrm{EI}} - \frac{ax^5}{120\mathrm{EI}L}. \quad (16)$$

Further, the classical Bernoulli-Euler beam model is recovered for $c = 0$ and the displacement $v_{2\infty}$ for $\alpha \to +\infty$ is provided by $v_{2\infty}(x) = -a(12c^2 - L^2)x^2/48\mathrm{EI}$.

Finally the expression of the bending moment for the considered coupled model can be recovered from the variational formulation and is given by

$$M = M_0 - \alpha^2 c^2 M_1^{(1)} + c^2 q \quad (17)$$

or equivalently in terms of displacements

$$M = EIv^{(2)} - EI\alpha^2 c^2 v^{(4)} + c^2 q. \quad (18)$$

B. Example

We introduce the following dimensionless quantities

$$\xi = \frac{x}{L}, \quad \tau = \frac{c}{L}, \quad v^*(\xi) = v_i(x)\frac{EI}{aL^4}, \quad (19)$$

and the dimensionless length-scale parameter τ assumes the values $\{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$.

Analysis of the results for displacements is plotted in Fig. 1 to show that the nanocantilever becomes stiffer for increasing values of the dimensionless length-scale parameter $\tau \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. The participation factor α ranges in the interval $[0,2]$.

Clearly, different values of nonlocal parameters are the source of different distribution patterns.

The maximum dimensionless deflections of the proposed nonlocal model, of the Eringen model and of the local model are plotted together for comparison. The maximum dimensionless deflection of the local model is $v_{\mathcal{L}}^*(1) = 0.03333$.

For sake of clarity, the maximum dimensionless deflection of the gradient elasticity model is not plotted in Fig. 1. It does not depends on the participation factor and is given by 0.032087 for $\tau = 0.1$, 0.029645 for $\tau = 0.2$, 0.027329 $\tau = 0.3$, 0.025585 for $\tau = 0.4$ and 0.024366 for $\tau = 0.5$.

The maximum dimensionless deflection obtained by the coupled model for a given τ and for a vanishing participation factor, i.e. α=0, coincides to the one evaluated by the Eringen model (EM).

The nonlocal model collapses to the local one (L) for τ=0 and the deflection is evidently independent of α.

Fig. 1 and above values show that, for any value of τ and α, the Eringen model, the gradient elasticity method and the proposed nonlocal model are stiffer than the local model. The deflection of the nanocantilever tip can become negative (i.e. the nanocantilever tip moves in the opposite direction of the applied load) for given values of the parameters α and τ.

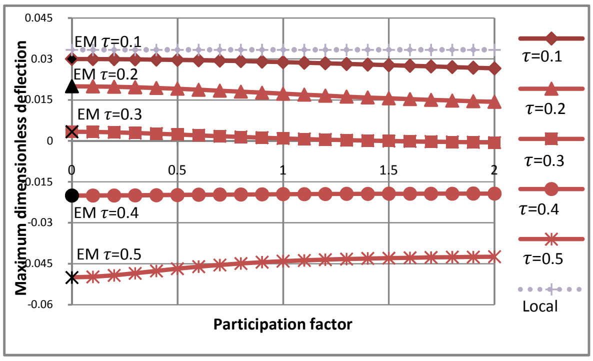

Fig. 1 Maximum dimensionless displacements of a nanocantilever under a linearly decreasing load

VI. Conclusion

The present paper presents a nonlocal model for Bernoulli-Euler nanobeams based on Eringen nonlocal theory. The model derives from the consistent thermodynamic framework thus enabling straightforward extensions to more complicated situations. The effectiveness of the formulation is verified by means of an example.

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