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Prediction of the Torsional Vibration Characteristics of a Rotor-Shaft System Using Its Scale Model and Scaling Laws

Jia-Jang Wu

***Abstract*—**This paper presents the scaling laws that provide the criteria of geometry and dynamic similitude between the full-size rotor-shaft system and its scale model, and can be used to predict the torsional vibration characteristics of the full-size rotor-shaft system by manipulating the corresponding data of its scale model. The scaling factors, which play fundamental roles in predicting the geometry and dynamic relationships between the full-size rotor-shaft system and its scale model, for torsional free vibration problems between scale and full-size rotor-shaft systems are firstly obtained from the equation of motion of torsional free vibration. Then, the scaling factor of external force (i.e., torque) required for the torsional forced vibration problems is determined based on the Newton's second law. Numerical results show that the torsional free and forced vibration characteristics of a full-size rotor-shaft system can be accurately predicted from those of its scale models by using the foregoing scaling factors. For this reason, it is believed that the presented approach will be significant for investigating the relevant phenomenon in the scale model tests.

***Keywords*—**Torsional vibration, full-size model, scale model, scaling laws.

I. Introduction

TORSIONAL vibration problems are important research topic of rotor-shaft systems. Hence, a lot of researchers have studied the relating problems. For example, Koser and Pasin [1] have studied the torsional vibrations of the drive shafts and mechanisms by means of analytical approach. Khulief and Mohiuddin [2] have investigated the torsional dynamic behaviour of a rotor-bearing system using finite element method and modal reduction technique. Yan and Zhang [3] have studied the dynamic problems of multi-spans rotor system, consists of rotors, bearings, oil film, supports, etc., using discrete element method and experiments. Aleyaasin et al. [4] has used the transfer matrix method to perform the flexural vibration analysis of a rotor mounted on fluid film bearings. Brusa et al. [5] have performed the torsional vibration analysis of a crankshaft system. Drew and Stone [6] have measured the torsional vibration characteristics of rotating machines by experiments. Wu and Chen [7] have presented a technique for replacing the gear-branched system with an equivalent straight-geared system, and then used the last model to study the torsional vibration characteristics of gear-branched system with finite element method. Qing and Cheng [8] have studied the coupled torsional and lateral vibrations of rotor-shaft systems using finite element method and [9], [10] have investigated those of unbalanced rotors by means of Lagrangian dynamics. In the foregoing researches, the theoretical torsional vibration analyses of the rotor-shaft systems were performed based on the full-size models of the rotor-shaft systems. However, if the size of the rotor-shaft system is very large and the theoretical analysis results must be validated by experiments, a full-size model test is usually expensive and time-consuming.

In general, the inherent vibration characteristics of a full-size structural system cannot be accurately predicted from the relevant features observed in its scale model if the latter is not properly scaled. Therefore, several researchers have investigated the relating problems. For example, [11] has investigated the physical modelling and similitude of marine structures. Qian et al. [12] have studied the scaling laws for impact damage in fibre composites. Rezaeepazhand and Simitses et al. [13]-[15] have used the similitude theory to establish the similarity conditions between the chosen structural systems, and then the scaling laws are derived and used to predict the vibration responses of the full-size structures from those of their scale models. Wu et al. [16] have derived the scaling laws for the vibration characteristics between a full-size crane structure and its scale model. Later on, [17], [18] have further derived the scaling laws for predicting the dynamic behaviour of a full-size plate subjected to multiple moving loads from those of its scale model. In these researches, the scaling laws were obtained by means of the similitude theory [19] and the dimensional analysis [20].

From the above-mentioned literature, it is seen that the researchers usually tackle the scaling issues by using the similitude theory and the scaling laws. In which, the similitude theory [19] is first employed to establish the similarity conditions between the full-size system and its scale model. Next, the scaling laws are derived based on the last similarity conditions and dimensional analysis theory [20]. Finally, the scaling laws are used to predict the dynamic characteristics of the full-size system from the corresponding ones of its scale model. Since the last approach has not been applied to the prediction of torsional vibration characteristics of rotor-shaft system, the title problem is studied here.

Jia-Jang Wu is with the Department of Marine Engineering, National Kaohsiung Marine University, No. 482, Jhong-Jhou 3rd Road, Cijin District, Kaohsiung City 80543, Taiwan (phone: 886-7-810-0888 ext. 5230; fax: 886-7-572-1035; e-mail: jjangwu@mail.nkmu.edu.tw).

II. SCALING LAWS FOR FREE VIBRATION OF A ROTOR-SHAFT SYSTEM

The equation of motion for a rotor-shaft system free vibrating in its torsional direction takes the form [21]

$$[\bar{J}_\theta^*]\{\ddot{\theta}(t)\}+[\bar{C}_\theta^*]\{\dot{\theta}(t)\}+[\bar{K}_\theta^*]\{\theta(t)\}=\{0\} \qquad (1)$$

where $[\bar{J}_\theta^*]$, $[\bar{C}_\theta^*]$ and $[\bar{K}_\theta^*]$ are respectively the overall mass, damping and stiffness matrices, while $\{\ddot{\theta}(t)\}$, $\{\dot{\theta}(t)\}$ and $\{\theta(t)\}$ are respectively the rotational acceleration, velocity and displacement vectors at any time *t*.

Perform the conventional modal analysis [21] to (1), one obtains

$$J_{ii}\frac{d^2\theta_i}{dt}+2J_{ii}\xi_i\omega_i\frac{d\theta_i}{dt}+J_{ii}\omega_i^2\theta_i=0 \qquad (2)$$

where J_{ii} and ξ_i are, respectively, the generalized mass and damping ratio, whereas $\ddot{\theta}_i$, $\dot{\theta}_i$ and θ_i are, respectively, the generalized angular acceleration, velocity, and displacement, each quantity corresponding to the i^{th} mode with natural frequency ω_i.

For a scaled rotor-shaft system vibrating in its i^{th} mode, (2) can be re-written as

$$J_{iis}\frac{d^2\theta_{is}}{dt_s^2}+2J_{iis}\xi_{is}\omega_{is}\frac{d\theta_{is}}{dt_s}+J_{iis}\omega_{is}^2\theta_{is}=0 \qquad (3)$$

where the subscript *s* denotes the *scaled* rotor-shaft system.

Similarly, for a full-size rotor-shaft system vibrating in its i^{th} mode, its equation of motion can be written by

$$J_{iiF}\frac{d^2\theta_{iF}}{dt_F^2}+2J_{iiF}\xi_{iF}\omega_{iF}\frac{d\theta_{iF}}{dt_F}+J_{iiF}\omega_{iF}^2\theta_{iF}=0 \qquad (4)$$

where the subscript *F* denotes the *full-size* rotor-shaft system.

Under the assumption that the scaling factors (λ_{ix}) are defined as the physical parameters of the scale model divided by the corresponding ones of the full-size model, the variables of (3) and (4) have the relations

$$\lambda_{i\theta}=\theta_{is}/\theta_{iF}\ ,\ \lambda_t=t_s/t_F\ ,\ \lambda_{i\xi}=\xi_{is}/\xi_{iF}\ ,$$
$$\lambda_{iJ}=J_{iis}/J_{iiF}\ ,\ \lambda_{i\omega}=\omega_{is}/\omega_{iF} \qquad (5)$$

Substituting (5) into (3) yields

$$(\lambda_{iJ}J_{iiF})\frac{d^2(\lambda_{i\theta}\theta_{iF})}{d(\lambda_t t_F)^2}+2(\lambda_{iJ}J_{iiF})(\lambda_{i\xi}\xi_{iF})(\lambda_{i\omega}\omega_{iF})\frac{d(\lambda_{i\theta}\theta_{iF})}{d(\lambda_t t_F)}+(\lambda_{iJ}J_{iiF})(\lambda_{i\omega}\omega_{iF})^2(\lambda_{i\theta}\theta_{iF})=0 \qquad (6)$$

Rearranging (6) leads to

$$(\frac{\lambda_{iJ}\lambda_{i\theta}}{\lambda_t^2})J_{iiF}\frac{d^2\theta_{iF}}{dt_F^2}+(\frac{\lambda_{iJ}\lambda_{i\xi}\lambda_{i\omega}\lambda_{i\theta}}{\lambda_t})2J_{iiF}\xi_{iF}\omega_{iF}\frac{d\theta_{iF}}{dt_F}+(\lambda_{iJ}\lambda_{i\omega}^2\lambda_{i\theta})J_{iiF}\omega_{iF}^2\theta_{iF}=0 \qquad (7)$$

Because (7) is obtained according to the equation of motion of the scaled rotor-shaft system and the scaling factors between the scaled and the full-size models, (4) and (7) are equivalent. For this reason, the terms in the parentheses of (7) are equal to each other, thus,

$$\frac{\lambda_{iJ}\lambda_{i\theta}}{\lambda_t^2}=\frac{\lambda_{iJ}\lambda_{i\xi}\lambda_{i\omega}\lambda_{i\theta}}{\lambda_t}=\lambda_{iJ}\lambda_{i\omega}^2\lambda_{i\theta} \qquad (8)$$

The last equation is the requirement for the dynamic similarity between scaled and full-size rotor-shaft system free vibrating in their i^{th} mode. Hence, based on the theory of mode superposition method [21], (8) can be re-written as

$$\frac{\lambda_J\lambda_\theta}{\lambda_t^2}=\frac{\lambda_J\lambda_\xi\lambda_\omega\lambda_\theta}{\lambda_t}=\lambda_J\lambda_\omega^2\lambda_\theta \qquad (9)$$

Equation (9) is the scaling laws between the scaled and full-size rotor-shaft system studied in this paper.

III. SCALING FACTORS FOR FREE TORSIONAL VIBRATION

Since the natural frequency (ω) is the reciprocal of time (*t*), therefore,

$$\lambda_\omega=\frac{1}{\lambda_t} \qquad (10)$$

Introducing (10) to the first two terms of (9), one obtains

$$\lambda_\xi=1 \qquad (11)$$

In such a case, (9) can be simplified as

$$\frac{\lambda_J\lambda_\theta}{\lambda_t^2}=\frac{\lambda_J\lambda_\omega\lambda_\theta}{\lambda_t}=\lambda_J\lambda_\omega^2\lambda_\theta \qquad (12)$$

Since (12) is an *equation of equality*, several sets of λ_θ, λ_ω, λ_J and λ_t may be obtained to satisfy the last equation. However, if the geometry similarity between scaled and full-size rotor-shaft systems is completely achieved, one may easily obtain a set of scaling factors (λ_θ, λ_ω, λ_J and λ_t) according to the fundamental physics concept. If the scaled and full-size rotor-shaft system possess completely geometry similitude, then one may define the scaling factor for length as a constant ratio, for example, λ_η. In such a case, the scaling factors for length (ℓ) and diameter (d) of shaft element, λ_ℓ and λ_d, are given by

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$$\lambda_\ell = \lambda_\eta \tag{13}$$

$$\lambda_d = \lambda_\eta \tag{14}$$

If the scaled and full-size rotor-shaft systems are made of the same material (i.e., the mass density ρ of both system are the same), one has

$$[\bar{J}_\theta^*]_s = \lambda_\eta^5 [\bar{J}_\theta^*]_F \tag{15}$$

where the $[\bar{J}_\theta^*]_s$ and $[\bar{J}_\theta^*]_F$ represent the mass matrices for scaled and full-size models, respectively.

Thus, the scaling factor for mass moment of inertia, λ_J , is given by

$$\lambda_J = \lambda_\eta^5 \tag{16}$$

Because the geometry between the scaled and full-size rotor-shaft systems is assumed to be completely similar in this paper, the scaling factor for angular displacement is taken to be 1, i.e.

$$\lambda_\theta = 1 \tag{17}$$

Substituting (10), (16) and (17) into (12) yields the scaling factor for time (t) and frequency (ω).

$$\lambda_t = \lambda_\eta \tag{18}$$

$$\lambda_\omega = 1/\lambda_\eta \tag{19}$$

Equations (13) and (14) are the scaling factors for complete geometry similarity between the scaled and full-size rotor-shaft systems, while (11) and (16)-(19) are those for complete dynamic similarity between the last structures in free vibration conditions. It is noted that if the scaling factors for dynamic similarity are completely achieved, those for geometry similarity must also be completely achieved. This is because the scaling factors for dynamic similarity are derived based on the requirement of complete geometry similarity, as one may see from (13)-(19).

IV. Scaling Factors for Forced Torsional Vibration

According to [19], if the scaled and the full-size rotor-shaft systems possess geometric, kinematic and dynamic similarity, and the scaling factors of mass, length and time for a free vibration system are λ_J , λ_η and λ_t , then the scaling factor of external force (i.e., torque) for the forced vibration system, λ_T , may be determined from the relation

$$\lambda_T = \lambda_J \lambda_\theta / \lambda_t^2 \tag{20}$$

The last expression is derived from Newton's second law $T(t) = J\ddot{\theta}(t)$ and has been validated by numerical examples [19], where $T(t)$, J and $\ddot{\theta}(t)$ are external force (or torque), mass moment of inertia and angular acceleration of the vibration system, respectively.

Now, the scaling factor of the external torque (λ_T) for the forced vibration system can be obtained by substituting (16)-(18) into (20).

$$\lambda_T = \lambda_\eta^5 / \lambda_\eta^2 = \lambda_\eta^3 \tag{21}$$

V. Scaling Factor for Spring Stiffness of Rotational Spring

Under the assumption that the material of the scaled and full-size models is the same (i.e., the shear modulus G of both models are the same), one obtains the relation

$$[\bar{K}_\theta^*]_s = \lambda_\eta^3 [\bar{K}_\theta^*]_F \tag{22}$$

where the $[\bar{K}_\theta^*]_s$ and $[\bar{K}_\theta^*]_F$ represent the stiffness matrices for scaled and full-size models, respectively.

Thus, the scaling factor for the spring stiffness of rotational spring is given by

$$\lambda_{K_\theta} = \lambda_{i\eta}^3 \tag{23}$$

VI. Scaling Factor for Damping Coefficient of Rotational Damper

The relation between the force (i.e., torque) and rotational damper is given by

$$T_c = C_r \dot{\theta}(t) \tag{24}$$

where T_c is the force (i.e., torque) due to rotational damper, C_r is the damping coefficient of the rotational damper, while $\dot{\theta}(t)$ is the angular velocity.

Therefore, the scaling factor for damping coefficient of rotational damper can be determined from the relation

$$\lambda_c = \frac{\lambda_T}{(\lambda_\theta / \lambda_t)} \tag{25}$$

Substituting (17), (18) and (20) into (25), one obtains

$$\lambda_c = \lambda_\eta^4 \tag{26}$$

VII. Numerical Results and Discussions

A. Validation of Scaling Factors for Torsional Free Vibration

In this subsection, a full-size clamped-free shaft carrying a tip disk and its scale model (c.f. Fig. 1) are studied. Where the physical properties for the full-size rotor-shaft system are:

shear modulus G_F =8.01× 10^{10} N/m^2, mass density ρ_F =7820 kg/m^3, diameter of shaft d_F = 0.115 m, polar moment of inertia of shaft cross-section area $I_{pF} = \pi d_F^4/32$ =1.716 × 10^{-5} m^4, mass moment of inertia per unit length $J_{pF} = \rho_F I_{pF} \times 1$ = 1.342 × 10^{-1} $kg{\cdot}m$, length of the shaft L_F =6.0 m, and mass moment of inertia of the disk $J_{DF} = J_{pF} \times L_F$ =8.052× 10^{-1} $kg{\cdot}m^2$. On the other hand, those for the 1/5 scale shaft are: shear modulus G_s =8.01× 10^{10} N/m^2, mass density ρ_s =7820 kg/m^3, diameter of shaft d_s = 0.023 m, polar moment of inertia of shaft cross-section area $I_{ps} = \pi d_s^4/32$ =2.746 × 10^{-8} m^4, mass moment of inertia per unit length $J_{ps} = \rho_s I_{ps} \times 1$ = 2.147× 10^{-4} $kg{\cdot}m$, length of the shaft L_s =1.2 m, and mass moment of inertia of the disk $J_{Ds} = J_{ps} \times L_s$ =2.577× 10^{-4} $kg{\cdot}m^2$. It is worthy of mention that the subscripts s and F respectively represent the scale and full-size rotor-shaft system. Besides, either the full-size rotor-shaft system or its scale mode is subdivided into 20 identical shaft elements.

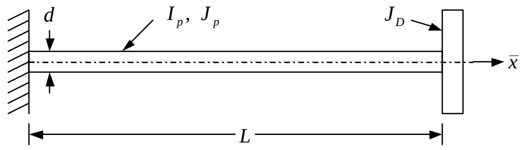

Fig. 1 A clamped-free shaft carrying a tip disk with mass moment of inertia $J_D = J_p L$

For convenience, the scale shaft and its full-size model are respectively called *C-shaft1* and *C-shaft5* hereafter. If length is the key parameter for the scaling between *C-shaft1* and *C-shaft5*, then the scaling factor for length should be $\frac{1}{5}$, i.e.,

$$\lambda_\eta = \tfrac{1}{5} \tag{27}$$

Substituting (27) into (11), (13), (14) and (16)-(19), the scaling factors relating to the dynamic similarity are determined.

$$\begin{aligned} &\lambda_\ell = \tfrac{1}{5}, \lambda_d = \tfrac{1}{5}, \lambda_t = \tfrac{1}{5}, \lambda_\omega = 5, \\ &\lambda_\xi = 1, \lambda_J = \tfrac{1}{3125}, \lambda_\theta = 1, \lambda_T = \tfrac{1}{125} \end{aligned} \tag{28}$$

From the last equation, one sees that if the total length and diameter of *C-shaft1* is $\frac{1}{5}$ of *C-shaft5* (i.e., $\lambda_\ell = \lambda_d = \frac{1}{5}$), then the scaling factors for time (λ_t), natural frequency (λ_ω), damping ratio (λ_ξ), mass moment of inertia (λ_J), angular displacement (λ_θ) and torque (λ_T) are equal to $\frac{1}{5}$, 5, 1, $\frac{1}{3125}$, 1 and $\frac{1}{125}$, respectively. In other words, although *C-shaft1* and *C-shaft5* are made of the same material and the scaling factor for the length (λ_η) is equal to $\frac{1}{5}$, the values of most the other scaling factors (such as λ_t, λ_ω, λ_ξ, λ_J, λ_θ and λ_T) are quite different from $\frac{1}{5}$. This is because, in addition to the conditions for the geometric similarity required by the static problem, the conditions for the kinematic and dynamic similarity required by the dynamic problem must also be satisfied.

From the descriptions of the physical properties for *C-shaft1* and *C-shaft5*, one sees that the scaling factor for length (λ_ℓ), diameter (λ_d) and mass moment of inertia (λ_J) are respectively

$$\lambda_\ell = L_s/L_F = \tfrac{1}{5},\ \lambda_d = d_s/d_F = \tfrac{1}{5},\ \lambda_J = J_{Ds}/J_{DF} \approx (\tfrac{1}{5})^5 \tag{29}$$

Table I lists the first five natural frequencies of the full-size rotor-shaft system, ω_{iF} (i =1 to 5), and those of its 1/5 scale model, ω_{is} (i =1 to 5). From the final column of the table, one sees that the scaling factor for natural frequencies ($\lambda_\omega = \omega_s/\omega_F$) is very close to 5, i.e.

$$\lambda_\omega = 5.01 \approx 5 \tag{30}$$

Since (29) and (30) agree with the scaling factors for length, diameter, mass moment of inertia and natural frequency, given by (28), it is believed that the presented scaling laws and scaling factors is viable for the torsional free vibration of the rotor-shaft system.

TABLE I
FIRST FIVE NATURAL FREQUENCIES ω_i (i =1 TO 5) OF THE FULL-SIZE AND SCALE CLAMPED-FREE SHAFT CARRYING A TIP DISK AND ITS SCALE MODEL

Natural frequencies, ω_i (rad/s)	Scale model, ω_{is}	Full-size model, ω_{iF}	Scaling factor, $\lambda_{i\omega} = \omega_{is}/\omega_{iF}$
ω_1	2294.6067	458.9056	5.00
ω_2	9146.4821	1829.1090	5.00
ω_3	17240.7555	3447.7575	5.00
ω_4	25653.0886	5130.0180	5.00
ω_5	34285.4273	6856.2765	5.00
Average	-------	-------	5.00

B. Validation of Scaling Factors for Torsional Forced Vibration

The scaling factors for torsional free vibrations are validated in the last subsection. This section will validate the scaling factors for torque, damping ratio and time by using the forced torsional vibration characteristics of preceding two rotor-shaft systems with their tip disks subjected to a torque, rotates about $\bar{x}$ axis, respectively. The values of torque, damping ratio and time for the last two rotor-shaft systems are listed in Table II. For simplicity, the initial conditions for both the full-size rotor-shaft system and its scale model are assumed to be "at rest" in this paper.

Figs. 2 (a) and (b) show the time histories for angular displacements (i.e. torsional angles) of tip disk of the *C-shaft1* and *C-shaft5*, $\theta_s^{(1)}(t)$ and $\theta_F^{(5)}(t)$. From the figures, one sees

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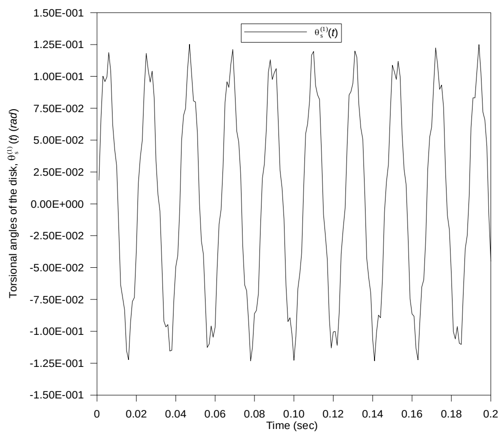
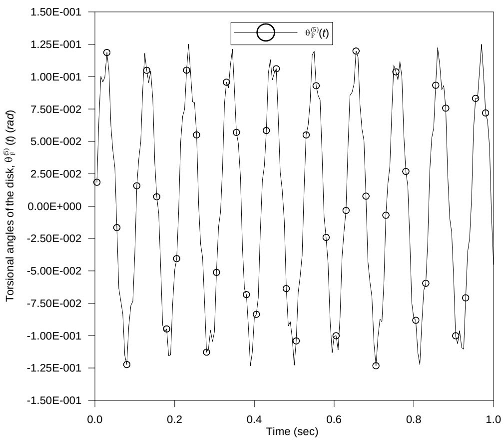
that the time histories for the angular displacements of tip disk of the scale rotor-shaft system, $\theta_s^{(1)}(t)$, are exactly similar to the corresponding ones of its full-size model, $\theta_F^{(5)}(t)$; however, the scale ratios of Figs. 2 (a) to (b) are 1 to 5 and 1 to 1, respectively, for the time axis (abscissa) and for angular displacement (i.e., torsional angle) axis (ordinate), i.e., $\lambda_t = \frac{1}{5}$ and $\lambda_\theta = 1$. Therefore, the scaling factor for time and angular displacement are respectively, $\lambda_t = \frac{1}{5}$ and $\lambda_\theta = 1$. This agrees with (28).

From the given data and numerical results shown in Table I and Fig. 2, it can be shown that all values of the scaling factors that provide dynamic similarities in (28) are satisfied. Because the scaling factors given by (28) do provide satisfactory dynamic similarity between the full-size rotor-shaft system and its scale model, the scaling factors given by (10), (21) and (13)-(19) should be viable for the rotor-shaft system studied in this paper.



(a)



(b)

Fig. 2 Torsional angles for the disk of (a) *C-shaft1*, $\theta_s^{(1)}(t)$ and (b) *C-shaft5*, $\theta_F^{(5)}(t)$

TABLE II
VALUES OF TORQUE, DAMPING RATIO AND TIME FOR FULL-SIZE CLAMPED-FREE ROTOR-SHAFT SYSTEM AND ITS SCALE MODEL

Parameters	Scale rotor-shaft system (*C-shaft1*)	Scaling factors	Full-size rotor-shaft system (*C-shaft5*)
Torque (*Nm*)	$T_s = T_0 \sin(\Omega t)$ =200sin(300*t*)	$\lambda_T = \frac{1}{125}$ $\lambda_\Omega = \lambda_\omega = 5$	$T_F = T_s/\lambda_T =$ $(T_0/\lambda_T)\sin((\Omega/\lambda_\Omega)t)$ =25000sin(60t)
Time (*s*)	$t_s = 0.2$	$\lambda_t = \frac{1}{5}$	$t_F = t_s/\lambda_t = 1.0$
Damping ratio	$\xi_s = 0.001$	$\lambda_\xi = 1$	$\xi_F = \xi_s/\lambda_\xi = 0.001$

VIII. CONCLUSIONS

The scaling laws and scaling factors for free and forced torsional vibration of rotor-shaft systems are presented in this paper. Based on the last scaling laws and scaling factors, one may successfully predict the dynamic characteristics of a full-size rotor-shaft system based on the relevant phenomenon of its scale model. From the numerical examples, one may conclude that if the total length and diameter of the scale shaft is $\frac{1}{5}$ of its full-size model (i.e., $\lambda_\ell = \lambda_d = \frac{1}{5}$), then the scaling factors for time (λ_t), natural frequency (λ_ω), damping ratio (λ_ξ), mass moment of inertia (λ_J), angular displacement (λ_θ) and torque (λ_T) are equal to $\frac{1}{5}$, 5, 1, $\frac{1}{3125}$, 1 and $\frac{1}{125}$, respectively. In other words, although the scale and the full-size rotor-shaft systems are made of the same material and the scaling factor for the length (λ_η) is equal to $\frac{1}{5}$, the values of most the other scaling factors (such as λ_t, λ_ω, λ_ξ, λ_J, λ_θ and λ_T) are quite different from $\frac{1}{5}$. This is because, in addition to the conditions for the geometric similarity required by the static problem, the conditions for the kinematic and dynamic similarity required by the dynamic problem must also be satisfied.

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