

Some New Bounds for a Real Power of the Normalized Laplacian Eigenvalues

Ayşe Dilek Maden

$$K'_f(G) = \sum_{i-j} d_i d_j r_{ij}$$

Abstract—For a given a simple connected graph, we present some new bounds via a new approach for a special topological index given by the sum of the real number power of the non-zero normalized Laplacian eigenvalues. To use this approach presents an advantage not only to derive old and new bounds on this topic but also gives an idea how some previous results in similar area can be developed.

Keywords—Degree Kirchhoff index, normalized Laplacian eigenvalue, spanning tree.

I. INTRODUCTION

THROUGHOUT this paper G will denote a simple connected graph with n vertices (labelled by v_1, v_2, \dots, v_n) and m edges. Moreover, for $1 \leq i \leq n$, the degree of each vertex v_i will be denoted by d_i .

Among various indices in mathematical chemistry, the Kirchhoff index $K_f(G)$ and a relative of it, the close degree Kirchhoff index $K'_f(G)$, have received a great deal of attention, recently. For a connected undirected graph G , the Kirchhoff index was defined by Klein and Randić ([16]) as

$$K_f(G) = \sum_{i < j} r_{ij},$$

where r_{ij} is the effective resistance of the edge $v_i v_j$. We refer the reader to [1], [16], [17], [21], and their bibliographies, to get a taste of the variety of approaches used to study this descriptor. In [28], Zhou et al. studied the extremal graphs with given matching number, connectivity and the minimal Kirchhoff index. Also in [23], [25] and [26] the authors determined independently the extremality on the unicyclic graphs with respect to the Kirchhoff index. Moreover, in [27], Zhou et al. presented some lower bounds for the Kirchhoff index of a connected (molecular) graph via the number of vertices (atoms), the number of edges (bands), valency (maximum vertex degree), connectivity and chromatic number.

The degree Kirchhoff index was proposed by Chen and Zhang in [7], defined as

The degree Kirchhoff index has been taken attention as much as the Kirchhoff index. For instance, in [12], the authors have been recently characterized unicyclic graphs having maximum, second-maximum, minimum and second-minimum degree Kirchhoff index. One can depict [7] for some bounds over the degree Kirchhoff index and for some relations between degree Kirchhoff and Kirchhoff indices. We finally refer [19] for further studies over degree Kirchhoff index.

For the adjacency matrix $A(G)$ and the diagonal matrix $D(G)$ of the vertex degrees of G , let us consider the Laplacian matrix $L(G) = D(G) - A(G)$. It is known that the eigenvalues of $L(G)$ are named as the Laplacian eigenvalues of G . Suppose $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$ are the Laplacian eigenvalues of G . By [13], we know that the multiplicity of $\mu_n = 0$ is equal to the number of connected components of G . We refer [6], [18] for more and some other details on Laplacian eigenvalues. We just want to remind the expression of Kirchhoff index in terms of the Laplacian eigenvalues (see [15], [22], [29]) as in the equality

$$K_f(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}. \quad (1)$$

Other than Laplacian matrix, there also exists the normalized Laplacian matrix $l(G) = D(G)^{-\frac{1}{2}} L(G) D(G)^{-\frac{1}{2}}$ of G , where $D(G)^{-\frac{1}{2}}$ is the matrix obtained by taking $\left(-\frac{1}{2}\right)$ -power of the each entry of $D(G)$. Similarly as Laplacian eigenvalues, the normalized Laplacian eigenvalues of G are the eigenvalues of $l(G)$. So let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$ be the normalized Laplacian eigenvalues matrix of G . By [8], the multiplicity of $\lambda_n = 0$ is actually equal to the number of connected components of G . We may refer [8], [11] for whole detailed information on normalized Laplacian eigenvalues. In [7], by considering normalized Laplacian eigenvalues, the degree Kirchhoff index is defined as

$$K'_f(G) = 2m \sum_{i=1}^{n-1} \frac{1}{\lambda_i}. \quad (2)$$

Ayşe Dilek Maden is with the Department of Mathematics, Science Faculty, University of Selçuk, Campus, 42075, Konya, Turkey (phone: +90 55529931340; fax: +90 3322412499; e-mail: aysedilekmeden@selcuk.edu.tr).

The author is also partially supported by Tubitak and supported by the Scientific Research Project Office (BAP) of Selçuk University.

Hence, by taking into account (1) and (2), we can easily conclude that the degree Kirchhoff index is the normalized Laplacian analogue of the ordinary Kirchhoff index.

This latter expression was the source of inspiration for a whole new family of descriptors, in terms of the sum of the α -th powers of normalized Laplacian eigenvalues as in the form

$$s'_\alpha = s'_\alpha(G) = \sum_{i=1}^h \lambda_i^\alpha, \tag{3}$$

defined in [5]. These authors found a number of bounds for arbitrary α and particularly for $\alpha = -1$, which is the case of the degree Kirchhoff index. We note that $\alpha = 1$ implies the trivial case $s'_1 = n$, and for $\alpha = 2$, we obtain

$$s'_2 = \text{trace}(l^2). \tag{4}$$

There exists a closed relation between s'_α and the general Randic index of G defined by

$$R_\alpha = R_\alpha(G) = \sum_{i-j} (d_i d_j)^\alpha,$$

where the summation is over all (unordered) edges $v_i v_j$ in G and $\alpha \neq 0$ is a fixed real number (see [4]). By (4), it is shown that

$$s'_2 = n + 2 \sum_{i-j} \frac{1}{d_i d_j} = n + 2R_{-1}$$

(cf. [30]). We may refer [2], [9], [24] for the detailed knowledge of the parameter R_{-1} and the useage of this into the normalized Laplacian eigenvalues.

For a non-zero real number α , one can think about the sum of the α -th powers of non-zero Laplacian eigenvalues. In fact this sum has been defined by Zhou in [27] as in the form

$$s_\alpha(G) = \sum_{i=1}^h \mu_i^\alpha,$$

where h is the number of non-zero Laplacian eigenvalues of G .

In this paper, for the graph G , we present some lower and upper bounds on $s'_\alpha(G)$ (where $\alpha \neq 0, 1$) in terms of mainly n, m, t (the number of spanning trees), Δ (see Lemma 1) and R_{-1} .

II. PRELIMINARY RESULTS

We seperate this section to express some assistant results which will be needed to construct our main theories.

Lemma 1 ([10]): The number of spanning trees of G is given by

$$t = \frac{\Delta}{2m} \prod_{i=1}^{n-1} \lambda_i,$$

where $\Delta = \prod_{i=1}^n d_i$.

Lemma 2 ([8]): Suppose that the normalized Laplacian eigenvalues of G are given by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$. Then

$$\lambda_1 \geq \frac{n}{n-1}. \tag{5}$$

Moreover the equality holds in (5) if and only if $G \cong K_n$.

Under the same assumptions on G as in Lemma 2, Chung also presented the following lemma about the normalized Laplacian eigenvalues.

Lemma 3 ([8]): Let the normalized Laplacian eigenvalues of G be given as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$. Then

$$0 \leq \lambda_1 \leq 2.$$

Moreover $\lambda_1 = 2$ if and only if G has a connected bipartite and nontrivial component.

Lemma 4 ([11]): Let us consider again the normalized Laplacian eigenvalues in Lemma 2, and let

$$P = 1 + \sqrt{\frac{2}{n(n-1)} R_{-1}}.$$

We then have

$$\lambda_1 \geq P. \tag{6}$$

Moreover the equality holds in (6) if and only if $G \cong K_n$.

We note that Lemma 4 implies the lower bound expressed in (6) is always better than the bound in (5).

Again, by according to the [11], we have the following two lemmas.

Lemma 5([11]): For a connected graph G of order $n > 2$, it is true that $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.

Lemma 6([11]): Let G be a graph of order n without isolated vertices. Then $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$ if and only if $G \cong K_n$.

Let a_1, a_2, \dots, a_r be positive real numbers. For a positive number k among the values $1 \leq k \leq r$, let us suppose that each P_k is defined as in the following:

$$\begin{aligned} P_1 &= \frac{a_1 + a_2 + \dots + a_r}{r}, \\ P_2 &= \frac{a_1 a_2 + a_1 a_3 + \dots + a_1 a_r + a_2 a_3 + \dots + a_{r-1} a_r}{\frac{1}{2} r(r-1)}, \\ &\vdots \\ P_{r-1} &= \frac{a_1 a_2 \dots a_{r-1} + a_1 a_2 \dots a_{r-2} a_r + \dots + a_2 a_3 \dots a_{r-1} a_r}{r}, \\ P_r &= a_1 a_2 \dots a_r. \end{aligned}$$

Hence the *arithmetic mean* is simply P_1 while the *geometric mean* is $P_r^{1/r}$. In fact the following famous lemma (see [3],[14],[20]) gives a relationship among them.

Lemma 7: (Maclaurin's Symmetric Mean Inequality) For $a_1, a_2, \dots, a_r \in \mathbb{R}^+$, it is true that

$$P_1 \geq P_2^{1/2} \geq P_3^{1/3} \geq \dots \geq P_r^{1/r}.$$

Equality among them holds if and only if $a_1 = a_2 = \dots = a_r$.

We purpose to obtain some better bounds by using this fruitful inequality (in Lemma 7) technique on this new family of descriptors (given before this lemma).

After all above material, we are ready to present our results on the bounds of the sum of the α -th power of normalized Laplacian eigenvalues $s'_\alpha(G)$ as defined in (3).

III. MAIN RESULTS

We recall that

$$R_{-1}(G) = \sum_{i-j} \frac{1}{d_i d_j}.$$

The first result of this paper is the following.

Theorem 1: Let α be a real number with $\alpha \neq 0, 1$, and let G be a connected graph with $n \geq 3$ vertices, m edges and having t spanning trees. Thus we have a lower bound

$$s'_\alpha(G) \geq P^\alpha + (n-2) \left(\frac{2mt}{\Delta P} \right)^{\alpha/(n-2)}, \quad (7)$$

where P is defined as in Lemma 4. Moreover equality in (7) holds if and only if $G \cong K_n$.

Proof: By Lemma 1, we have

$$\frac{\Delta \lambda_1^{n-1}}{2mt} = \prod_{i=2}^{n-1} \frac{\lambda_1}{\lambda_i} \quad \text{as } \lambda_1 \geq \lambda_i, i = 2, 3, \dots, n-1,$$

that is,

$$\lambda_1^{n-1} \geq \frac{2mt}{\Delta}. \quad (8)$$

Setting $r = n-2$ and $a_i = \lambda_i^{-\alpha}$ in Lemma 7, we obtain

$$P_{n-3}^{1/(n-3)} \geq P_{n-2}^{1/(n-2)}$$

such that $P_{n-2} = \prod_{j=2}^{n-1} \lambda_j^{-\alpha}$ and

$$P_{n-3} = \frac{\sum_{i=2}^{n-1} \sum_{j=2, j \neq n-i+1}^{n-1} \lambda_j^{-\alpha}}{n-2} = \frac{\prod_{j=2}^{n-1} \lambda_j^{-\alpha}}{n-2} \times \sum_{i=2}^{n-1} \frac{1}{\lambda_j^{-\alpha}} = \left(\frac{\Delta \lambda_1}{2mt} \right)^\alpha (s'_\alpha - \lambda_1^\alpha).$$

From this, we then get

$$\left(\frac{\Delta \lambda_1}{2mt} \right)^\alpha (s'_\alpha - \lambda_1^\alpha) \geq \left(\frac{\Delta \lambda_1}{2mt} \right)^\alpha.$$

In other words,

$$s'_\alpha \geq \lambda_1^\alpha + (n-2) \left(\frac{2mt}{\Delta \lambda_1} \right)^{\alpha/(n-2)}.$$

Let us now consider a function

$$g(x) = x^\alpha + (n-2) \left(\frac{2mt}{\Delta x} \right)^{\alpha/(n-2)}$$

such that $x \geq P$ and $x^{n-1} \geq \frac{2mt}{\Delta}$. It is clear that

$$g'(x) = \alpha \left[x^{\alpha-1} - x^{-\frac{\alpha+n-2}{n-2}} \left(\frac{2mt}{\Delta} \right)^{\alpha/(n-2)} \right] \geq \alpha \left[x^{\alpha-1} - x^{-\frac{\alpha+n-2}{n-2}} x^{\frac{\alpha(n-1)}{n-2}} \right] = 0 \quad \text{as } x^{n-1} \geq \frac{2mt}{\Delta},$$

and so $g(x)$ is an increasing function on $x \geq P$ and $x^{n-1} \geq \frac{2mt}{\Delta}$.

Hence we have

$$g(x) \geq P^\alpha + (n-2) \left(\frac{2mt}{\Delta} \right)^{\alpha/(n-2)}$$

which gives the required lower bound in (7).

Now let us suppose that the equalities in both sides of (7) hold. Then all inequalities in the above processes must become equalities. The lower bound equality will be implied that $\lambda_1 = P$ and $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$ by Lemma 7. In addition, by Lemmas 4 and 5, we have $G \cong K_n$, as required. The converse part is quite clear.

For $i = 1, 2, 3, \dots, n-1$, by taking $a_i = \lambda_i$ in Lemma 7, and using similar technique as in the proof of Theorem 1, we obtain the following result.

Theorem 2: Let G be a connected graph with $n \geq 3$ vertices, m edges and t spanning trees. Also let P be assumed as in Theorem 1. Thus we have a lower bound

$$K'_f(G) \geq \frac{2m}{P} + 2(n-2)m \left(\frac{\Delta P}{2mt} \right)^{1/(n-2)}$$

with equality if and only if $G \cong K_n$.

The next two corollaries are the consequences of Theorem 1 and 2.

Corollary 1: Let T be a tree of order n . Then

$$s'_\alpha(T) \geq P^\alpha + (n-2) \left(\frac{2m}{\Delta P} \right)^{\alpha/(n-2)}$$

and

$$K'_f(T) \geq \frac{2m}{P} + 2(n-2)m \left(\frac{\Delta P}{2m} \right)^{1/(n-2)}$$

Proof: Since T is a tree, it is clear that $t=1$. Thus, from Theorems 1 and 2, we get the result.

Corollary 2: Let U be a connected unicyclic graph of order n . Then

$$s'_\alpha(U) \geq P^\alpha + (n-2) \left(\frac{6m}{\Delta P} \right)^{\alpha/(n-2)}$$

and

$$K'_f(U) \geq \frac{2m}{P} + 2(n-2)m \left(\frac{\Delta P}{2mn} \right)^{1/(n-2)}$$

Equalities hold if and only if $U \cong K_3$.

Proof: For any unicyclic graph U of order n , we certainly have $3 \leq t \leq n$. Again by Theorems 1 and 2, we obtain the required lower bounds. On the other hand, the same theorems imply the necessary and sufficient equality condition on these lower bounds.

Remark 1: Let us point out that in Theorems 1 and 2, we recover the same bounds as in Theorem 1 and Corollary 2 in the paper [5], through a different approach. We actually improve them in the next theorem (see Theorem 3).

Theorem 3: Let G be a connected graph with $n \geq 3$ vertices, m edges and t spanning trees. Hence we have the lower and upper bounds

$$\left. \begin{aligned} s'_\alpha(G) &\geq (n-1) \left[\left(\frac{2mt}{\Delta} \right)^\alpha \frac{(s'_{-\alpha})^2 - s'_{-2\alpha}}{(n-1)(n-2)} \right]^{1/(n-3)} \\ \text{and} \\ s'_\alpha(G) &\leq \sqrt{s'_{2\alpha} + (n-1)(n-2) \left(\frac{2mt}{\Delta} \right)^\alpha \left(\frac{s'_{-\alpha}}{n-1} \right)^{n-3}} \end{aligned} \right\} \quad (9)$$

respectively.

Moreover, equality holds if and only if $G \cong K_n$.

Proof: Setting $r=n-1$ and $a_i = \lambda_i^\alpha$ (for $i=1,2,\dots,n-1$) in Lemma 7, we obtain

$$P_1 \geq P_{n-3}^{1/(n-3)},$$

where $P_1 = \frac{\sum_{i=1}^{n-1} \lambda_i^\alpha}{n-1}$, and also

$$P_{n-3} = \frac{\prod_{i=1}^{n-1} \lambda_i^\alpha}{(n-1)(n-2)} \left\{ \left(\sum_{i=1}^{n-1} \frac{1}{\lambda_i^\alpha} \right)^2 - \sum_{i=1}^{n-1} \frac{1}{\lambda_i^{2\alpha}} \right\}.$$

We hence obtain

$$s'_\alpha(G) \geq (n-1) \left[\left(\frac{2mt}{\Delta} \right)^\alpha \frac{(s'_{-\alpha})^2 - s'_{-2\alpha}}{(n-1)(n-2)} \right]^{1/(n-3)}.$$

On the other hand, by taking $r=n-1$ and $a_i = \lambda_i^{-\alpha}$ (for $i=1,2,\dots,n-1$) in Lemma 7, we get

$$s'_\alpha(G) \leq \sqrt{s'_{2\alpha} + (n-1)(n-2) \left(\frac{2mt}{\Delta} \right)^\alpha \left(\frac{s'_{-\alpha}}{n-1} \right)^{n-3}}.$$

The equality holds in (9) if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$ from Lemma 7. Also, by Lemma 6, $G \cong K_n$. Conversely the equality follows easily.

Remark 2: Although we managed to see that the equalities in (9) always hold on special examples (see Example 1), it is still remained to see it in the general case.

Using similar arguments as in Theorem 3, one can see the truthness of the following result for $K'_f(G)$.

Theorem 4: Under the same assumptions with Theorem 3, we have

$$\left. \begin{aligned} K'_f(G) &\geq 2m(n-1) \left[\left(\frac{2mt}{\Delta} \right)^\alpha \frac{n^2 - n - 2R_1}{(n-1)(n-2)} \right]^{n-3} \\ \text{and} \\ K'_f(G) &\leq 2m \sqrt{s'_{2\alpha} + (n-1)(n-2) \left(\frac{2mt}{\Delta} \right)^\alpha \left(\frac{n}{n-1} \right)^{n-3}} \end{aligned} \right\} \quad (10)$$

respectively, with equality holding if and only if $G \cong K_n$.

Notice that bounds $K'_f(G) = 2ms'_{-1}$ can be also easily derived by taking into account the bounds for s'_α with $\alpha = -1$.

Remark 3: As a first consequence of Theorems 3 and 4, we can easily express the results on a tree T and an unicyclic graph U .

Remark 4: Note that if G is a k -regular graph, then $\lambda_i = \frac{\mu_i}{k}$ for $i=1,2,\dots,n$ (see [8]). Hence we have $s_\alpha = k^\alpha s'_\alpha$ for any k -regular graph. Therefore, in the case of G is regular, results obtained for s'_α can be immediately re-stated for s_α .

In the following, we give a lower and an upper bound over s'_α for connected bipartite graphs.

Theorem 5: Let G be a connected bipartite graphs with $n > 2$ vertices, m edges and t spanning trees. Then

$$s'_\alpha \geq 2^\alpha + \left(\frac{mt}{\Delta}\right)^{\alpha/(n-2)} (n-2)$$

and

$$s'_\alpha \leq 2^\alpha + \left(s'_{-\alpha} - \frac{1}{2^\alpha}\right)^{n-3} \frac{1}{(n-2)^{n-4}} \left(\frac{mt}{\Delta}\right)^\alpha$$

with equality if and only if $G \cong K_{p,q}$.

Proof:

Lower Bound: Now, in Lemma 7, let us take $r = n - 2$ for both cases $a_i = \lambda_i^\alpha$ and $a_i = \lambda_i^{-\alpha}$ such that $i = 1, 2, \dots, n - 1$, respectively. Therefore we write

$$P_1 \geq P_{n-2}^{1/(n-2)} \text{ for } a_i = \lambda_i^\alpha$$

and

$$P_1 \geq P_{n-3}^{1/(n-3)} \text{ for } a_i = \lambda_i^{-\alpha},$$

where

$$P_1 = \frac{\sum_{i=2}^{n-1} \lambda_i^\alpha}{n-2} = \frac{s'_\alpha - \lambda_1^\alpha}{n-2},$$

$$P_{n-2} = \prod_{i=2}^{n-1} \lambda_i^\alpha = \left(\frac{2mt}{\Delta \lambda_1}\right)^\alpha$$

and

$$P_{n-3} = \frac{\prod_{j=2}^{n-1} \lambda_j^{-\alpha}}{n-2} \sum_{i=2}^{n-1} \lambda_i^\alpha = \left(\frac{\Delta \lambda_1}{2mt}\right)^\alpha \frac{s'_\alpha - \lambda_1^\alpha}{n-2}.$$

Since G is connected bipartite graph, we have $\lambda_1 = 2$ and hence the result follows. All equalities hold if and only if $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$.

Now we suppose that all equalities hold. Then, by Lemma 5, we conclude that $G \cong K_{p,q}$.

Conversely, we can easily see that the equalities hold for the complete bipartite graph $K_{p,q}$.

As a consequence of Theorem 5, we obtain the following corollary.

Corollary 3: There exist the lower and upper bounds

$$m + 2m(n-2) \left(\frac{\Delta}{mt}\right)^{1/(n-2)} \leq K'_f(G) \leq m + \frac{2\Delta}{t(n-2)}$$

with equality if and only if $G \cong K_{p,q}$.

Example 1: For the complete bipartite graph $K_{1,3}$, the normalized Laplacian spectrum is $\{0, 1, 1, 2\}$. For $\alpha = 2$, while the lower bound in (7) gives $s'_2 \geq 5.98$, the both lower and upper bounds in (9) gives a unique value $s'_2 = 6$. Even this example itself enough to show that the bounds obtained in

Theorem 3 would be better than the bounds in Theorem 1 and so [5].

REFERENCES

- [1] D. Banchev, A.T. Balaban, X. Liu, D.J. Klein, "Molecular cyclicity and centrality of polycyclic graphs. I. Cyclicity based on resistance distances or reciprocal distances", *Int. J. Quantum Chem.*, vol. 50, pp. 2978-2981, 1994.
- [2] M. Bianchi, A. Cornaro, J.L. Palacios, A. Torriero, "Bounding the sum of powers of normalized Laplacian eigenvalues of graphs through majorization methods", *MATCH Commun. Math. Comput. Chem.*, vol. 70, pp. 707-716, 2013.
- [3] P. Biler, A. Witkowski, *Problems in Mathematical Analysis*, New York, 1990.
- [4] B. Bollobas, P. Erdős, "Graphs of extremal weights", *ArsCombin.*, vol. 50, pp. 225-233, 1998.
- [5] S.B. Bozkurt, D. Bozkurt, "On the sum of powers of normalized Laplacian eigenvalues of graphs", *MATCH Commun. Math. Comput. Chem.*, vol. 68, pp. 917-930, 2012.
- [6] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, California, 1990.
- [7] H. Chen, F. Zhang, "Resistance distance and the normalized Laplacian Spectrum", *Discrete App. Math.*, vol. 155, pp. 654-661, 2007.
- [8] F.R.K. Chung, *Spectral Graph Theory*, CBMS Lecture Notes, Providence, 1997.
- [9] M. Covers, S. Fallat, S. Kirkland, "On the normalized Laplacian energy and the general Randic index of graphs", *Lin. Algebra Appl.*, vol. 433, pp. 172-190, 2010.
- [10] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.
- [11] K.Ch. Das, A.D. Maden (Güngör), S.B. Bozkurt, "On the normalized Laplacian eigenvalues of graphs", *ArsCombin.*, to be published.
- [12] L. Feng, I. Gutman, G. Yu, "Degree Kirchhoff index of unicyclic graphs", *MATCH Commun. Math. Comput. Chem.*, vol. 69, no.3, pp. 629-643, 2013.
- [13] M. Fiedler, "Algebraic connectivity of graphs", *Czech. Math. J.*, vol. 23, pp. 298-305, 1973.
- [14] S. Furuichi, "On refined Young inequalities and reverse inequalities", vol. 5, no. 1, pp. 21-31, 2011.
- [15] I. Gutman, B. Mohar, "The quasi-Wiener and the Kirchhoff indices Coincide", *J. Chem. Inf. Comput. Sci.*, vol. 36, pp. 982-985, 1996.
- [16] D.J. Klein, M. Randic, "Resistance distance. Applied graph theory and discrete mathematics in chemistry (Saskatoon, SK, 1991)", *J. Math. Chem.*, vol. 12, no. 1-4, pp. 81-95, 1993.
- [17] D.J. Klein, "Graph geometry, graph metrics & Wiener Fifty years of the Wiener index", *MATCH Commun. Math. Comput. Chem.*, vol. 35, pp. 7-27, 1997.
- [18] R. Merris, "Laplacian matrices of graphs. A Survey", *Lin. Algebra Appl.* vol. 197, pp. 143-176, 1994.
- [19] J. Palacios, J.M. Renom, "Another look at the degree Kirchhoff index", *Int. J. Quantum Chem.*, vol. 111, pp. 3453-3455, 2011.
- [20] S. Rosset, "Normalized symmetric functions", *Newton's inequalities and a new set of stronger inequalities*, vol. 96, no. 9, pp. 815-819, 1989.
- [21] W. Xiao, I. Gutman, "On resistance matrices", *MATCH Commun. Math. Comput. Chem.*, vol. 49, pp. 67-81, 2003.
- [22] W. Xiao, I. Gutman, "Resistance distance and Laplacian spectrum", *Theor. Chem. ACC.*, vol. 110, pp. 284-289, 2003.
- [23] Y.J. Yang, X.Y. Jiang, "Unicyclic graphs with extremal Kirchhoff index", *MATCH Commun. Math. Comput. Chem.*, vol. 60, pp. 107-120, 2008.
- [24] G. Yu, L. Feng, "Randić index and eigenvalues of graphs", *Rocky Mount. J. Math.*, vol. 40, pp. 713-721, 2010.
- [25] W. Zhang, H. Deng, "The second maximal and minimal Kirchhoff indices of unicyclic graphs", *MATCH Commun. Math. Comput. Chem.*, vol. 61, pp. 683-695, 2009.
- [26] B. Zhou, N. Trinajstić, "A note on Kirchhoff index", *Chem. Phys. Lett.*, vol. 455, pp. 120-123, 2008.
- [27] B. Zhou, "On sum of powers of the Laplacian eigenvalues of graphs", *Lin. Algebra Appl.*, vol. 429, pp. 2239-2246, 2008.
- [28] B. Zhou, N. Trinajstić, "The Kirchhoff index and the matching number", *Int. J. Quantum Chem.*, vol. 109, pp. 2978-2981, 2009.

- [29] B. Zhou, N. Trinajstić, "On resistance distance and Kirchhoff index", *J. Math. Chem.*, vol. 46, pp. 283-289, 2009.
- [30] P. Zumstein, Comparison of spectral methods through the adjacency matrix and the Laplacian of a graph, Diploma Thesis, ETH Zürich, 2005.



Ayşe Dilek Maden was born on 02 October, 1975 in Antakya, Turkey. The author went to Department of Mathematics of Selcuk University, Science Faculty (Konya, Turkey) as an undergraduate student and again at same university completed her MsC in 1999 and her PhD in 2004. Her PhD thesis is on matrix theory. Her academic degree changed as Associate Professor Doctor in 2010. Now she is still working with same academic degree in the same university.

In addition the author became membership of the Investigation Ph.D. Thesis Development Council in 2008 and Associated Director of Selcuk University, Research Center and Applied Mathematics in 2009. The author's major fields of study are linear algebra, matrix theory and graph theory (spectral graph theory).

Moreover she became referee for some important journals such as Hacettepe Journal of Mathematics & Statistics, Applied Mathematics and Computation, Journal of Inequalities and Applications.

Her some papers are in the following:

- [1] A . D. Maden (Güngör), K. Ch. Das,, "Improved upper and lower bounds for the spectral radius of digraphs", *Applied Mathematics and Computation*, vol. 216,pp. 791-799, 2010.
- [2] A. D. Maden (Güngör), A. S. Cevik, "On the Harary energy and Harary Estrada index of a graph" , *MATCH – Communications in Mathematical and in Computer Chemistry*, vol. 64, no. 1, pp. 281-296, 2010.
- [3] A. D. Maden (Güngör), A. S. Cevik, "A generalization for the clique and independence numbers", *ELA – The Electronic Journal of Linear Algebra*, vol.23, pp. 164-170,2012.
- [4] A. D. Maden, I. Gutman, A. S. Cevik, "Bounds for Resistance-Distance Spectral Radius", *Hacettepe Journal of Mathematics and Statistics*, vol. 42, no. 1, pp. 43-50, 2013.