# The Bent and Hyper-Bent Properties of a Class of Boolean Functions 

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#### Abstract

This paper considers the bent and hyper-bent properties of a class of Boolean functions. For one case, we present a detailed description for them to be hyper-bent functions, and give a necessary condition for them to be bent functions for another case.


Keywords-Boolean functions, bent functions, hyper-bent functions, character sums.

## I. Introduction

BENT function is a class of Boolean functions with even variables and with the maximal distance to all affine functions. In fact, the distance of an $n$-variable bent function to any affine function equals $2^{n-1}-2^{\frac{n}{2}-1}$. Bent function was introduction by Rothaus [9] in 1976, later in 2001 Youssef et al [10] found a subclass of bent functions with even better cryptographic properties, which was named as hyper-bent functions. Thanks to their applications in cryptography, coding theory and combinatorial design, many interests have been put in bent and hyper-bent functions recently[2], [3], [4], [6], [7], [8].

In this paper, we consider a class of Boolean functions defined on $\mathbb{F}_{2^{n}}$ of the form:

$$
\begin{equation*}
f_{a, b}^{(r)}(x):=\operatorname{Tr}_{1}^{n}\left(a x^{r\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{4}\left(b x^{\frac{2^{n}-1}{5}}\right) \tag{1}
\end{equation*}
$$

where $n=2 m, m \equiv 2 k(\bmod 4), k \in\{0,1\}, a \in \mathbb{F}_{2^{n}}$ and $b \in \mathbb{F}_{16}$. When $m=2(\bmod 4)$, with the help of the factorization of $x^{5}+x+a^{-1}$ and Kloosterman sums, this paper characterizes the cases for $f_{a, b}^{(r)}$ to be hyper-bent. Further more , for $a \in \mathbb{F}_{2 \frac{m}{2}}$, we list all the hyper-bent functions of the form of $f_{a, b}^{(r)}$. When $m=0(\bmod 4)$, we give a necessary condition for $f_{a, b}^{(r)}$ to be bent.

The rest of paper is organized as follows. In Section II, we give some notations and recall some basic knowledge for this paper. Then we describe the hyper-bent properties of $f_{a, b}^{(r)}$ when $m \equiv 2(\bmod 4)$ and study the bent properties of $f_{a, b}^{(r)}$ when $m \equiv 0(\bmod 4)$ in Section III and Section IV respectively. Finally, we conclude our work in Section V.

## II. Preliminaries

The sign function of Boolean function $f$ is $\chi(f):=(-1)^{f}$. Definition 1: A Boolean function $f: \mathbb{F}_{2^{n}} \rightarrow \mathbb{F}_{2}$ is called a bent function, if $\widehat{\chi}_{f}(w)=\sum_{x \in \mathbb{F}_{2^{n}}}(-1)^{f(x)+\operatorname{Tr}_{1}^{n}(w x)}=$
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$\pm 2^{\frac{n}{2}}\left(\forall w \in \mathbb{F}_{2^{n}}\right)$, where $\operatorname{Tr}_{1}^{n}$ is the absolute trace function defined as $\operatorname{Tr}_{1}^{n}(x):=x+x^{2}+x^{2^{2}}+\cdots+x^{2^{n-1}}$.
Hyper-bent function is an important subclass of bent functions defined as

Definition 2: A bent function $f: \mathbb{F}_{2^{n}} \rightarrow \mathbb{F}_{2}$ is called a hyper-bent function, if, for any $i$ satisfying $\left(i, 2^{n}-1\right)=1$, $f\left(x^{i}\right)$ is also a bent function.

Charpin and Gong [4] gave the following property to determine a hyper-bent function.
Proposition 1: Let $n=2 m, \alpha$ be a primitive element of $\mathbb{F}_{2^{n}}$ and $f$ be a Boolean function over $\mathbb{F}_{2^{n}}$ satisfying $f\left(\alpha^{2^{m+1}} x\right)=f(x)\left(\forall x \in \mathbb{F}_{2^{n}}\right)$ and $f(0)=0$. Let $\xi$ be a primitive $2^{m}+1$-th root in $\mathbb{F}_{2^{n}}^{*}$. Then $f$ is a hyper-bent function if and only if the cardinality of the set $\left\{i \mid f\left(\xi^{i}\right)=\right.$ $\left.1,0 \leq i \leq 2^{m}\right\}$ is $2^{m-1}$.

Kloosterman sum is a powerful tool to study the hyper-bent properties of some classes of boolean functions.

Kloosterman sums on $\mathbb{F}_{2^{n}}$ are defined as

$$
K_{m}(a):=\sum_{x \in \mathbb{F}_{2^{m}}} \chi\left(\operatorname{Tr}_{1}^{m}\left(a x+\frac{1}{x}\right)\right), \quad a \in \mathbb{F}_{2^{m}}
$$

Some properties of Kloosterman sums are given by the following proposition.

Proposition 2: ([5],Theorem 3.4]) Let $a \in \mathbb{F}_{2^{m}}$. Then $K_{m}(a) \in\left[1-2^{(m+2) / 2}, 1+2^{(m+2) / 2}\right]$ and $4 \mid K_{m}(a)$.
Quintic Weil sums on $\mathbb{F}_{2^{m}}$ are

$$
Q_{m}(a):=\sum_{x \in \mathbb{F}_{2^{m}}} \chi\left(\operatorname{Tr}_{1}^{m}\left(a\left(x^{5}+x^{3}+x\right)\right)\right), \quad a \in \mathbb{F}_{2^{m}}
$$

And the value of $Q_{m}(a)$ is related to the factorization of the polynomial $P(x)=x^{5}+x+a^{-1}$ [1].

When $a \in \mathbb{F}_{2}^{* m_{1}}, m=2 m_{1}, K_{m}(a)$ and $Q_{m}(a)$ have the following properties
Proposition 3: (Lemma 3 [1]) If $a \in \mathbb{F}_{2}^{*} m_{1}, m=2 m_{1}$,
(1) $1-K_{m}(a)=\left(1-K_{m_{1}}(a)\right)^{2}-2 \cdot 2^{m_{1}}$.
(1) if $m_{1} \equiv 1(\bmod 2)$, then $Q_{m}(a) \in\left\{0,2 \cdot 2^{m / 2},-4\right.$. $\left.2^{m / 2}\right\}$.

Proposition 4: [11] The Ramanujan-Nagell equation $x^{2}-$ $D=2^{n+2}$ has at most 4 solutions $(x, n)$, which are
$(x, n):=\left(2^{k}-3,1\right),\left(2^{k}-1, k\right),\left(2^{k}+1, k+1\right),\left(3 \cdot 2^{k}-1,2 k+1\right)$,
where $k \in \mathbb{N}$ and $D \in \mathbb{N}$ is odd.
With the help of the solutions of Ramanujan-Nagell equation,
Lemma 1: If $a \in \mathbb{F}_{2^{m_{1}}}, m=2 m_{1}, m_{1}>1$, then $K_{m}(a) \neq$ -4 .

Proof: By Propostion 3, if $K_{m}(a)=-4$,

$$
\begin{equation*}
\left(1-K_{m_{1}}(a)\right)^{2}=2 \cdot 2^{m_{1}}+5 \tag{2}
\end{equation*}
$$

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It is easy to check that when $m_{1}<5,2 \cdot 2^{m_{1}}+5$ is not a square. By Propostion 4, (2) has at most 4 solutions $\left(\left|\left(1-K_{m_{1}}(a)\right)\right|\right.$ ,$n$ ), which are
$\left(\left|\left(1-K_{m_{1}}(a)\right)\right|, m_{1}-1\right)=$
$\left(2^{k}-3,1\right),\left(2^{k}-1, k\right),\left(2^{k}+1, k+1\right),\left(3 \cdot 2^{k}-1,2 k+1\right)$,
where $k \in \mathbb{N}$. We can check all the 4 solutions can not satisfy (2). For example, if $\left(\left|\left(1-K_{m_{1}}(a)\right)\right|, m_{1}-1\right)=\left(3 \cdot 2^{k}-\right.$ $1,2 k+1$ ), then

$$
\begin{equation*}
\left(3 \cdot 2^{k}-1\right)^{2}=2^{2 k+1+2}+5 . \tag{3}
\end{equation*}
$$

When $k=1,2,\left(3 \cdot 2^{k}-1\right)^{2} \neq 2^{2 k+1+2}+5$. When $k \geq 3$, $\left(3 \cdot 2^{k}-1\right)^{2}>2^{2 k+1+2}+5$. Thus (3) has no integral solution, therefore (2) has no integral solution either, which concludes the proof.
III. The hyper-bent property of $f_{a, b}^{(r)}$ When $m=2$ $(\bmod 4)$
In the this section, we consider the Boolean function $f_{a, b}^{(r)}$ defined by (1), where $n=2 m, m \equiv 2(\bmod 4), a \in \mathbb{F}_{2^{n}}$ and $b \in \mathbb{F}_{16}$. As the cyclotomic coset of 2 module $2^{n}-1$ containing $\frac{2^{n}-1}{5}$ is

$$
\left\{\frac{2^{n}-1}{5}, 2 \cdot \frac{2^{n}-1}{5}, 2^{2} \cdot \frac{2^{n}-1}{5}, 2^{3} \cdot \frac{2^{n}-1}{5}\right\} .
$$

Its size is 4 , or $o\left(\frac{2^{n}-1}{5}\right)=4$, which means $f_{a, b}^{(r)}$ is neither in the class considered by Charpin and Gong [4] nor in the class studied by Mesanager [6], [7].
Let $\alpha$ be a primitive element of $\mathbb{F}_{2^{n}}, \beta=\alpha^{\frac{2^{n}-1}{5}}, \xi=$ $\alpha^{2^{m}-1}, U=<\xi>, V=<\xi^{5}>$. Since $5 \mid\left(2^{m}+1\right), V$ is the subgroup of $U$ and $\# V=\frac{2^{m}+1}{5}$.
For any $i \in \mathbb{F}_{2^{m}}$, define

$$
\begin{aligned}
S_{i} & =\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{i\left(2^{m}-1\right)} v\right)\right) \\
& =\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{-2 i} v\right)\right)=\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{-5 i+3 i} v\right)\right) \\
& =\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{3 i} v\right)\right) . \quad\left(\text { as } \xi^{-5 i} \in V\right)
\end{aligned}
$$

From the definition of $S_{i}$,

$$
\begin{equation*}
S_{i}=S_{i} \quad(\bmod 5) . \tag{4}
\end{equation*}
$$

To study the hyper-bent properties of $f_{a, b}^{(r)}$, we define the following character sum

$$
\begin{equation*}
\Lambda_{r}(a, b):=\sum_{u \in U} \chi\left(f_{a, b}^{(r)}(u)\right) . \tag{5}
\end{equation*}
$$

Similar to the proof of Proposition 9 in [1], the hyper-bent properties of $f_{a, b}^{(r)}$ can be described as

Proposition 5: $f_{a, b}^{(r)}$ is a hyper-bent function if and only if $\Lambda_{r}(a, b)=1$.

Before our work on $f_{a, b}^{(r)}$, let us consider a general case of $f_{a, b}^{(r)}$ which is defined as

$$
\begin{equation*}
f_{a, b}^{(r, k)}:=\operatorname{Tr}_{1}^{n}\left(a x^{r\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{4}\left(b x^{k^{2^{n}-1}} 5\right), \tag{6}
\end{equation*}
$$

where $a, b$ is defined as above and $k \in \mathbb{N}$.
When $k \equiv 0(\bmod 5), f_{a, b}^{(r, k)}=\operatorname{Tr}_{1}^{n}\left(a x^{r\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{4}(b)$ is a special case studied by Charpin and Gong in [4]. In this paper we only consider the case of $k \not \equiv 0(\bmod 5)$.
Proposition 6: The hyper-bent properties of $f_{a, b}^{(r, k)}$ can be represented by that of $f_{a, b}^{(r)}$ efficiently, where $a \in \mathbb{F}_{2^{n}}, b \in \mathbb{F}_{16}$, $k \not \equiv 0(\bmod 5)$.

Proof: For $b \in \mathbb{F}_{16}^{*}, b$ can be written as $b=\omega \beta^{j}$, where $\omega^{3}=1,0 \leq j \leq 4$. Thus

$$
\operatorname{Tr}_{1}^{4}\left(b x^{\frac{2^{n}-1}{5}}\right)=\operatorname{Tr}_{1}^{4}\left(\omega \beta^{j} x^{k \frac{2^{n}-1}{5}}\right)=\operatorname{Tr}_{1}^{4}\left(\omega\left(\beta^{\frac{j}{k}} x^{\frac{2^{n}-1}{5}}\right)^{k}\right) .
$$

It is easy to check,

$$
\begin{aligned}
\operatorname{Tr}_{1}^{4}\left(\omega x^{\frac{2^{n}-1}{5}}\right) & =\operatorname{Tr}_{1}^{4}\left(\omega^{2} x^{2^{\frac{2^{n}-1}{5}}}\right) \\
& =\operatorname{Tr}_{1}^{4}\left(\omega x^{4 \frac{2^{n}-1}{5}}\right)=\operatorname{Tr}_{1}^{4}\left(\omega^{2} x^{3 \frac{2^{n}-1}{5}}\right) .
\end{aligned}
$$

Then $\operatorname{Tr}_{1}^{4}\left(b x^{k \frac{2^{n}-1}{5}}\right)=\operatorname{Tr}_{1}^{4}\left(b^{\prime} x^{\frac{2^{n}-1}{5}}\right)$, where $b^{\prime} \in \mathbb{F}_{16}^{*}$.
Hence the result stands.
A step further, $f_{a, b}^{(r)}$ has following proposition.
Proposition 7: Let $f_{a, b}^{(r)}$ be defined as (1) and $(r, 5)=1$, then $f_{a, b}^{(r)}$ is a hyper-bent function if and only if $f_{a^{\prime}, b^{\prime}}^{(r)}$ is a hyper-bent one, where $a=a^{\prime} \xi^{i} \in \mathbb{F}_{2^{n}}, a^{\prime} \in \mathbb{F}_{2^{m}}, b, b^{\prime}=$ $b \alpha^{-\frac{i}{r} \frac{2^{n}-1}{5}} \in \mathbb{F}_{16}$.

Proof: Notice that $\forall a \in \mathbb{F}_{2^{n}}, a=a^{\prime} \xi^{i}$, where $a^{\prime} \in \mathbb{F}_{2^{m}}$, $\xi=\alpha^{2^{m}-1}$ is a primitive $2^{m}+1$-th root of unity in $\mathbb{F}_{2^{n}}$ and $0 \leq i \leq 2^{m}$. We have

$$
\begin{aligned}
f_{a, b}^{(r)}(x) & =\operatorname{Tr}_{1}^{n}\left(a x^{r\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{4}\left(b x^{\frac{2^{n}-1}{5}}\right) \\
& =\operatorname{Tr}_{1}^{n}\left(a^{\prime}\left(\alpha^{\frac{i}{r}} x\right)^{r\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{4}\left(b \alpha^{-\frac{i}{r} \frac{2^{n}-1}{5}}\left(\alpha^{\frac{i}{r}} x\right)^{\frac{2^{n}-1}{5}}\right) \\
& =f_{a^{\prime}, b^{\prime}}^{(r)}\left(\alpha^{-\frac{i}{r}} x\right),
\end{aligned}
$$

where $b^{\prime}=b \alpha^{-\frac{i}{r} \frac{2^{n}-1}{5}} \in \mathbb{F}_{16}$.
Thus $f_{a, b}^{(r)}$ is linearly equivalent to $f_{a^{\prime}, b^{\prime}}^{(r)}$, that is to say, $f_{a, b}^{(r)}$ is a hyper-bent function if and only if $f_{a^{\prime}, b^{\prime}}^{(r)}$ is a hyper-bent one.

By Proposition 7, if $a=a^{\prime} \xi^{i}$, and $\beta=\alpha^{\frac{2^{n}-1}{5}}$, we have the following results

- $f_{a, b}^{(1)}$ is linearly equivalent to $f_{a^{\prime} b \beta^{4 i}}^{(1)}$.
- $f_{a, b}^{(2)}$ is linearly equivalent to $f_{a^{\prime}}^{(2)}{ }^{2}{ }^{22}$.
- $f_{a, b}^{(3)}$ is linearly equivalent to $f_{a^{\prime} b \beta^{3 i}}^{(3)}$.
- $f_{a, b}^{(4)}$ is linearly equivalent to $f_{a^{\prime}, b \beta^{i}}^{(4)}$.

By Proposition 7 and Proposition 6, when $a \in \mathbb{F}_{2^{n}}, k \in$ $\mathbb{N}, b \in \mathbb{F}_{16}$, the hyper-bent properties of $f_{a, b}^{(r, k)}$ can be fully represented by that of $f_{a, b}^{(r)}$, where $a \in \mathbb{F}_{2^{m}}, b \in \mathbb{F}_{16}$. Since the hyper-bent properties of $f_{a, b}^{(1)}$ had been studied elaborately in [1], in the following parts of this Section we only consider the rest cases of $r$.
A. The Case of $r=5$

1) The hyper-bent properties of $f_{a, b}^{(5)}$, where $a \in \mathbb{F}_{2^{m}}$ :

Proposition 8: Let $n=2 m$ and $m \equiv \pm 2, \pm 6(\bmod 20)$,
If $b \in\{0\} \bigcup\left\{\beta^{i} \mid i=0,1,2,3,4\right\}$, then the Boolean function

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$f_{a, b}^{(5)}$ is not a hyper-bent function. Further, if $b \in \mathbb{F}_{16}^{*} \backslash\left\{\beta^{i} \mid 0 \leq\right.$ $i \leq 4\}, f_{a, b}^{(5)}$ is a hyper-bent function if and only if

$$
\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)=1
$$

Proof: By (5),

$$
\begin{aligned}
\Lambda_{5}(a, b) & =\sum_{u \in U} \chi\left(f_{a, b}^{(5)}(u)\right) \\
& =\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a u^{5\left(2^{m}-1\right)}\right)\right) \chi\left(\operatorname{Tr}_{1}^{4}\left(b u^{\frac{2^{n}-1}{5}}\right)\right)
\end{aligned}
$$

Notice that $U=<\xi>, V=<\xi^{5}>$ and $U=$ $\xi^{0} V \bigcup \xi^{1} V \bigcup \xi^{2} V \bigcup \xi^{3} V \bigcup \xi^{4} V$. Then,

$$
\begin{align*}
& \Lambda_{5}(a, b)=  \tag{7}\\
& \sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(\xi^{i} v\right)^{\frac{2^{n}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a\left(\xi^{i} v\right)^{5\left(2^{m}-1\right)}\right)\right) \\
= & \sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(\xi^{i} v\right)^{\frac{2^{n}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a\left(\xi^{5 i}\right)^{2^{m}-1} v^{5\left(2^{m}-1\right)}\right)\right) \tag{8}
\end{align*}
$$

Since $\left(\xi^{5 i}\right)^{2^{m}-1} \in V$ and $m \equiv \pm 2, \pm 6(\bmod 20),\left(5\left(2^{m}-\right.\right.$ $1), \# V)=\left(5, \frac{2^{m}+1}{5}\right)=1$. Then $v \longmapsto\left(\xi^{5 i}\right)^{2^{m}-1} v^{5\left(2^{m}-1\right)}$ is a permutation of $V$. Hence,

$$
\begin{aligned}
\Lambda_{5}(a, b) & =\sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(\xi^{i} v\right)^{\frac{2^{n}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right) \\
& =\left(\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i \frac{2^{n}-1}{5}}\right)\right)\right)\left(\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)\right)
\end{aligned}
$$

As $\xi^{\frac{2^{n}-1}{5}}=\left(\alpha^{2^{m}-1}\right)^{\frac{\left(2^{m}-1\right)\left(2^{m}+1\right)}{5}}=\beta^{2^{m}-1}=\beta^{2^{m}+1-2}=$ $\beta^{3}$,

$$
\begin{align*}
\Lambda_{5}(a, b) & =\left(\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{3 i}\right)\right)\left(\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)\right)\right. \\
& =\left(\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i}\right)\right)\left(\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)\right) .\right. \tag{9}
\end{align*}
$$

By (9), when $b=0, \Lambda_{5}(a, 0)=5 \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)$, and thus $\Lambda_{5}(a, 0) \neq 1$. By Proposition $5, f_{a, 0}^{(5)}$ is not a hyper-bent function.

When $b \neq 0, b$ can be represented as $b=\omega \beta^{j}$, where $\omega^{3}=1$ and $0 \leq j \leq 4$. Then

$$
\begin{equation*}
\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i}\right)\right)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \beta^{i+j}\right)\right)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \beta^{i}\right)\right) \tag{10}
\end{equation*}
$$

Since $\omega^{3}=1$ and $\omega^{4}=\omega$, we have

$$
\operatorname{Tr}_{1}^{4}\left(\omega \beta^{i}\right)=\operatorname{Tr}_{1}^{4}\left(\omega^{4} \beta^{4 i}\right)=\operatorname{Tr}_{1}^{4}\left(\omega \beta^{4 i}\right)
$$

If $\omega=1, \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i}\right)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\beta^{i}\right)\right)\right.$. As $\beta$ satisfies $\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=0, \operatorname{Tr}_{1}^{4}\left(\beta^{i}\right)=1, i \neq 0$. Then

$$
\begin{aligned}
\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i}\right)\right) & =-3 . \text { Therefore } \\
\Lambda_{5}(a, b) & =-3 \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right), b=\beta^{j}, 0 \leq j \leq 4
\end{aligned}
$$

By Proposition 5, $f_{a, \beta^{j}}^{(5)}$ is not a hyper-bent function. When $\omega \neq 1$, we have

$$
\begin{aligned}
& \operatorname{Tr}_{1}^{4}(\omega \beta)+\operatorname{Tr}_{1}^{4}\left(\omega \beta^{2}\right)=\operatorname{Tr}_{1}^{4}\left(\omega\left(\beta+\beta^{2}\right)\right) \\
& =\omega\left(\beta+\beta^{2}+\beta^{3}+\beta^{4}\right)+\omega^{2}\left(\beta+\beta^{2}+\beta^{3}+\beta^{4}\right) \\
& =1
\end{aligned}
$$

Then $\chi\left(\operatorname{Tr}_{1}^{4}(\omega \beta)\right)+\chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \beta^{2}\right)\right)=0$. Similarly, $\chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \beta^{3}\right)\right)+\chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \beta^{4}\right)\right)=0$. Therefore,
$\Lambda_{5}(a, b)=\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right), b=\omega \beta^{j}, 0 \leq j \leq 4, \omega^{3}=1, \omega \neq 1$.
By Proposition 5, the second part of this proposition follows.

In Proposition 8, we consider the hyper-bent properties of the Boolean function $f_{a, b}^{(5)}$ for $m \equiv \pm 2, \pm 6(\bmod 20)$. The proposition below discusses the hyper-bent properties of $f_{a, b}^{(5)}$ for $m \equiv 10(\bmod 20)$.

Proposition 9: Let $n=2 m, m \equiv 10(\bmod 20), a \in \mathbb{F}_{2^{m}}$, $b \in \mathbb{F}_{16}$. then the Boolean function $f_{a, b}^{(5)}$ is not a hyper-bent function.

Proof: Notice that $\Lambda_{5}(a, b)=$ $\sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i \frac{2^{n}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a\left(\xi^{5 i}\right)^{2^{m}-1} v^{5\left(2^{m}-1\right)}\right)\right)$. Since $m \equiv 10(\bmod 20), \quad 25 \mid\left(2^{m}+1\right) \quad$ and $\left(5\left(2^{m}-1\right), \frac{2^{m}+1}{5}\right)=5$. Then $v \longmapsto v^{5\left(2^{m}-1\right)}$ is a 5 to 1 morphism from $V$ to $V^{5}:=\left\{v^{5} \mid v \in V\right\}$. Therefore,
$\Lambda_{5}(a, b)=5 \sum_{i=0}^{4} \sum_{v \in V^{5}} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{\frac{2^{2}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a\left(\xi^{5 i}\right)^{2^{m}-1} v\right)\right)$.
Hence, $5 \mid \Lambda_{5}(a, b)$ and $\Lambda_{5}(a, b)$ is not equal to 1 , By Proposition 5, $f_{a, b}^{(5)}$ is not a hyper-bent function.

By Proposition 8,

$$
\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)=\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a v^{2^{m}-1}\right)\right)
$$

Notice that $\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)=S_{0}$ in [1]. By Proposition 15 in [1],

$$
\begin{equation*}
\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right)=\frac{1}{5}\left[1-K_{m}(a)+2 Q_{m}(a)\right] \tag{11}
\end{equation*}
$$

Further, By Proposition 16 and 18 in [1], we have the following results.

Proposition 10: Let $n=2 m, m \equiv \pm 2, \pm 6(\bmod 20)$, $m \geq 6$ and $b \in \mathbb{F}_{16}^{*} \backslash\left\{\beta^{i} \mid 0 \leq i \leq 4\right\}$, then $f_{a, b}^{(5)}$ is a hyper-bent function if and only if one of the assertions (1) and (2) holds.
(1) $Q_{m}(a)=0, K_{m}(a)=-4$.
(2) $Q_{m}(a)=2^{m_{1}}, K_{m}(a)=2 \cdot 2^{m_{1}}-4$.
2) The hyper-bent properties of $f_{a, b}^{(5)}$ where $a \in \mathbb{F}_{2^{n}}$ : In this part, we always assume $n=2 m, m=2 m_{1}, m_{1} \in \mathbb{N}$.
Lemma 2: Let $b \in \mathbb{F}_{16}^{*}, \gamma \in\left\{z \in \mathbb{F}_{2^{n}}: z^{5}=1, z \neq 1\right\}=<$ $\alpha^{\frac{2^{n}-1}{5}}>$, then

$$
\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)= \begin{cases}1, & b^{5} \neq 1 \\ -3, & b^{5}=1\end{cases}
$$

Proof: Firstly, if $b^{5}=1$,

$$
\begin{aligned}
\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right) & =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\gamma^{i}\right)\right)=1+\sum_{i=0}^{3} \chi\left(\operatorname{Tr}_{1}^{4}\left(\gamma^{2^{i}}\right)\right) \\
& =1+4 \chi\left(\operatorname{Tr}_{1}^{4}(\gamma)\right)=-3
\end{aligned}
$$

Secondly, if $b^{5} \neq 1$,

$$
\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b^{2} \gamma^{2 i}\right)\right)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b^{2} \gamma^{i}\right)\right)
$$

Since $\forall b \in \mathbb{F}_{16}^{*}, b=\omega^{j} \gamma^{i}, 0 \leq j \leq 2,0 \leq i \leq 4$, we have
$\sum_{b \in \mathbb{F}_{16}} \chi\left(\operatorname{Tr}_{1}^{4}(b)\right)=1+\sum_{b \in \mathbb{F}_{16}^{*}} \chi\left(\operatorname{Tr}_{1}^{4}(b)\right)$
$=1+\sum_{j=0}^{2} \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\omega^{j} \gamma^{i}\right)\right)$
$=1+\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\gamma^{i}\right)\right)+\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \gamma^{i}\right)\right)+\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\omega^{2} \gamma^{i}\right)\right)$
$=1+(-3)+2 \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(\omega \gamma^{i}\right)\right)$.
Notice that $\sum_{b \in \mathbb{F}_{16}} \chi\left(\operatorname{Tr}_{1}^{4}(b)\right)=0$, hence $\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)=1$, and the conclusion stands.

Theorem 1: If $a=a^{\prime} \xi^{i}, a^{\prime} \in \mathbb{F}_{2^{m}}$, the hyper-bent properties of $f_{a, b}^{(5)}$ can be described as follows:
(1) when $m \equiv 10(\bmod 20), f_{a, b}^{(5)}$ is not hyper-bent.
(2) when $m \equiv \pm 2, \pm 6(\bmod 20), f_{a, b}^{(5)}$ is hyper-bent if and only if $S_{2 i}=1$.

Proof: To the character sum of $f_{a, b}^{(5)}$ :

$$
\begin{align*}
& \Lambda\left(a^{\prime} \xi^{i}, b\right)=\sum_{u \in U} \chi\left(f_{a^{\prime} \xi^{i}, b}^{(5)}(u)\right) \\
& =\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} \xi^{i} u^{5\left(2^{m}-1\right)}\right)\right) \chi\left(\operatorname{Tr}_{1}^{4}\left(b u^{\frac{2^{n}-1}{5}}\right)\right) \\
& =\sum_{j=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} \xi^{i}\left(\xi^{j} v\right)^{5\left(2^{m}-1\right)}\right)\right) \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(\xi^{j} v\right)^{\frac{2^{n}-1}{5}}\right)\right) \\
& =\sum_{j=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{j^{\frac{2^{n}-1}{5}}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} \xi^{i} \xi^{5 j\left(2^{m}-1\right)} v^{5\left(2^{m}-1\right)}\right)\right) . \tag{12}
\end{align*}
$$

If $m \quad \equiv \quad 10(\bmod 20), \quad$ then $(5, \# V) \quad=\quad 5 . \quad$ By (12), $\quad \Lambda\left(a^{\prime} \xi^{i}, b\right)=$ $5 \sum_{j=0}^{4} \sum_{v^{\prime} \in V^{5}} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{j \frac{2^{n}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} \xi^{i} \xi^{5 j\left(2^{m}-1\right)} v^{\prime}\right)\right)$, where $V^{5}=\left\{v^{5} \mid v \in V\right\}, v \mapsto v^{5\left(2^{m}-1\right)}$ is a 5 to 1
morphism from $V$ to $V^{5}$. Thus $\Lambda\left(a^{\prime} \xi^{i}, b\right) \neq 1$, and $f_{a, b}^{(5)}$ is not a hyper-bent function.
If $m \equiv \pm 2, \pm 6(\bmod 20)$, then $(5, \# V)=1$. By (12) and (9),

$$
\begin{aligned}
& \Lambda\left(a^{\prime} \xi^{i}, b\right)=\sum_{j=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{j}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} \xi^{i} v\right)\right) \\
& =\left(\sum_{j=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{j}\right)\right)\right)\left(\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime}\left(\xi^{\frac{i}{2^{m}-1}}\right)^{2^{m}-1} v\right)\right)\right)
\end{aligned}
$$

where $\beta=\alpha^{\frac{2^{n}-1}{5}}, \xi^{\frac{2^{n}-1}{5}}=\beta^{3}$. Since $\frac{1}{2^{m}-1} \equiv 2(\bmod 5)$, then by (4),

$$
\begin{aligned}
\Lambda\left(a^{\prime} \xi^{i}, b\right) & =\left(\sum_{j=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{j}\right)\right)\right)\left(\sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime}\left(\xi^{2 i}\right)^{2^{m}-1} v\right)\right)\right) \\
& =\left(\sum_{j=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{j}\right)\right)\right) S_{2 i}
\end{aligned}
$$

By Lemma 2,

$$
\Lambda\left(a^{\prime} \xi^{i}, b\right)= \begin{cases}S_{2 i}, & b^{5} \neq 1 \\ -3 S_{2 i}, & b^{5}=1\end{cases}
$$

If $b^{5}=1,3 \mid \Lambda\left(a^{\prime} \xi^{i}, b\right)$. Thus $f_{a^{\prime} \xi^{i}, b}^{(5)}$ is not a hyper-bent function.
If $b^{5} \neq 1$, then $f_{a^{\prime} \xi^{i}, b}^{(5)}$ is a hyper-bent function if and only if $S_{2 i}=1$.

## B. The Case of $r=2$

When $b=0$, the hyper-bent propriety of $f_{a, 0}^{(2)}$ has been studied by Canteaut et al in [2]. We consider the case of $b \neq 0$. Proposition 11: Let $a \in \mathbb{F}_{2^{m}}, b \in \mathbb{F}_{16}^{*}$, we have
(1) if $b=1$, then $\Lambda_{2}(a, b)=S_{0}-2\left(S_{1}+S_{2}\right)=2 S_{0}-$ $\Lambda_{2}(a, 0)$.
(2) if $b \in\left\{\beta+\beta^{2}, \beta+\beta^{3}, \beta^{2}+\beta^{4}, \beta^{3}+\beta^{4}\right\}$, then $\Lambda_{2}(a, b)=$ $S_{0}$.
(3) if $b=\beta$ or $\beta^{4}$, then $\Lambda_{2}(a, b)=-S_{0}-2 S_{2}$.
(4) if $b=\beta^{2}$ or $\beta^{3}$, then $\Lambda_{2}(a, b)=-S_{0}-2 S_{1}$.
(5) if $b=1+\beta$ or $1+\beta^{4}$, then $\Lambda_{2}(a, b)=-S_{0}+2 S_{2}$.
(6) if $b=1+\beta^{2}$ or $1+\beta^{3}$, then $\Lambda_{2}(a, b)=-S_{0}+2 S_{1}$.
(7) if $b=\beta+\beta^{4}$, then $\Lambda_{2}(a, b)=S_{0}+2 S_{2}-2 S_{1}$.
(8) if $b=\beta^{2}+\beta^{3}$, then $\Lambda_{2}(a, b)=S_{0}-2 S_{2}+2 S_{1}$.

Proof: Similar to proof of Proposition 13 in [1] the results hold.

Corollary 1: Let $a \in \mathbb{F}_{2^{m}}, b \in \mathbb{F}_{16}^{*}$, we have
(1) $f_{a, b}^{(2)}$ holds the same hyper-bent propertyies as $f_{a, b^{2}}^{(1)}$.
(2) if $b$ satisfies $(b+1)\left(b^{4}+b+1\right)=0$, then $f_{a, b}^{(2)}$ holds the same hyper-bent properties as $f_{a, b}^{(1)}$.

Proof: (1) By Proposition 11 and Proposition 13 in [1],

$$
\Lambda_{2}(a, b)=\Lambda_{1}\left(a, b^{2}\right)
$$

Hence $f_{a, b}^{(2)}$ is a hyper-bent function if and only if $f_{a, b^{2}}^{(1)}$ is.
(2) Similarly, if $b$ satisfying $(b+1)\left(b^{4}+b+1\right)=0$, then,

$$
\Lambda_{2}(a, b)=\Lambda_{1}(a, b) .
$$

Thus $f_{a, b}^{(2)}$ holds the same hyper-bent properties as $f_{a, b}^{(1)}$.

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## C. The General Case of $r$

Theorem 2: Let $n=2 m, m \equiv 2(\bmod 4), a \in \mathbb{F}_{2^{m}}$ and $b \in \mathbb{F}_{16}$. If $\left(r, \frac{2^{m}+1}{5}\right)>1$, then $f_{a, b}^{(r)}$ is not a hyper-bent function. Further, if $\left(r, \frac{2^{m}+1}{5}\right)=1$, then
(1) If $r \equiv 0(\bmod 5)$, then $f_{a, b}^{(r)}$ and $f_{a, b}^{(5)}$ has the same hyper-bent properties.
(2) If $r \equiv \pm 1(\bmod 5)$, then $f_{a, b}^{(r)}$ and $f_{a, b}^{(1)}$ has the same hyper-bent properties.
(3) If $r \equiv \pm 2(\bmod 5)$, then $f_{a, b}^{(r)}$ and $f_{a, b}^{(2)}$ has the same hyper-bent properties.

Proof: Notice that
$\Lambda_{r}(a, b)=\sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(\xi^{i} v\right)^{\frac{2^{n}-1}{5}}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a\left(\xi^{i} v\right)^{r\left(2^{m}-1\right)}\right)\right)$ $=\sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i^{\frac{2}{}}-1}\right)\right) \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{r i\left(2^{m}-1\right)} v^{r\left(2^{m}-1\right)}\right)\right)$.
Let $d=\left(r\left(2^{m}-1\right), \# V\right)=\left(r, \frac{2^{m}+1}{5}\right)$, then $\Lambda_{r}(a, b)=$ $d \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i^{\frac{2^{n}}{5}}} \mathbf{5}\right)\right) \sum_{v \in V^{d}} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{r i\left(2^{m}-1\right)} v^{r\left(2^{m}-1\right)}\right)\right)$, where $V^{d}=\left\{v^{d} \mid v \in V\right\}$. If $d=\left(r, \frac{2^{m}+1}{5}\right)>1, d \mid \Lambda_{r}(a, b)$ and $\Lambda_{r}(a, b) \neq 1$. Hence, $f_{a, b}^{(r)}$ is not a hyper-bent function.

When $d=\left(r, \frac{2^{m}+1}{5}\right)=1$,

$$
\begin{equation*}
\Lambda_{r}(a, b)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i^{\frac{2^{n}-1}{5}}}\right)\right) \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{r i\left(2^{m}-1\right)} v\right)\right) \tag{13}
\end{equation*}
$$

If $r \equiv 0(\bmod 5)$, from $\xi^{\frac{2^{n}-1}{5}}=\beta^{3}$, we have

$$
\begin{aligned}
\Lambda_{r}(a, b) & =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{3 i}\right)\right) \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{r i\left(2^{m}-1\right)} v\right)\right) \\
& =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i}\right)\right) \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}(a v)\right) .
\end{aligned}
$$

Then $\Lambda_{r}(a, b)=\Lambda_{5}(a, b)$. Therefore, $f_{a, b}^{(r)}$ and $f_{a, b}^{(5)}$ has the same hyper-bent properties.

If $r \equiv 1(\bmod 5)$, then

$$
\Lambda_{r}(a, b)=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i^{\frac{2^{n}-1}{5}}}\right)\right) \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{i\left(2^{m}-1\right)} v\right)\right) .
$$

By Proposition 10 in [1], $\Lambda_{r}(a, b)=\Lambda_{1}(a, b)$. Hence, $f_{a, b}^{(r)}$ and $f_{a, b}^{(1)}$ has the same hyper-bent properties.

If $r \equiv 2(\bmod 5)$, then

$$
\begin{aligned}
\Lambda_{r}(a, b) & =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i^{\frac{2^{n}-1}{5}}}\right)\right) \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{2 i\left(2^{m}-1\right)} v\right)\right) \\
& =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{3 i}\right)\right) S_{2 i} \\
& =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{9 i}\right)\right) S_{6 i}=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{4 i}\right)\right) S_{i} .
\end{aligned}
$$

By Lemma 1 in [1],

$$
\begin{align*}
\Lambda_{r}(a, b)= & \chi\left(\operatorname{Tr}_{1}^{4}(b)\right) S_{0}+\left(\chi\left(\operatorname{Tr}_{1}^{4}(b \beta)\right)+\chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{4}\right)\right)\right) S_{1} \\
& +\left(\chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{2}\right)\right)+\chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{3}\right)\right)\right) S_{2} . \tag{14}
\end{align*}
$$

Hence, $\Lambda_{r}(a, b)=\Lambda_{2}(a, b) . f_{a, b}^{(r)}$ and $f_{a, b}^{(2)}$ has the same hyper-bent properties.

If $r \equiv 3(\bmod 5)$,

$$
\begin{aligned}
\Lambda_{r}(a, b) & =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \xi^{i^{\frac{2^{n}-1}{5}}}\right)\right) \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a \xi^{3 i\left(2^{m}-1\right)} v\right)\right) \\
& =\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{3 i}\right)\right) S_{3 i}=\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i}\right)\right) S_{i} .
\end{aligned}
$$

From Lemma 1 in [1],

$$
\begin{align*}
\Lambda_{r}(a, b)= & \chi\left(\operatorname{Tr}_{1}^{4}(b)\right) S_{0}+\left(\chi\left(\operatorname{Tr}_{1}^{4}(b \beta)\right)+\chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{4}\right)\right)\right) S_{1} \\
& +\left(\chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{2}\right)\right)+\chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{3}\right)\right)\right) S_{2} . \tag{15}
\end{align*}
$$

Hence, $\Lambda_{r}(a, b)=\Lambda_{3}(a, b)$. From (14) and (15), we have $\Lambda_{2}(a, b)=\Lambda_{3}(a, b)$. Thus, $f_{a, b}^{(r)}$ and $f_{a, b}^{(2)}$ have the same hyper-bent properties.

Similarly, if $r \equiv 4(\bmod 5)$, then $\Lambda_{r}(a, b)=\Lambda_{4}(a, b)=$ $\Lambda_{1}(a, b)$. Thus, $f_{a, b}^{(r)}$ and $f_{a, b}^{(1)}$ have the same hyper-bent properties.

Above all, the results stand.
From the above discussion, we have the following results on $f_{a, b}^{(r)}$.

Proposition 12: Let $a \in \mathbb{F}_{2^{m}}$ and $\left(r, \frac{2^{m}+1}{5}\right)=1$, then
(1) If $\frac{1}{5}\left[1-K_{m}(a)+2 Q_{m}(a)\right]=1$, then the following Boolean functions
(a) $f_{a, b}^{(r)}, b \in \mathbb{F}_{16}^{*} \backslash\left\{\beta^{i} \mid i=0,1,2,3,4\right\}, r \equiv 0(\bmod 5)$.
(b) $f_{a, b}^{(r)}, r \not \equiv 0(\bmod 5), b^{4}+b+1=0$.
are hyper-bent functions.
(2) If $-\frac{1}{5}\left[3\left(1-K_{m}(a)\right)-4 Q_{m}(a)\right]=1$, then the Boolean function $f_{a, 1}^{(r)}(r \not \equiv 0(\bmod 5))$ is a hyper-bent function.

Proof: By Theorem 2, (11), Proposition 8 and Proposition 16 in [1], this proposition follows.

With Proposition 12, we can generalize Theorem 3 in [1] to the following theorem.
Theorem 3: Let $n=2 m, m=2 m_{1}, m_{1} \equiv 1(\bmod 2)$, $m_{1} \geq 3$ and $\left(r, \frac{2^{m}+1}{5}\right)=1$, If one of two assertions (1) and (2) holds,
(1) $p(x)=x^{5}+x+a^{-1}$ over $\mathbb{F}_{2^{m}}$ is (1)(2) ${ }^{2}$ and $K_{m}(a)=$ -4.
(2) $p(x)=x^{5}+x+a^{-1}$ is irreducible over $\mathbb{F}_{2^{m}}$. The quadratic form $\mathfrak{q}(x)=\operatorname{Tr}_{1}^{m}\left(x\left(a x^{4}+a x^{2}+a^{2} x\right)\right)$ over $\mathbb{F}_{2^{m}}$ is even. $K_{m}(a)=2 \cdot 2^{m_{1}}-4$.

Then the Boolean functions
(a) $f_{a, b}^{(r)}, b \in \mathbb{F}_{16}^{*} \backslash\left\{\beta^{i} \mid i=0,1,2,3,4\right\}, r \equiv 0(\bmod 5)$.
(b) $f_{a, b}^{(r)}, r \not \equiv 0(\bmod 5), b^{4}+b+1=0$.
are hyper-bent functions.
Proof: By Proposition 16 and Theorem 3 in [1] and Proposition 12, this theorem follows.
By Proposition 16 , Proposition 12 and Theorem 2 in [1], we have the following results for the hyper-bent properties of $f_{a, b}^{(r)}$ :

Theorem 4: Let $n=2 m, m=2 m_{1}, m_{1} \equiv 1(\bmod 2)$, $m_{1} \geq 3,\left(r, \frac{2^{m}+1}{5}\right)=1$ and $r \not \equiv 0(\bmod 5)$, then $f_{a, 1}^{(r)}$ is a hyper-bent function if and only if the following assertions holds.

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(1) $p(x)=x^{5}+x+a^{-1}$ is irreducible over $\mathbb{F}_{2^{m}}$.
(2) The quadratic form $\mathfrak{q}(x)=\operatorname{Tr}_{1}^{m}\left(x\left(a x^{4}+a x^{2}+a^{2} x\right)\right)$ over $\mathbb{F}_{2^{m}}$ is even.
(3) $K_{m}(a)=\frac{4}{3}\left(2-2^{m_{1}}\right)$.

If $a \in \mathbb{F}_{2} \frac{m}{2}$, the hyper-bent properties of $f_{a, b}^{(r)}$ is
Theorem 5: Let $n=2 m, m=2 m_{1}, m_{1} \equiv 1(\bmod 2)$ and $m_{1} \geq 3$. If $n \neq 12,28$, any Boolean function in

$$
\begin{equation*}
\left\{f_{a, b}^{(r)} \left\lvert\, a \in \mathbb{F}_{2 \frac{m}{2}}\right., b \in \mathbb{F}_{16}\right\} \tag{16}
\end{equation*}
$$

is not a hyper-bent function. Further, if $n=12$, all the hyper-bent functions in (16) are $\operatorname{Tr}_{1}^{12}\left(a x^{r\left(2^{6}-1\right)}\right)+$ $\operatorname{Tr}_{1}^{4}\left(b x^{\frac{2^{12}-1}{5}}\right)$, where $r \not \equiv 0(\bmod 5),\left(r, \frac{2^{m}+1}{5}\right)=1,(a+$ 1) $\left(a^{3}+a^{2}+1\right)=0$ and $b=\beta^{i}, i=1,2,3,4$. If $n=28$, all the hyper-bent functions in (16) are $\left.\operatorname{Tr}_{1}^{28}\left(a x^{r\left(2^{14}-1\right.}\right)\right)+$ $\operatorname{Tr}_{1}^{4}\left(b x^{\frac{2^{28}-1}{5}}\right)$, where $r \not \equiv 0(\bmod 5),\left(r, \frac{2^{m}+1}{5}\right)=1,(a+$ 1) $\left(a^{7}+a^{6}+a^{5}+a^{4}+a^{3}+a^{2}+1\right)=0$ and $b=\beta^{i}, i=1,2,3,4$.

Proof: Notice that $a \in \mathbb{F}_{2} \frac{m}{2}$. By Theorem 2, if $f_{a, b}^{(r)}$ is a hyper-bent function, $\left(r, \frac{2^{m}+1}{5}\right)=1$.

Suppose $\left(r, \frac{2^{m}+1}{5}\right)=1$. we first prove that $f_{a, 0}^{(r)}$ is not a hyper-bent function when $r \equiv 0(\bmod 5)$. By Theorem 2, $f_{a, b}^{(r)}$ is a hyper-bent function if and only if $f_{a, b}^{(5)}$ is a hyper-bent function. If $b=0$,
$\Lambda_{5}(a, 0)=\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a u^{5\left(2^{m}-1\right)}\right)\right)=5 \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{n}\left(a v^{2^{m}-1}\right)\right)$. Hence, $5 \mid \Lambda_{5}(a, 0)$ and $\Lambda_{5}(a, 0) \neq 1$. Therefore, $f_{a, 0}^{(5)}$ is not a hyper-bent function. Then $f_{a, 0}^{(r)}$ is not a hyper-bent function.
When $b \neq 0$, by Theorem $3, f_{a, b}^{(r)}$ is a hyper-bent function if and only if $f_{a, b^{\prime}}^{(1)}\left(b^{\prime 4}+b^{\prime}+1=0\right)$ is a hyper-bent function. By Theorem 5 in [1], $f_{a, b^{\prime}}^{(1)}\left(b^{\prime 4}+b^{\prime}+1=0\right)$ is not a hyper-bent function. Hence, $f_{a, b}^{(r)}$ is not a hyper-bent function when $r \equiv 0$ $(\bmod 5)$.
Now we discuss the case $r \equiv \pm 1(\bmod 5)$ and $\left(r, \frac{2^{m}+1}{5}\right)=$ 1. By Theorem $2, f_{a, b}^{(r)}$ is a hyper-bent function if and only if $f_{a, b}^{(1)}$ is a hyper-bent function. By Theorem 5 in [1], there are only two cases. The first case is $n=12$, where $a$ and $b$ satisfy

$$
(a+1)\left(a^{3}+a^{2}+1\right)=0, b=\beta^{i}, i=1,2,3,4 .
$$

The second case is $n=28$, where $a$ and $b$ satisfy $(a+1)\left(a^{7}+a^{6}+a^{5}+a^{4}+a^{3}+a^{2}+1\right)=0, b=\beta^{i}, i=1,2,3,4$.
When $r \equiv \pm 2(\bmod 5)$ and $\left(r, \frac{2^{m}+1}{5}\right)=1$, we have similar results.

Above all, this theorem follows.
IV. The bent property of $f_{a, b}^{(r)}$ when $m=0(\bmod 4)$

In this section we consider the bent properties of $f_{a, b}^{(r)}$, where $m \equiv 0(\bmod 4), a \in \mathbb{F}_{2^{n}}, b \in \mathbb{F}_{16}$.

Proposition 13: Let $a=a^{\prime} \xi^{k} \in \mathbb{F}_{2^{n}}^{*}, b \in \mathbb{F}_{16}^{*}, a^{\prime} \in \mathbb{F}_{2^{m}}^{*}$, $0 \leq k \leq 2^{m}, m \equiv 0(\bmod 4), m=2 m_{1}$. One necessary condition for $f_{a, b}^{(r)}$ to be a bent function is: $\left(r, 2^{m}+1\right)=1$, $a^{\prime} \in \mathbb{F}_{2^{m}} \backslash \mathbb{F}_{2^{m_{1}}}, b^{5} \neq 1, \widehat{\chi}_{f_{a, b}^{(r)}}(0)=2^{m}$ and $K_{m}\left(a^{\prime}\right)=-4$.

Proof: Notice that $\forall x \in \mathbb{F}_{2^{n}}^{*}, x=y u$, where $y \in \mathbb{F}_{2^{m}}^{*}$, $u \in U=<\alpha^{2^{m}-1}>$. Since $m \equiv 0(\bmod 4), 5 \mid 2^{m}-1$.

Thus $u^{\frac{2^{n}-1}{5}}=\left(u^{2^{m}+1}\right)^{\frac{2^{m}-1}{5}}=1$. Now, consider the Walsh spectrum of $f_{a, b}^{(r)}$ at 0 , which is

$$
\left.\left.\begin{array}{l}
\widehat{\chi}_{f_{a, b}^{(r)}}(0)=\sum_{x \in \mathbb{F}_{2^{n}}} \chi\left(f_{a, b}^{(r)}(x)\right)=1+\sum_{u \in U} \sum_{y \in \mathbb{F}_{2}^{*} m} \chi\left(f_{a, b}^{(r)}(y u)\right) \\
=1+\sum_{u \in U} \sum_{y \in \mathbb{F}_{2}^{*} m} \chi\left(\operatorname{Tr}_{1}^{n}\left(a(y u)^{r\left(2^{m}-1\right)}\right)\right) \chi\left(\operatorname { T r } _ { 1 } ^ { 4 } \left(b(y u)^{\frac{2}{}^{n}-1} 5\right.\right.
\end{array}\right)\right), \sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a u^{r\left(2^{m}-1\right)}\right)\right) \sum_{y \in \mathbb{F}_{2}^{*} m} \chi\left(\operatorname{Tr}_{1}^{4}\left(b y^{\frac{2^{n}-1}{5}}\right)\right) \quad \text { (17) }
$$

$\mathbb{F}_{2^{m}}^{*}$ can be written as $\mathbb{F}_{2^{m}}^{*}=\bigcup_{i=0}^{4} \beta^{i} V$, where $V=\left\{z^{5} \mid\right.$ $\left.z \in \mathbb{F}_{2^{m}}^{*}\right\}, \beta \in \mathbb{F}_{2^{m}}^{*} \backslash V$.

If $\left(r\left(2^{m}-1\right), 2^{m}+1\right)=1$, by (17),
$\widehat{\chi}_{f_{a, b}^{(r)}}(0)=$
$1+\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} \xi^{k} u^{r\left(2^{m}-1\right)}\right)\right) \sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(v \beta^{i}\right)^{\frac{2^{n}-1}{5}}\right)\right)$
$=1+\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} u\right)\right) \sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \beta^{i \frac{2^{n}-1}{5}}\right)\right)$
$=1+\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a^{\prime} u\right)\right) \sum_{v \in V} \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)$
$=1+\left(1-K_{m}\left(a^{\prime}\right)\right) \frac{2^{m}-1}{5} \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)$,
$\left(r\left(2^{m}-1\right), 2^{m}+1\right)=1, u \mapsto \xi^{k} u^{r\left(2^{m}-1\right)}$ is a permutation in $U, \sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a u^{2^{m}-1}\right)\right)=1-K_{m}(a) . \gamma=\beta^{\frac{2^{n}-1}{5}} \neq 1$ is a 5-th primitive root of unity in $\mathbb{F}_{2^{n}}$. If $f_{a, b}^{(r)}$ is a bent function,
$\widehat{\chi}_{f_{a, b}^{(r)}}(0)=1+\left(K_{m}\left(a^{\prime}\right)-1\right)\left(\frac{2^{m}-1}{5}\right) \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)= \pm 2^{m}$. By Lemma 2,
(1) if $\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)=-3$, then $K_{m}\left(a^{\prime}\right)=\frac{8}{3}$ or $3\left(2^{m}-\right.$ 1) $\left(K_{m}\left(a^{\prime}\right)-1\right)=-5\left(2^{m}+1\right)$. Since $K_{m}\left(a^{\prime}\right)$ is an integer, however $\left(\frac{2^{m}-1}{5}, 2^{m}+1\right)=1$, Neither of the two equations stands, thus $f_{a, b}^{(r)}$ is not a bent function.
(2) if $\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)=1$, which means $K_{m}\left(a^{\prime}\right)=-4$, $\widehat{\chi}_{f_{a, b}^{(r)}}(0)=2^{m}$, or $\left(2^{m}-1\right)\left(K_{m}\left(a^{\prime}\right)-1\right)=5\left(2^{m}+1\right)$, $\widehat{\chi}_{f_{a, b}^{(r)}}^{(r)}(0)=-2^{m}$. Since $\left(\frac{2^{m}-1}{5}, 2^{m}+1\right)=1$, the last group of equations can not stand. By Lemma 1, if $a^{\prime} \in \mathbb{F}_{2^{m_{1}}}$, then $K_{m}\left(a^{\prime}\right) \neq-4$.

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If $\left(r\left(2^{m}-1\right), 2^{m}+1\right)=d>1$. Since $5 \mid 2^{m}-1,5 \nmid d$. By (17),

$$
\begin{aligned}
& \widehat{\chi}_{f_{a, b}^{(r)}}(0)= \\
& 1+\sum_{u \in U} \chi\left(\operatorname{Tr}_{1}^{n}\left(a u^{r\left(2^{m}-1\right)}\right)\right) \sum_{i=0}^{4} \sum_{v \in V} \chi\left(\operatorname{Tr}_{1}^{4}\left(b\left(v \beta^{i}\right)^{\frac{2^{n}-1}{5}}\right)\right) \\
& =1+d \sum_{u^{\prime} \in U^{d}} \chi\left(\operatorname{Tr}_{1}^{n}(a u)\right) \frac{2^{m}-1}{5} \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right) \\
& =1+d h \frac{2^{m}-1}{5} \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right),
\end{aligned}
$$

where $U^{d}=\left\{u^{d} \mid u \in U\right\}, u \mapsto u^{r\left(2^{m}-1\right)}$ is a $d$ to 1 morphism from $U$ to $U^{d}, h=\sum_{u^{\prime} \in U^{d}} \chi\left(\operatorname{Tr}_{1}^{n}(a u)\right)$. If $f_{a, b}^{(r)}$ is a bent function,

$$
\widehat{\chi}_{f_{a, b}^{(r)}}(0)=1+d h\left(\frac{2^{m}-1}{5}\right) \sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)= \pm 2^{m} .
$$

By Lemma 2,
(1) if $\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)=-3$, then $3 d h=-5$ or $3 d h\left(2^{m}-\right.$ 1) $=5\left(2^{m}+1\right)$.
(2) if $\sum_{i=0}^{4} \chi\left(\operatorname{Tr}_{1}^{4}\left(b \gamma^{i}\right)\right)=1$, then $d h=5$ or $d h\left(2^{m}-1\right)=$ $-5\left(2^{m}+1\right)$.
Notice that $d>1,5 \nmid d, 3 \nmid 2^{m}+1,\left(2^{m}-1,2^{m}+1\right)=1$, all of the above equations can not stand.

Above all, the results follow.

## V. Conclusion

This paper considers the bent and hyper-bent properties of the Boolean functions $f_{a, b}^{(r)}$ of the form $f_{a, b}^{(r)}:=$ $\operatorname{Tr}_{1}^{n}\left(a x^{r\left(2^{m}-1\right)}\right)+\operatorname{Tr}_{1}^{4}\left(b x^{\frac{2^{n}-1}{5}}\right)$, where $n=2 m, m=2 k$ $(\bmod 4), k \in\{0,1\}, a \in \mathbb{F}_{2^{n}}$ and $b \in \mathbb{F}_{16}$. When $m=2$ $(\bmod 4)$, we give a detailed description of the hyper-bent properties of $f_{a, b}^{(r)}$, and prove that the hyper-bent properties of $f_{a, b}^{(r)}$ can be characterized by that of $f_{a^{\prime}, b^{\prime}}^{(r)}$, where $a=a^{\prime} \xi^{i} \in$ $\mathbb{F}_{2^{n}}, a^{\prime} \in \mathbb{F}_{2^{m}}, b, b^{\prime}=b \alpha^{-\frac{i}{r} \frac{2^{n}-1}{5}} \in \mathbb{F}_{16}$. We also prove that $f_{a, b}^{(r)}$ is not a hyper-bent function unless $n=12$ or $n=28$ when $a \in \mathbb{F}_{2} \frac{m}{2}$. Further, we give all the hyper-bent functions for $n=12$ or $n=28$. When $m=0(\bmod 4)$, we give a necessary condition for $f_{a, b}^{(r)}$ to be a bent function. To those strict restrictions, it seems $f_{a, b}^{(r)}$ can not be bent. In fact with the help of computer, we have checked all of the functions which satisfy Proposition 13 for $m=4,8$, and find that none of them is bent. Thus we guess when $m=0(\bmod 4), f_{a, b}^{(r)}$ can not be bent.

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