Numerical Solution for Integro-Differential Equations by Using Quartic B-Spline Wavelet and Operational Matrices

Khosrow Maleknejad, Yaser Rostami

Abstract—In this paper, Semi-orthogonal B-spline scaling functions and wavelets and their dual functions are presented to approximate the solutions of integro-differential equations.The B-spline scaling functions and wavelets, their properties and the operational matrices of derivative for this function are presented to reduce the solution of integro-differential equations to the solution of algebraic equations. Here we compute B-spline scaling functions of degree 4 and their dual, then we will show that by using them we have better approximation results for the solution of integro-differential equations in comparison with less degrees of scaling functions

Keywords—Integro-differential equations, Quartic B-spline wavelet, Operational matrices.

I. Introduction

THE integral equation is a mathematical model of many evolutionary problems with memory arising from biology, chemistry, physics, engineering[3]. In recent years, many different basic functions have been used to estimate the solution of integral equations[9]-[21], such as orthogonal bases and wavelets. A differential equation can be replaced by an integral equation which incorporates its boundary conditions [1], as such each solution of the integral equation automatically satisfies these boundary conditions[3]. In application to discrete data sets, wavelets may be considered as basis functions generated by dilations and translations of a signal function [2].In this paper, we develop a non- orthogonal (semi-orthogonal) wavelet using B-spline specially constructed for the bounded interval[17]-[16], this wavelet can be represented in a closed-form. Integro-Differential Equations(IDE) have applications in natural sciences and engineering . Recent work in the context of solution of these type of problems include spline approximation method [25], collocation method [12], wavelet basis method [4]-[5],hybrid Legendre polynomials and block-pulse functions approach [18]-[22], wavelet-Galerkin method [20],hybrid Taylor polynomials and block-pulse functions approach [19],Chebyshev collocation method [23]and Petrov-Galerkin method[30] etc.
Wavelet analysis has been applied in a wide range of engineering disciplines; particularly, wavelets are very

K. Maleknejad is with the Department of Mathematics , Fandanesh Institute of Higher Education (FDIHE), Saveh, 39157-14774, Iran. (e-mail: maleknejad@iust.ac.ir, HTTP://webpages.iust.ac.ir/maleknejad).
Y. Rostami is with Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad University, Alborz, Iran. (e-mail: rostamiysr@gmail.com).

successfully used in signal analysis, time frequency analysis and fast algorithms for easy implementation [10]. Orthogonal wavelet either have infinite support or a nonsymmetric and in some cases,fractal nature. These properties can make them a poor choice for characterization of a function[11]-[29]. In contrast, the Semi-Orthogonal (SO) wavelets have finite support, both even and odd symmetry and simple analytical expressions,ideal attributes of a basis function[10]-[16].
In this present paper, we apply compactly supported quartic(five order) (SO) B-spline wavelets, specially constructed for the bounded interval $[0, 1]$ to solve the first Order Fredholm integro- differential equation of the form:

$$\begin{cases} g(x)y'(x) = f(x) + \int\limits_0^1 k(x,t)y(t)dt \\ y(0) = y_0 \end{cases} \quad (1)$$

where f, g and k are given continuous functions and y is an unknown function to be determined.
The use of SO compactly supported spline wavelets is justified by their interesting properties[14]-[28].Thes wavelets are smoothness,the larger are their supports in time(space).The order of vanishing moments usually increases with smoothness.Total positivity properties of splines have certain desirable properties from an approximation points of view[24].

The last decade demonstrates an augmentation of interest of B-spline wavelets.Solving integral and integro-differential equations using linear B-spline[15],quadratic B-spline[13] and cubic B-spline scaling functions[7]-[8] This wavelet family also found and application in image processing[2]. The efficiency of approximation using B-spline wavelets was compared with other families of orthogonal wavelets[26].The obtained results indicate the effectiveness of high-order B-spline in fault detection and localization.

II. B-spline scaling functions and wavelets on[0,1]

We generate a doubly-indexed family of wavelets form ψ by dilating and translating: [11]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi(\frac{t-b}{a}), \qquad a \neq 0, \qquad a,b \in R \quad (2)$$

The wavelets have been grouped in different families . These wavelets have an particularly desirable property that they are

zero in everywhere except of determined closed and bounded interval that we say so-called function has compact support. We can say generally that jth generation of daughters will have 2^j wavelets defined by:

$$\psi_{j,k}(t) = \psi(2^j t - k), \qquad 0 \le k \le 2^j - 1 \qquad (3)$$

The members of this generation will be constant on intervals of length $2^{-(j+1)}$. The first idea in studying of wavelets was this matter that we can write functions as linear combinations of the Father and Mother wavelets and first generation of daughters, This basis denote by B_j .There is another basis of wavelets that is called sons wavelet . Here, we can define generations of sons wavelet by the following relation :

$$\phi_{j,k}(t) = \psi(2^j t - k), \qquad 0 \le k \le 2^j - 1 \qquad (4)$$

Now, assume S_j denote the set of 2^j functions $\{\phi_{j,k}(t)\}_{k=0}^{2^j-1}$. Therefore, S_j will account as a basis for the inner product space V_j. Vector space V_j with the basis S_j , forms a nested sequence of subspaces $V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$ and using the basis B_j for V_j , and orthogonal decomposition theorem we will have :

$$V_j = V_{j-1} \oplus V_{j-1}^\perp = (V_{j-2} \oplus V_{j-2}^\perp) \oplus V_{j-1}^\perp =$$
$$\ldots = V_0 \oplus V_0^\perp \oplus V_1^\perp \oplus \ldots V_{j-1}^\perp \qquad (5)$$

The wavelets have especial particularities that all of them is gathered in a collection of Multi Resolution Analysis (MRA). Multi Resolution Analysis of $L^2(\mathbf{R})$ is defined as a sequence of closed subspaces V_j of $L^2(\mathbf{R})$, $j \in Z$, with the following properties [11]:

1) $V_j \subset V_{j+1}$
2) $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$
3) $f(x) \in V_0 \Longleftrightarrow f(x+1) \in V_0$
4) $\bigcup V_j$ is dense in $L^2(\mathbf{R})$ and $\bigcap V_j = \phi$
5) A scaling function $\varphi \in V_0$,with a non vanishing integral, exists such that the collection φ is Riesz basis of V_0

Some of the important properties relevant to the present work are given below:

1. *vanishing moments*: a wavelet is said to be have a vanishing moment of order m if

$$\int_{-\infty}^{+\infty} x^p \psi(x) dx = 0; \qquad p = 0, ..., m - 1 \qquad (6)$$

All wavelets must satisfy the above condition for $p = 0$.
2. *Semi − orthogonality*: the wavelets $\psi_{j,k}$ form a semi-orthogonal basis if

$$\langle \psi_{j,k}, \psi_{i,s} \rangle = 0; \quad i \ne j; \quad \forall i, j, k, s \in Z \qquad (7)$$

The generalization to biorthogonal wavelets has been considered to gain more flexibility. Here,a dual scaling function $\tilde{\varphi}$ and a dual wavelet $\tilde{\psi}$ exist that generate a Dual Multi Resolution Analysis (DMRA) with subspaces $\widetilde{V_j}$ and $\widetilde{W_j}$, such that

$$\widetilde{V_j} \perp W_j \quad and \quad V_j \perp \widetilde{W_j} \qquad (8)$$

and consequently

$$\widetilde{W_j} \perp W_{j'} \quad for \quad j \ne j' \qquad (9)$$

When semi-orthogonal wavelets are constructed from B-spline of order m, the lowest octave level $j = j_0$ is determined in[10] by

$$2^{j_o} \ge 2m - 1 \qquad (10)$$

so as to give a minimum of one complete wavelet on the interval [0,1].In this paper we will use a wavelet generated by a quartic B-spline$(m = 5)$ cardinal B-spline function. From(10), the five order B-spline lowest level, which must be an integer,is determined to $j_0 = 4$. For each level $j \ge j_0$ this constrains all octave levels to $j \ge 4$.

A. Definition

Let m and n be two positive integers and

$$a = x_{-m+1} = ... = x_0 < x_1 < ... < x_n$$
$$= x_{n+1} = ... = x_{n+m-1} = b \qquad (11)$$

be an equally spaced knots sequence. The functions

$$B_{m,j,X}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} B_{m-1,j,X}(x)$$
$$+ \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} B_{m-1,j+1,X}(x) \qquad (12)$$

and

$$B_{1,j,X}(x) = \begin{cases} 1 & x \in [x_j, x_{j+1}) \\ 0 & else \end{cases} \qquad (13)$$

are called cardinal B-spline functions of order $m \ge 2$ for the knot sequence $X = \left\{ x_k^j \right\}_{k=1-m}^{2^j+m-1}$ and $Supp[B_{m,j,X}(x)] = [x_j, x_{j+m}] \bigcap [a, b]$.
For the sake of simplicity, suppose $[a, b] = [0, n]$ and $x_j = j, j = 0, ...n$. The $B_{m,j,X}(x) = B_m(x - k), j = 0, ..., n - m$, are interior B-spline functions , while the remaining $B_{m,j,X}(x), j = -m + 1, ..., -1$ and $j = n - m + 1, ..., n - 1$ are boundary B-spline functions[14], for the bounded interval $[0, n]$. Since the boundary B-spline functions at 0 are symmetric reflections of those at n, it is sufficient to construct only the first half functions by simply replacing x with $n - x$.
By considering the interval $[a, b] = [0, 1]$, at any level $j \in Z^+$,the discrete step is 2^{-j} , and this generates $n = 2^j$ number of segments in $[0, 1]$ with knot sequence[31]

$$X^j = \begin{cases} x_{1-m}^j = x_{2-m}^j = ... = x_0^j = 0 \\ x_k^j = k2^{-j} \qquad k = 1, ..., 2^j - 1 \\ x_{2^j}^j = x_{2^j+1}^j = ... = x_{2^j+m-1}^j = 0 \end{cases} \qquad (14)$$

For each level $j \ge j_0$ the scaling function of order m can be defined as allows:

$$\varphi_{m,j,k}(x) =$$
$$\begin{cases} B_{m,j_0,k}(2^{j-j_\circ}x) & k = 1-m, ..., -1 \\ B_{m,j_0,2^j-m-k}(1-2^{j-j_\circ}x) & k = 2^j-m+1, ..., 2^j-1 \\ B_{m,j_0,0}(2^{j-j_\circ}x - 2^{-j_\circ}k) & k = 0, ..., 2^j-m \end{cases}$$
$$(15)$$

And the two-scale relation for the m-order semi-orthogonal compactly supported B-spline wavelet are defined as follows:

$$\psi_{m,j,k-m}(x) = \sum_{j=k}^{2k+2m-2} q_{k,j} B_{m,j,k-m}(x), k = 1, ..., m-1$$
$$(16)$$

$$\psi_{m,j,k-m}(x) = \sum_{j=2k-m}^{2k+2m-2} q_{k,j} B_{m,j,k-m}(x), k = m, ..., n-m+1$$
$$(17)$$

$$\psi_{m,j,k-m}(x) = \sum_{j=2k-m}^{n+k+m-1} q_{k,j} B_{m,j,k-m}(x), k = n-m+2, ..., n$$
$$(18)$$

where $q_{k,j} = q_{j-2k}$.

Hence, there are $2(m-1)$ boundary wavelets and $(n - 2m + 2)$ inner wavelet in boundary interval $[a, b]$. Finally by considering the level j with $j \geq j_0$, the B-spline wavelet in $[0, 1]$ can be expressed as follows:

$$\psi_{m,j,k}(x) =$$
$$\begin{cases} \psi_{m,j_0,k}(2^{j-j_\circ}x) & k = -m+1, ..., -1 \\ \psi_{m,j_0,2^j-2m+1-k}(1-2^{j-j_\circ}x) & k = 2^j-2m+2, ..., 2^j-m \\ \psi_{m,j_0,0}(2^{j-j_\circ}x - 2^{-j_\circ}k) & k = 0, ..., 2^j-2m+1 \end{cases}$$
$$(19)$$

The scaling functions $\varphi_{m,j,k}(x)$,occupy m segments and the wavelet functions $\psi_{m,j,k}(x)$ occupy $2m-1$ segments.

III. GENERAL ORDER B-SPLINE WAVELETS

The B-spline wavelet can be defined recursively by the convolution[10]:

$$\varphi_m(x) = \int_{-\infty}^{+\infty} \varphi_{m-1}(x-t)\varphi_1(t)dt = \int_0^1 \varphi_{m-1}(x-t)dt$$
$$(20)$$

where

$$\varphi_1(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & else \end{cases}$$
$$(21)$$

The construction of the scaling function of m-th order B-spline wavelet is based on the two scale relation:

$$\varphi_m(x) = \sum_{k=0}^{m} p_k \varphi_m(2x - k)$$
$$(22)$$

where p_k is the two scale sequence and can be expressed as a combination:

$$p_k = 2^{1-m} \binom{m}{k}, 0 \leq k < m$$
$$(23)$$

The two-scale relation for m-th order B-spline wavelets is given by:

$$\psi_m = \sum_{k=0}^{3m-2} q_k \varphi_m(2x - k)$$
$$(24)$$

$$q_k = (-1)^k 2^{1-m} \sum_{l=0}^{m} \binom{m}{l} \varphi_{2m}(k-l+1)$$
$$(25)$$

The decomposition relation for m-th order B-spline wavelet is:

$$\varphi_m(2x-l) = \sum_k \big(a_{l-2k}\varphi(x-k) + b_{l-2k}\psi(x-k)\big), l \in Z$$
$$(26)$$

where decomposition sequences $\{a_k\}$ and $\{b_k\}$ are as follows:

$$a_k = \frac{(-1)^{k+1}}{2} \sum_l q_{-k+2m-2l-1} c_{l,2m}$$
$$(27)$$

$$b_k = -\frac{(-1)^{k+1}}{2} \sum_l p_{-k+2m-2l-1} c_{l,2m}$$
$$(28)$$

In (27)and (28) the coefficents sequence $\{c_{k,m}\}$ is presented by m-th order fundamental cardinal spline functions:

$$L_m(x) = \sum_{k=-\infty}^{\infty} c_{k,m}\varphi_m(\frac{m}{2} + x - k)$$
$$(29)$$

To obtain the coefficient sequences,using an analytical relation for B-spline wavelets with order $m < 3$. For higher values of m obtaining the analytical solutions became very difficult,and for values of m greater than 5, it is impossible in the light of Abel-Ruffini theorem. Therefore, the analytical formula was omitted here. Another way of obtaining the coefficient sequences is to form the bi-infinite system of equations as follows:

$$\sum_{k=-\infty}^{\infty} c_{k,m}\varphi_m(\frac{m}{2} + j - k) = \delta_{j,0}, j \in Z$$
$$(30)$$

The coefficients sequence $\{c_{k,m}\}$ is infinite for $m \geq 3$, so that (29)does not vanish identically outside any compact set.However, these coefficients decay to zero exponentially fast as $k \to \infty$, which implies decaying to zero of (29) as $x \to \pm\infty$.

IV. QUARTIC B-SPLINE WAVELET ($m = 5$)

Quartic B-spline $\varphi_5(x)$ scaling function is given by the next recursive relation:
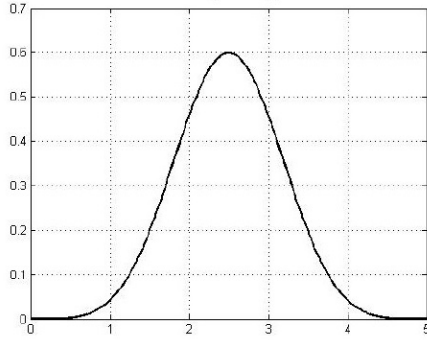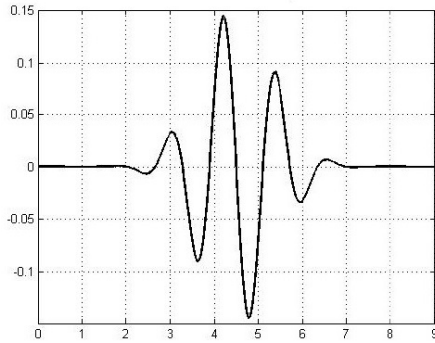
Fig. 1.   Scaling functions of quartic B-spline wavelet

Fig. 2.   Wavelet functions of quartic B-spline wavelet

$$\varphi_5(x) = \begin{cases} \dfrac{x^4}{24} & 0 \le x < 1 \\ -\dfrac{x^4}{6} + \dfrac{5x^3}{6} - \dfrac{5x^2}{4} + \dfrac{5x}{6} - \dfrac{5}{24} & 1 \le x < 2 \\ \dfrac{x^4}{4} - \dfrac{5x^3}{2} + \dfrac{35x^2}{4} - \dfrac{25x}{2} + \dfrac{155}{24} & 2 \le x < 3 \\ -\dfrac{x^4}{6} + \dfrac{5x^3}{2} - \dfrac{55x^2}{4} + \dfrac{65x}{2} - \dfrac{655}{24} & 3 \le x < 4 \\ \dfrac{x^4}{24} - \dfrac{5x^3}{6} + \dfrac{25x^2}{4} - \dfrac{125x}{6} + \dfrac{625}{24} & 4 \le x < 5 \\ 0 & else \end{cases} \tag{31}$$

where the compact support in the range $[0, m]$ referring to the property B-spline scaling functions. Two scale sequences $\{p_k\}_{k=0}^{5}$ and $\{q_k\}_{k=0}^{13}$ are as follow . Based on them two scale relations for $\varphi_5(x)$ and $\psi_5(x)$ can be constructed using(22) and(24)respectively.

$$\{p_k\}_{k=0}^{5} = \left\{ \frac{1}{16}, \frac{5}{16}, \frac{5}{8}, \frac{5}{8}, \frac{5}{16}, \frac{1}{16} \right\} \tag{32}$$

$$\{q_k\}_{k=0}^{13} = \{ \frac{1}{5806080}, -\frac{169}{1935360}, \frac{2141}{725760}, -\frac{5197}{181440},$$
$$\frac{149693}{1161216}, -\frac{54289}{165888}, \frac{74339}{145152}, -\frac{74339}{145152}, \frac{54289}{165888}, -\frac{149693}{1161216},$$
$$\frac{5197}{181440}, -\frac{2141}{725760}, \frac{169}{1935360}, \frac{1}{5806080} \} \tag{33}$$

Figs. 1 and 2 show the scaling and wavelet function for quartic B-spline wavelet.

So the corresponding scaling function is:

$$\varphi_{j,k}(x) = \begin{cases} \dfrac{(x_j-k)^4}{24} & k \le x_j < k+1 \\ -\dfrac{(x_j-k)^4}{6} + \dfrac{5(x_j-k)^3}{6} - \dfrac{5(x_j-k)^2}{4} + \\ \quad \dfrac{5(x_j-k)}{6} - \dfrac{5}{24} & k+1 \le x_j < k+2 \\ \dfrac{(x_j-k)^4}{4} - \dfrac{5(x_j-k)^3}{2} + \dfrac{35(x_j-k)^2}{4} - \\ \quad \dfrac{25(x_j-k)}{2} + \dfrac{155}{24} & k+2 \le x_j < k+3 \\ -\dfrac{(x_j-k)^4}{6} + \dfrac{5(x_j-k)^3}{2} - \dfrac{55(x_j-k)^2}{4} + \\ \quad \dfrac{65(x_j-k)}{2} - \dfrac{655}{24} & k+3 \le x_j < k+4 \\ \dfrac{(x_j-k)^4}{24} - \dfrac{5(x_j-k)^3}{6} + \dfrac{25(x_j-k)^2}{4} - \\ \quad \dfrac{125(x_j-k)}{6} + \dfrac{625}{24} & k+4 \le x_j < k+5 \\ 0 & else \end{cases} \tag{34}$$

With the respective left and right hand side boundary scaling function.The actual coordinate position x is related to x_j according to $x_j = 2^j x$.

$$\varphi_5(2x-k) = \begin{cases} \dfrac{(2x-k)^4}{24} & k/2 \le x < k/2+1/2 \\ -\dfrac{(2x-k)^4}{6} + \dfrac{5(2x-k)^3}{6} - \dfrac{5(2x-k)^2}{4} + \\ \quad \dfrac{5(2x-k)}{6} - \dfrac{5}{24} & k/2+1/2 \le x < k/2+1 \\ \dfrac{(2x-k)^4}{4} - \dfrac{5(2x-k)^3}{2} + \dfrac{35(2x-k)^2}{4} - \\ \quad \dfrac{25(2x-k)}{2} + \dfrac{155}{24} & k/2+1 \le x < k/2+3/2 \\ -\dfrac{(2x-k)^4}{6} + \dfrac{5(2x-k)^3}{2} - \dfrac{55(2x-k)^2}{4} + \\ \quad \dfrac{65(2x-k)}{2} - \dfrac{655}{24} & k/2+3/2 \le x < k/2+2 \\ \dfrac{(2x-k)^4}{24} - \dfrac{5(2x-k)^3}{6} + \dfrac{25(2x-k)^2}{4} - \\ \quad \dfrac{125(2x-k)}{6} + \dfrac{625}{24} & k/2+2 \le x < k/2+5/2 \\ 0 & else \end{cases} \tag{35}$$

$$\varphi(x) = \frac{1}{16}\varphi(2x) + \frac{5}{16}\varphi(2x-1) + \frac{5}{8}\varphi(2x-2) +$$
$$\frac{5}{8}\varphi(2x-3) + \frac{5}{16}\varphi(2x-4) + \frac{1}{16}\varphi(2x-5) \tag{36}$$

$$\varphi_{5,-4}(x) = \begin{cases} \dfrac{(16x+4)^4}{24} - \dfrac{5(16x+4)^3}{6} + \dfrac{25(16x+4)^2}{4} - \\ \quad \dfrac{125(16x+4)}{6} + \dfrac{625}{24} & 0 \le x < 1/16 \\ 0 & else \end{cases} \tag{37}$$

$$\varphi_{5,-3}(x) = \begin{cases} -\dfrac{(16x+3)^4}{6} + \dfrac{5(16x+3)^3}{2} - \dfrac{55(16x+3)^2}{4} + \\ \quad \dfrac{65(16x+3)}{2} - \dfrac{655}{24} & 0 \le x < 1/16 \\ \dfrac{(16x+3)^4}{24} - \dfrac{5(16x+3)^3}{6} + \dfrac{25(16x+3)^2}{4} - \\ \quad \dfrac{125(16x+3)}{6} + \dfrac{625}{24} & 1/16 \le x < 1/8 \\ 0 & else \end{cases} \tag{38}$$

$$\varphi_{5,-2}(x) = \begin{cases} \dfrac{(16x+2)^4}{4} - \dfrac{5(16x+2)^3}{2} + \dfrac{35(16x+2)^2}{4} - \\ \quad \dfrac{25(16x+2)}{2} + \dfrac{155}{24} & 0 \le x < 1/16 \\ -\dfrac{(16x+2)^4}{6} + \dfrac{5(16x+2)^3}{2} - \dfrac{55(16x+2)^2}{4} + \\ \quad \dfrac{65(16x+2)}{2} - \dfrac{655}{24} & 1/16 \le x < 1/8 \\ \dfrac{(16x+2)^4}{24} - \dfrac{5(16x+2)^3}{6} + \dfrac{25(16x+2)^2}{4} - \\ \quad \dfrac{125(16x+2)}{6} + \dfrac{625}{24} & 1/8 \le x < 3/16 \\ 0 & else \end{cases} \tag{39}$$

$\varphi_{5,-1}(x) =$

$$
\begin{cases}
-\frac{(16x+1)^4}{6} + \frac{5(16x+1)^3}{6} - \frac{5(16x+1)^2}{4} + \\
\quad \frac{5(16x+1)}{6} - \frac{5}{24} & 0 \le x < 1/16 \\
\frac{(16x+1)^4}{4} - \frac{5(16x+1)^3}{2} + \frac{35(16x+1)^2}{4} - \\
\quad \frac{25(16x+1)}{2} + \frac{155}{24} & 1/16 \le x < 1/8 \\
-\frac{(16x+1)^4}{6} + \frac{5(16x+1)^3}{2} - \frac{55(16x+1)^2}{4} + \\
\quad \frac{65(16x+1)}{2} - \frac{655}{24} & 1/8 \le x < 3/16 \\
\frac{(16x+1)^4}{24} - \frac{5(16x+1)^3}{6} + \frac{25(16x+1)^2}{4} - \\
\quad \frac{125(16x+1)}{6} + \frac{625}{24} & 3/16 \le x < 1/4 \\
0 & else
\end{cases}
$$

(40)

$\varphi_{5,12}(x) =$

$$
\begin{cases}
-\frac{(16x-12)^4}{6} + \frac{5(16x-12)^3}{2} - \frac{55(16x-12)^2}{4} + \\
\quad \frac{65(16x-12)}{2} - \frac{655}{24} & 15/16 \le x < 1 \\
\frac{(16x-12)^4}{4} - \frac{5(16x-12)^3}{2} + \frac{35(16x-12)^2}{4} - \\
\quad \frac{25(16x-12)}{2} + \frac{155}{24} & 7/8 \le x < 15/16 \\
-\frac{(16x-12)^4}{6} + \frac{5(16x-12)^3}{6} - \frac{5(16x-12)^2}{4} + \\
\quad \frac{5(16x-12)}{6} - \frac{5}{24} & 13/16 \le x < 7/8 \\
\frac{(16x-12)^4}{24} & 12/16 \le x < 13/16 \\
0 & else
\end{cases}
$$

(41)

$\varphi_{5,13}(x) =$

$$
\begin{cases}
\frac{(16x-13)^4}{4} - \frac{5(16x-13)^3}{2} + \frac{35(16x-13)^2}{4} - \\
\quad \frac{25(16x-13)}{2} + \frac{155}{24} & 15/16 \le x < 1 \\
-\frac{(16x-13)^4}{6} + \frac{5(16x-13)^3}{6} - \frac{5(16x-13)^2}{4} + \\
\quad \frac{5(16x-13)}{6} - \frac{5}{24} & 7/8 \le x < 15/16 \\
\frac{(16x-13)^4}{24} & 13/16 \le x < 7/8 \\
0 & else
\end{cases}
$$

(42)

$\varphi_{5,14}(x) =$

$$
\begin{cases}
-\frac{(16x-14)^4}{6} + \frac{5(16x-14)^3}{6} - \frac{5(16x-14)^2}{4} + \\
\quad \frac{5(16x-14)}{6} - \frac{5}{24} & 15/16 \le x < 1 \\
\frac{(16x-14)^4}{24} & 7/8 \le x < 15/16 \\
0 & else
\end{cases}
$$

(43)

$$
\varphi_{5,15}(x) =
\begin{cases}
\frac{(16x-15)^4}{24} & 15/16 \le x < 1 \\
0 & else
\end{cases}
$$

(44)

$$
\psi(x) = \sum_{k=0}^{13} q_k \varphi(2x - k)
$$

(45)

$\psi_5(x) =$

$$
\begin{cases}
\frac{1}{8709120}x^4 & 0 \le x < 1/2 \\
-\frac{73}{1244160}x^4 + \frac{1}{8505}x^3 - \frac{1}{11340}x^2 + \\
\quad \frac{1}{34020}x - \frac{1}{272160} & 1/2 \le x < 1 \\
\frac{9581}{4354560}x^4 - \frac{19417}{2177280}x^3 + \frac{1303}{96768}x^2 - \\
\quad \frac{19609}{2177280}x + \frac{6547}{2903040} & 1 \le x < 3/2 \\
-\frac{118927}{4354560}x^4 + \frac{366119}{2177280}x^3 - \frac{186253}{483840}x^2 + \\
\quad \frac{121121}{311040}x - \frac{427181}{2903040} & 3/2 \le x < 2 \\
\frac{759239}{4354560}x^4 - \frac{3146561}{2177280}x^3 + \frac{6466601}{1451520}x^2 - \\
\quad \frac{13202873}{2177280}x + \frac{26819897}{8709120} & 2 \le x < 5/2 \\
-\frac{2980409}{4354560}x^4 + \frac{5183893}{725760}x^3 - \frac{13426333}{483840}x^2 + \\
\quad \frac{134472511}{2903040}x - \frac{123314239}{4354560} & 5/2 \le x < 3 \\
\frac{7873577}{4354560}x^4 - \frac{16524079}{725760}x^3 + \frac{7385369}{69120}x^2 - \\
\quad \frac{17868671}{80640}x + \frac{497668543}{2903040} & 3 \le x < 7/2 \\
-\frac{14714327}{4354560}x^4 + \frac{108543091}{2177280}x^3 - \frac{56901557}{207360}x^2 + \\
\quad \frac{1454458651}{2177280}x - \frac{5286189059}{8709120} & 7/2 \le x < 4 \\
\frac{15619}{3402}x^4 - \frac{33822017}{435456}x^3 + \frac{15828929}{32256}x^2 - \\
\quad \frac{597598433}{435456}x + \frac{277413649}{193536} & 4 \le x < 9/2 \\
-\frac{15619}{3402}x^4 + \frac{38150335}{435456}x^3 - \frac{20157247}{32256}x^2 + \\
\quad \frac{859841695}{435456}x - \frac{64472345}{27648} & 9/2 \le x < 5 \\
\frac{14714327}{4354560}x^4 - \frac{4466137}{62208}x^3 + \frac{165651247}{290304}x^2 - \\
\quad \frac{875490655}{435456}x + \frac{4614904015}{1741824} & 5 \le x < 11/2 \\
-\frac{7873577}{4354560}x^4 + \frac{30717383}{725760}x^3 - \frac{179437319}{483840}x^2 + \\
\quad \frac{16606729}{11520}x - \frac{869722273}{414720} & 11/2 \le x < 6 \\
\frac{2980409}{4354560}x^4 - \frac{12698561}{725760}x^3 + \frac{16211669}{96768}x^2 - \\
\quad \frac{19138891}{26880}x + \frac{3289787993}{2903040} & 6 \le x < 13/2 \\
-\frac{379619}{2177280}x^4 + \frac{219161}{45360}x^3 - \frac{48550091}{967680}x^2 + \\
\quad \frac{134333585}{580608}x - \frac{206319601}{516096} & 13/2 \le x < 7 \\
\frac{118927}{4354560}x^4 - \frac{887291}{1088640}x^3 + \frac{26470067}{2903040}x^2 - \\
\quad \frac{78936197}{1741824}x + \frac{1176488101}{13934592} & 7 \le x < 15/2 \\
-\frac{1915}{870912}x^4 + \frac{38237}{544320}x^3 - \frac{2442883}{2903040}x^2 + \\
\quad \frac{7802653}{1741824}x - \frac{124594649}{13934592} & 15/2 \le x < 8 \\
\frac{503}{8709120}x^4 - \frac{1069}{544320}x^3 + \frac{72701}{2903040}x^2 - \\
\quad \frac{1236079}{8709120}x + \frac{21016259}{69672960} & 8 \le x < 17/2 \\
\frac{1}{8709120}x^4 - \frac{1}{241920}x^3 + \frac{1}{17920}x^2 - \\
\quad \frac{3}{8960}x + \frac{27}{35840} & 17/2 \le x < 9 \\
0 & else
\end{cases}
$$

V. FUNCTION APPROXIMATING USING SCALING FUNCTION

For any positive integer $M = j_0$, a function $f(x)$ defined over $[0, 1]$ may be represented by B-spline scaling functions as:

$$
f(x) = \sum_{k=-4}^{2^M - 1} s_k \varphi_{M,k} = S^T \Phi_M
$$

(46)

where

$$
S = [s_{-4}, s_{-3}, ..., s_{2^M - 1}]
$$
$$
\Phi_M = [\Phi_{M,-4}, \Phi_{M,-3}, ..., \Phi_{M,2^M-1}]
$$

(47)

with

$$
s_k = \int_0^1 f(x)\tilde{\varphi}_{M,k}(x)dx, \ k = -4, -3, ..., 2^M - 1
$$

(48)

where $\tilde{\varphi}_{M,k}(x)$ are dual functions of $\varphi_{M,k}(x)$.

$$\widetilde{\Phi} = T_\Phi \Phi, \qquad T_\Phi = (P_M)^{-1} \qquad (49)$$

These can be obtained by linear combinations of $\varphi_{M,k}(x)$,$k = -4, -3, ..., 2^M - 1$, as follows.Let $\tilde{\Phi}_M$ be the dual functions of Φ_M given by:

$$\tilde{\Phi}_M = [\tilde{\varphi}_{M,-4}, \tilde{\varphi}_{M,-3}, ..., \tilde{\varphi}_{M,2^M-1}] \qquad (50)$$

Using (47)and(48)we get:

$$\int_0^1 \tilde{\Phi}_M \Phi_M^T dx = I_1 \qquad (51)$$

where I_1 is$(2^M + 4) \times (2^M + 4)$ identity matrix. Let:

$$P_M = \int_0^1 \Phi_M \Phi_M^T dx \qquad (52)$$

The entry $(P_M)_{i,j}$ of the matrix P_M is calculated from:

$$\int_0^1 \varphi_{M,i}(x)\varphi_{M,j}(x)dx \qquad (53)$$

From(51)and(52) we get:

$$\tilde{\Phi}_M = (P_M)^{-1}\Phi_M \qquad (54)$$

Furthermore,a function $f(x)$ defined over$[0,1]$ may be represented by B-spline wavelets as:

$$f(x) = \sum_{k=-4}^{2^j-1} c_{m,k}\varphi_{5,k}(x) + \sum_{j=5}^{\infty} \sum_{k=-4}^{2^j-5} d_{j,k}\psi_{j,k}(x) \qquad (55)$$

If the infinite series in(55) is truncated at M, then(55) cam be written as:

$$f(x) \simeq \sum_{k=-4}^{15} c_{m,k}\varphi_{5,k}(x) + \sum_{j=5}^{M} \sum_{k=-4}^{2^j-5} d_{j,k}\psi_{j,k}(x) = C^T\Psi(x) \qquad (56)$$

where $\varphi_{5,k}$ and $\psi_{j,k}$ are scaling and wavelets functions, respectively C and Ψ are$(2^M + 4) \times 1$ vectors given by:

$$C = [c_{-4}, c_{-3}, ..., c_{15}, d_{5,-4}, d_{5,-3}$$
$$, ..., d_{5,7}, ..., d_{M,-M+1}, ..., d_{2^M,-5}] \qquad (57)$$

$$\Psi = [\varphi_{5,-4}, \varphi_{5,-3}, ..., \varphi_{5,15}, \psi_{5,-4},$$
$$\psi_{5,-3}, ..., \psi_{5,7}, ..., \psi_{M,-M+1}, ..., \psi_{M,2^M-5}] \qquad (58)$$

with

$$c_k = \int_0^1 f(x)\tilde{\varphi}_{5,k}(x)dx, \ k = -4, -3, ...15 \qquad (59)$$

$$d_{j,k} = \int_0^1 f(x)\tilde{\psi}_{j,k}(x)dx,$$
$$j = 5, 4, ...M, \ k = -4, -3, ..., 2^j - 5 \qquad (60)$$

where$\tilde{\varphi}_{5,k}(x)$ and $\tilde{\psi}_{j,k}(x)$ are dual functions of $\varphi_{5,k}(x)$ and $\psi_{j,k}(x)$ respectively.These can be obtained by linear combinations as follows:

$$\Phi = [\varphi_{5,-4}(x), \varphi_{5,-3}(x), ..., \varphi_{5,15}(x)]^T \qquad (61)$$

$$\Psi = [\psi_{5,-4}(x), \psi_{5,-3}(x), ..., \psi_{M,2^M-5}(x)]^T \qquad (62)$$

VI. THE OPERATIONAL MATRICES OF DERIVATIVE

The differentiation of vectors Φ_M in (47)can be expressed as:

$$\Phi'_M = D_\Phi \Phi_M \qquad (63)$$

where $D_\Phi$is $(2^M + 4) \times (2^M + 4)$ operational matrices of derivative for B-spline scaling functions.

$$D_\Phi = \int_0^1 \Phi'_M(t)\tilde{\Phi}_M^T(t)dt$$
$$= (\int_0^1 \Phi'_M(t)\Phi_M^T(t)dt)(P_M)^{-1} = F(P_M)^{-1} \qquad (64)$$

where

$$F = \int_0^1 \Phi'_M(t)\Phi_M^T(t)dt \qquad (65)$$

F is matrices $(2^M + 4) \times (2^M + 4)$ as follows:

$$\begin{bmatrix} \int_0^1 \varphi'_{M,-4}(t)\varphi_{M,-4}(t)dt & \cdots & \int_0^1 \varphi'_{M,-4}(t)\varphi_{M,2^M-1}(t)dt \\ \vdots & \ddots & \vdots \\ \int_0^1 \varphi'_{M,2^M-1}(t)\varphi_{M,-4}(t)dt & \cdots & \int_0^1 \varphi'_{M,2^M-1}(t)\varphi_{M,2^M-1}(t)dt \end{bmatrix} \qquad (66)$$

Since the element $\varphi_{M,k}$ in the vector Φ_M given in(47) is nonzero between $\frac{k}{2^M}$ and $\frac{k+5}{2^M}$,for any entries of $F_{j,k}$ we have:

$$F_{j,k} = \int_0^1 \varphi'_{M,j}(t)\varphi_{M,k}(t)dt = \int_{\frac{k}{2^M}}^{\frac{k+5}{2^M}} \varphi'_{M,j}(t)\varphi_{M,k}(t)dt \qquad (67)$$

VII. THE OPERATIONAL MATRICES OF DERIVATIVE USING WAVELETS

The differentiation of vectors Ψ in (58) can be expressed as:

$$\Psi' = D_\Psi \Psi \qquad (68)$$

where D_Ψ is $(2^M + 4) \times (2^M + 4)$ operational matrices of derivative for B-spline wavelets. The matrix D_Ψ can be obtained by considering:

$$\Psi = H\Phi_M \qquad (69)$$

where H is a $(2^M + 4) \times (2^M + 4)$ matrix, which can be calculated as follows:

$$\Phi_j = [\varphi_{j,-4}, \varphi_{j,-3}, ..., \varphi_{j,2^j-1}]^T$$
$$\Psi_j = [\psi_{j,-4}, \psi_{j,-3}, ..., \psi_{j,2^j-5}]^T \qquad (70)$$

Using (32) and (70) we get

$$\Phi_j = \alpha_j \Phi_{j+1} \qquad (71)$$

where $\alpha_j, j = 5, ...,$ is $(2^j + 4) \times (2^{j+1} + 4)$ matrix. From (33) and (70) we have:

$$\Psi_j = U_j \Phi_{j+1} \qquad (72)$$

where $U_j, j = 4, ...,$ is $(2^j) \times (2^{j+1}+4)$ matrix. Using (62), (71) and (72) matrix H obtain as follows:

$$\begin{bmatrix} \alpha_5 \times \alpha_4 \times ... \times \alpha_M \\ - - - - - - - - \\ U_5 \times \alpha_4 \times ... \times \alpha_M \\ - - - - - - - - \\ \vdots \\ U_{M-2} \times \alpha_{M-1} \times \alpha_M \\ - - - - - - - - \\ U_{M-1} \times \alpha_M \\ - - - - - - - - \\ U_M \end{bmatrix} \qquad (73)$$

From (64), (65) and (73) we get:

$$\Psi' = H\Phi'_M = HD_\Phi \Phi_M = HF(P_M)^{-1}\Phi_M = D_\Psi \Psi \quad (74)$$

VIII. FIRST ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

In this section we solve first order Fredholm integro-differential equations of the form (1) by using B-spline wavelets. Let:

$$w(x) = \int_0^1 k(x,t)y(t)dt \qquad (75)$$

For this purpose, we first approximate $y(x)$ and $w(x)$ to expand (56) as:

$$y(x) = C^T \Psi(x) \qquad (76)$$

$$w(x) = \int_0^1 k(x,t)C^T \Psi(t)dt \qquad (77)$$

where $\Psi(x)$ is defined in (58) and C is $(2^M+4) \times 1$ unknown vector defined similarly to C in (57). We can approximate (77) using quadrature Newton-Cotes integration techniques as:

$$w(x) = \int_0^1 k(x,t)C^T \Psi(t)dt = \sum_{i=1}^n \omega_i k(x,t_i)C^T \Psi(t_i) \quad (78)$$

where ω_i and t_i are weight and nodes of Newton-Cotes integration method. For approximate $y'(x)$ to expand (68) as:

$$y'(x) = C^T \Psi'(x) = C^T D_\Psi \Psi(x) \qquad (79)$$

From (1), (77) and (78), we get:

$$g(x)C^T D_\Psi \Psi(x) = f(x) + w(x) \qquad (80)$$

Also using boundary values in (1) and (76) we have

$$C^T \Psi(0) = y_0 \qquad (81)$$

To find the solution $y(x)$ in (76) we first collocate (80) in $x_i = (2i-1)/(2^{M+2}-1), i = 1, ..., 2^{M+1}$, the resulting equation generates 2^{M+1} algebraic equations. The total unknowns for vector C in (76) is $2^{M+1}+2$. These can be obtained by using (80), (81).

IX. ERROR CONSIDERATION

In this section, we found an error bound for the presented method.

A. Theorem 1

Assume that $f \in C^5[0,1]$ is represented by quartic B-spline wavelets as (56), where has 5 vanishing moments, then [10]

$$|d_{j,k}| \leq \mu\varepsilon \frac{2^{-6j}}{5!} \qquad (82)$$

where $\mu = \max |f^{(5)}(t)|_{t\in[0,1]}$ and $\varepsilon = \int_0^1 |x^5 \tilde{\psi}_5(x)|dx$.

B. Theorem 2

Consider the previous theorem assume that $e_j(x)$ be error of approximation in space V_j, then

$$|e_j(x)| = O(2^{-5j}) \qquad (83)$$

Thus, order of error depend on the level j. Obviously, for larger level of j, the error of approximation will be smaller.

C. Theorem 3

For the m-th order B-spline wavelet the approximation error decreases with the m-th power of the scale 2^j [6]

$$\|f - P_j f\| \le C2^{-jm} \left\|f^{(m)}\right\| \qquad (84)$$

Specifically we can derive the following asymptotic relation,

$$\lim_{j \to \infty} \|f - P_j f\| = C_m 2^{-jm} \left\|f^{(m)}\right\| \qquad (85)$$

Where the constant C_m is the same for all spline wavelet transforms of a given order m, and is given by [31]

$$C_m = \sqrt{\frac{B_{2m}}{(2m)!}} \qquad (86)$$

Where B_{2m} is Bernouillis number of order $2m$.

D. Theorem 4

Assume $K_1, K_2 \in L^2$ in two dimensional rectangle $[0,1] \times [0,1]$ and $g_1, g_2 \in ([0,1] \times [0,1])$. If y and y_j are the exact and approximate solution (obtained by m-order B-spline wavelet) of (1), respectively, then

$$\|y(x) - y_j(x)\| \le B2^{-jm} \left\|y^{(m)}\right\| \qquad (87)$$

X. NUMERICAL EXAMPLES

We applied the method presented in this paper and solved three examples . This method differs from the collocation procedure with chebyshev wavelets presented in [27] and hybrid taylor and block-pulse functions given in [19] and thus these could be used as a basis for comparison.

1) Example 1: Consider first-order Fredholm integro-differential equation

$$\begin{cases} y'(x) = -\cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2}\sin(4\pi x) \\ \quad + \int\limits_0^1 \sin(4\pi x + 2\pi t)y(t)dt + y(x) \\ \quad\quad\quad y(0) = 1 \end{cases} \qquad (88)$$

The exact solution of this problem is $y(x) = \cos(2\pi x)$. We applied the method presented and solved equation. The absolute error for $M = 4, 5$ are shown in Table I .

2) Example 2: Consider first-order Fredholm integro-differential equation

$$\begin{cases} y'(x) = \frac{1 - e^{1+x}}{1+x} + \int\limits_0^1 e^{xt}y(t)dt + y(x) \\ \quad\quad\quad y(0) = 1 \end{cases} \qquad (89)$$

The exact solution of this problem is $y(x) = e^x$. The absolute error for $M = 4, 5$ are shown in Table II .

TABLE I
EXACT AND ABSOLUTE ERROR VALUES WITH M=4,5

x_i	$M = 4$	$M = 5$	$Exact$
0	0.026×10^{-8}	0.011×10^{-8}	1.00000000
0.1	0.0194×10^{-8}	0.0168×10^{-8}	0.80901699
0.2	0.0166×10^{-8}	0.016×10^{-8}	0.30901699
0.3	0.012×10^{-8}	0.010×10^{-8}	-0.30901699
0.4	0.0194×10^{-8}	0.015×10^{-9}	-0.80901699
0.5	0.0786×10^{-8}	0.066×10^{-8}	-1.00000000
0.6	0.001×10^{-8}	0.012×10^{-9}	-0.80901699
0.7	0.0316×10^{-8}	0.029×10^{-9}	-0.30901699
0.8	0.017×10^{-8}	0.0116×10^{-9}	0.30901699
0.9	0.0293×10^{-8}	0.0066×10^{-8}	0.80901699
1	0.014×10^{-8}	0.0123×10^{-8}	1.00000000

TABLE II
EXACT AND ABSOLUTE ERROR VALUES WITH M=4,5

x_i	$M = 4$	$M = 5$	$Exact$
0	0.045×10^{-8}	0.023×10^{-10}	1.00000000
0.1	1.010×10^{-8}	0.0010×10^{-10}	1.10517092
0.2	1.0554×10^{-9}	0.024×10^{-10}	1.22140276
0.3	0.052×10^{-9}	0.022×10^{-10}	1.34985881
0.4	0.037×10^{-8}	0.007×10^{-10}	1.49182470
0.5	0.068×10^{-9}	0.018×10^{-10}	1.64872127
0.6	0.027×10^{-8}	0.015×10^{-10}	1.82211880
0.7	0.06×10^{-9}	0.006×10^{-10}	2.01375271
0.8	0.011×10^{-9}	0.0126×10^{-10}	2.22554093
0.9	0.0292×10^{-9}	0.0042×10^{-10}	2.45960311
1	0.01×10^{-9}	0.003×10^{-10}	2.71828183

TABLE III
EXACT AND ABSOLUTE ERROR VALUES WITH M=4,5

x_i	$M = 4$	$M = 5$	$Exact$
0	0.0093×10^{-8}	0.006×10^{-8}	0.0
0.1	0.0095×10^{-9}	0.0045×10^{-9}	0.09531018
0.2	0.012×10^{-9}	0.0036×10^{-9}	0.18232156
0.3	0.012×10^{-9}	0.01×10^{-10}	0.26236426
0.4	0.076×10^{-8}	0.092×10^{-9}	0.33647224
0.5	0.126×10^{-8}	0.05×10^{-10}	0.40546511
0.6	0.006×10^{-9}	0.003×10^{-9}	0.47000363
0.7	0.15×10^{-9}	0.076×10^{-9}	0.53062825
0.8	0.076×10^{-9}	0.004×10^{-9}	0.58778666
0.9	0.008×10^{-9}	0.001×10^{-9}	0.64185389
1	0.006×10^{-9}	0.0066×10^{-9}	0.69314718

3) Example 3:

$$\begin{cases} y'(x) = \frac{1}{1+x} - \frac{x}{2} - \ln(x+1) + \frac{1}{(\ln 2)^2} \int\limits_0^1 \frac{x}{1+t}y(t)dt + y(x) \\ \quad\quad\quad y(0) = 0 \end{cases} \qquad (90)$$

The exact solution of this problem is $y(x) = \ln(1 + x)$. The absolute error for $M = 4, 5$ are shown in Table III .

XI. Conclusion

In the present work, a technique has been developed for solving first-order Fredholm integro-differential equations. As the B-spline wavelets have the property of semi-orthogonal on the interval [0, 1], we can combine various bases with quartic B-spline wavelets to produce Hybrid functions, with the property of semi-orthogonality . The same approach can be used to solve other problems. The operational matrices of derivative for B-spline scaling functions and wavelets are given. The problem has been reduced to solving a system of algebraic equations and applications are demonstrated through numerical examples. The numerical examples show that the accuracy improves with increasing the M in quartic B-spline wavelets, and thus for better results, using the larger M is recommended.

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Professor Khosrow Maleknejad obtained his MSc degree in Applied Mathematics from Tehran University, Iran, in 1972 and received his PhD degree in Applied Mathematics in Numerical Analysis area from the University of Wales, Aberystwyth, UK in 1980. In September 1972, he joined the School of Mathematics at Iran University of Science and Technology(IUST), Narmak, Tehran,Iran. Where he was previously lecturer , assistant Professor and associate professor at IUST. He has been a professor since 2002 at IUST. He was a Visiting Professor at the University of California at Los Angeles in 1990. His research interests include numerical analysis in solving ill-posed problems and solving Fredholm and Volterra integral equations. He has authored as a Editor-in-chief of the International Journal of Mathematical Sciences, which publishers by Springer.He is a member of the AMS. External link: http://webpages.iust.ac.ir/maleknejad/.

Yaser Rostami obtained his MSc degree in Mathematics from the Islamic Azad University ,Karaj Branch,Iran(KIAU) in 2007, He is a PhD degree student in Applied Mathematics in Numerical Analysis area from the KIAU. In 2008, he joined the Faculty of the Department Basic Science at Islamic Azad University, Malard Branch, Iran(IAUMALARD). Where he was previously lecturer at IAUMALARD. His research interests include numerical analysis in solving Integral Equations and Wavelets. He has authored of several books of Calculus and Differential Equation .He is a member of the IMS.