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Stress Solitary Waves Generated by a Second-Order Polynomial Constitutive Equation

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***Abstract*—**In this paper, a nonlinear constitutive law and a curve fitting, two relationships between the stress-strain and the shear stress-strain for sandstone material were used to obtain a second-order polynomial constitutive equation. Based on the established polynomial constitutive equations and Newton's second law, a mathematical model of the non-homogeneous nonlinear wave equation under an external pressure was derived. The external pressure can be assumed as an impulse function to simulate a real earthquake source. A displacement response under nonlinear two-dimensional wave equation was determined by a numerical method and computer-aided software. The results show that a suit pressure in the sandstone generates the phenomenon of stress solitary waves.

***Keywords*—**Polynomial constitutive equation, solitary.

I. Introduction

THE nonlinear Schordinger equation (NLS) [1] was first derived from shallow water equation and used to describe one dimensional soliton in physics. Nowadays, solitons are widely used to physics, mathematics, biology, chemistry, and communications [2]-[7]. However, a soliton in the earthquake's investigation literature is still quite rare.

A construction of the nonlinear relationship between stress and strain for polymer material was used for the solution of the nonlinear stress-strain problem before a material function was obtained by experiment [8]. Yoshinaka [9] employed experiment method to non-linear stress and strain dependent behavior of soft rocks under cycle triaxial conditions. Typical stress-strain and shear stress-strain curves for sandstone in Lab. test were presented by Chang and Jeng [10]. In elastic region, the material was strongly non-linear curve. Then, two non-linear stress-strain and shear stress-strain constitute equations must be used to derive the wave equation of earthquake dynamics at the kilometer length scale.

Our paper used the curve fitting stress-strain curve of laboratory rock material to establish nonlinear wave equation with external source. The curve fitting stress-strain and shear stress-shear strain curves comes from Chang and Jeng [10]. One needs to find a curve to pass certain data points. Specifically, a polynomial curve-fitting algorithm is used to fit the stress-strain and shear stress-shear strain relation of sandstone from a laboratory text [10]. Based on Newton's second law and the stress-strain curve equation, a two-dimensional nonlinear wave equation was established. Through a numerical method and computer-aided software, a displacement response under nonlinear two-dimensional wave equation was evaluated. A type of external source was used to simulate the earthquake force. The results show that the sandstone can generate bright and darken stress soliton under the external force stimulus.

II. Mathematical Models of the Nonlinear Constitutive Relations

Typical stress-strain and shear stress-strain curves for sandstone in triaxial compressive test were presented by [10], as shown in Figs. 1 (a) and (b). In Figs. 1 (a) and (b), the chosen stress-strain and shear stress-strain curves were divided by ten points on these curves. Based on above information, a second order polynomial with three unknowns constant is used to fit stress-strain and shear stress-strain curves in nonlinear region of Figs. 1 (a) and (b). Then stress-strain relation and shear stress-strain relation can be represented in the form of

$$\sigma_x = a_0 + a_1\varepsilon_x + a_2\varepsilon_x^{\ 2}, \tag{1}$$

$$\sigma_y = b_0 + b_1\varepsilon_y + b_2\varepsilon_y^{\ 2}, \tag{2}$$

$$\tau_{xy} = c_0 + c_1\gamma + c_2\gamma^2, \tag{3}$$

$$\tau_{yx} = d_0 + d_1\gamma + d_2\gamma^2, \tag{4}$$

where σ_x and σ_y represent a stress at x-axis and y-axis directions, respectively. Symbols ε_x and ε_y were a strain represented by $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$, respectively. u=u(x, y, t) and v=v(x, y, t) were displacement at x-axis and y-axis directions, respectively. a_0, a_1, a_2, b_0, b_1, and b_2 were unknown constants and determined by curve fitting conditions. Similarly, τ represents a shear stress. The first subscript in the notation refers to the plane on which the shear stress acts; the second subscript is the direction of shear stress. The symbol γ is a shear strain represented by $\frac{1}{2}(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})$. c_0, c_1, and c_2 are unknown constants and determined by curve fitting conditions. For simplicity, the selected stress-strain curves in the x direction

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and y direction are the same. In other words, the coefficients b_0, b_1, and b_2 in (2) are equal to the coefficients a_0, a_1, and a_2 in (1). Based on curve fitting method, (1) and (3) must pass through the ten points in region $\overline{ab}$ and $\overline{cd}$ on the stress-strain and shear stress-strain curves in Figs. 1 (a) and (b), respectively.

Substituting ten points on Figs. 1 (a) and (b) into (1)~(4) and through *Mathematica* software package, the unknown coefficients from (1) to (4) can be evaluated and the obtained mathematical equations of nonlinear stress-strain and shear stress-strain curves were respectively shown in Figs. 2 and 3.

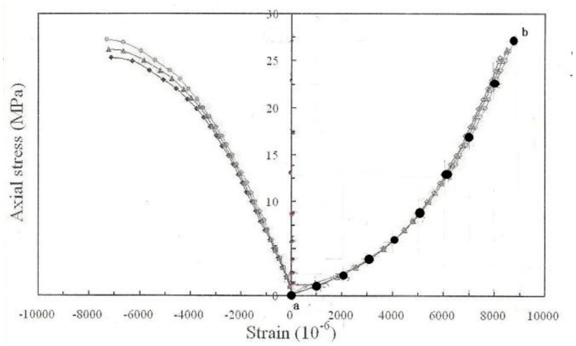


Fig. 1 (a) The relationship between stress and strain redraw from Chang and Jeng [10]

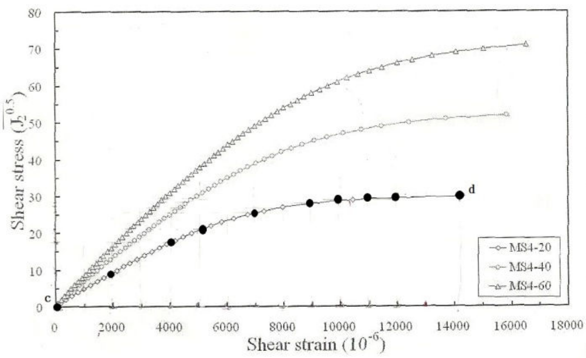


Fig. 1 (b) The relationship between shear stress and shear strain redraw from Chang and Jeng [10]

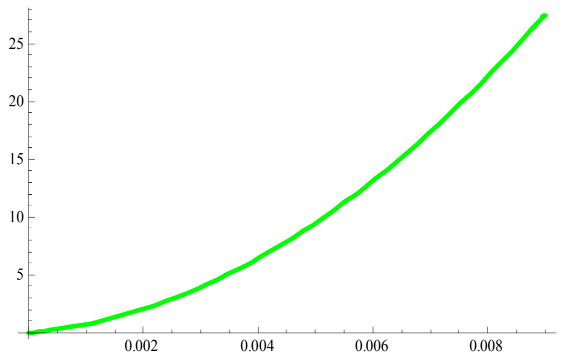


Fig. 2 The nonlinear stress-strain curves of the second-order polynomial

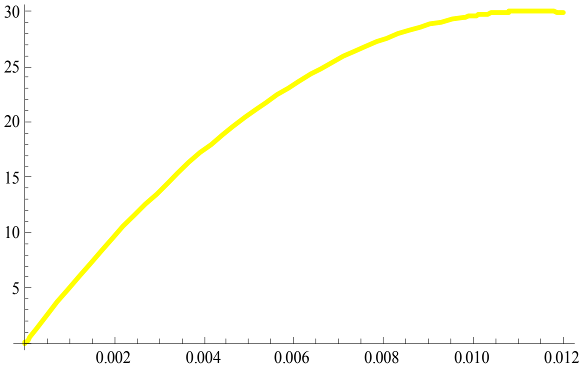


Fig. 3 The nonlinear shear stress-strain curves of the second-order polynomial

III. Mathematical Modeling

As shown in Fig. 4, the compression force can be an impulse function to simulate a real crust condition. One also assumes that σ_x, σ_y, τ_{xy}, and τ_{yx} are independent of z. A body force is neglect. This model in Fig. 4 has a unit depth. The driving equations of equilibrium by Newton's second law for the rectangular plate are given by

$$\frac{\partial \sigma_x}{\partial x}dxdy + \frac{\partial \tau_{xy}}{\partial y}dxdy + P\times U(x-x_1, x-x_2)\times U(y-y_1, y-y_2)\times \delta(t-t_1)dxdy = \rho\frac{\partial^2 u}{\partial t^2}dxdy \quad (5)$$

Since *dxdy* is not zero, the quantity in this term must vanish. Thus we can obtain

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + P\times U(x-x_1, x-x_2)\times U(y-y_1, y-y_2)\times \delta(t-t_1) = \rho\frac{\partial^2 u}{\partial t^2} \quad (6)$$

where u=$u(x, y, t)$ is displacement at x direction. U is an impulse function. P is an external force. τ_{xy} is a shear stress. The first subscript in the notation refers to the plane on which the shear stress acts; the second subscript is the direction of shear system. $\delta(t-t_1)$ is an impulse function. The normal stress on the x surface is defined by σ_x.

Using the Newton's second law for driving equations of equilibrium in the y direction yields the relation

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2} \quad (7)$$

where v=$v(x, y, t)$ is a displacement in the y direction. Substituting (1) and (3) into (6) we can obtain:

$$a_1\frac{\partial^2 u}{\partial x^2}+2\times a_2(\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x^2}+c_1(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2})+2\times c_2(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})$$
$$(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2})+P\times U(x-x_1,x-x_2)\times U(y-y_1,y-y_2)\times\delta(t-t_1) \quad (8)$$
$$=\rho\frac{\partial^2 u}{\partial t^2}$$

Similarly, substituting (2) and (4) into (18) we can obtain

$$b_1\frac{\partial^2 v}{\partial y^2}+2\times b_2(\frac{\partial v}{\partial y})\frac{\partial^2 v}{\partial y^2}+c_1(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2})+2\times c_2(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})$$
$$(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2})=\rho\frac{\partial^2 v}{\partial t^2} \quad (9)$$

The nonlinear feature of the plate of the two-dimensional sandstone is denoted by displacements $u=u(x, y, t)$ and $v=v(x, y, t)$. Equations (8) and (9) were nonlinear wave equation generated by the nonlinear region of the stress-strain curve and the shear stress-strain curve. The coefficients a_1, a_2, c_1, c_2 b_1, and b_2 were determined in Section II. The initial conditions were assumed by setting the displacement, $u(x,\ y,\ 0)=0$, $v(x,\ y,\ 0)=0$ and particle velocity, $\frac{\partial u(x,\ y,\ t)}{\partial t}|_{t=0}=0$ and $\frac{\partial v(x,\ y,\ t)}{\partial t}|_{t=0}=0$. The boundary conditions at the left and right faces of Fig. 5 were respectively given as $u(0, y, t)=0$ and $u(L, y, t)=0$. The boundary conditions at the top and bottom faces of Fig. 5 were respectively given as $v(x, 0, t)=0$ and $v(x, w, t)=0$. To determine whether the established mathematical modeling has solitary wave, therefore, the length and width of L=1000m and W=1000m were tested at the beginning of the study. Symbol ρ is a density. **P** is an external force.

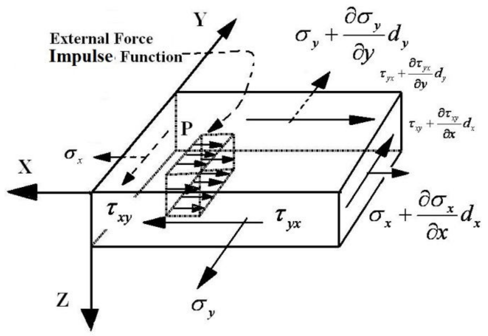


Fig. 4 The rectangular plate with unit thickness and edge lengths L and W

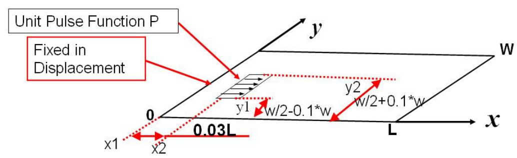


Fig. 5 The plate was subjected to a uniform compression force in the x-direction

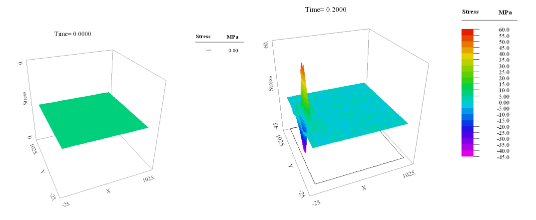
IV. Numerical Illustration

Based on the set models and assigned initial and boundary conditions, (8) and (9) were solved using finite element method and second-order implicit back difference formula. Applying a modified Newton-Raphson iteration procedure for time-stepping, a non-linear formula for (8) and (9) was obtained. In this research, the grid tablet comprised 2153 nodes and 1034 cells. In calculating the two dimensional nonlinear wave equation, the acceptable deviation was 10^{-7}.

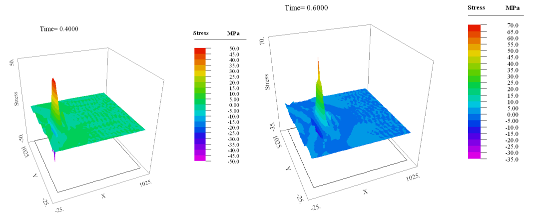
The form of external force in the impulse function is similar to the energy source from the instantaneous compression between tectonic plates. In this research, the magnitude of the initial external force was set at 4MPa, and the boundary and initial conditions were set as stated in the previous section. Using numerical methods, stress propagation through space is shown in Figs. 6 and 7, where stress propagation occurred at every 0.2s interval when t=0.0~1.4s. After calculations began, a maximum external stress of 60MPa was reached at the point of exertion around t=0.2s. As time progressed, stress value generally remained at 39MPa at the point of exertion. However, until x=1000m(right end), although the right propagating stress wave value showed slight changes, the patterns generally remained fixed and did not show repeated periodic fluctuations, therefore indicating non-regular periodic wave cycles. Before x=500m, stress value fluctuated between 60-70MPa; after x=500m, stress value was between 45-39MPa. With increasing propagation distance, the stress peak value showed a pattern of decline. The stress wave moved at a rate of 370m/s (0.37km/s), and maintained a relatively fixed pattern and velocity consistent with soliton, thereby suggesting that this stress wave is a stress soliton.

Fig. 7 was the bottom viewpoint of Fig. 6. From Figs. 6, 7 and during the propagation, only the peak stress values showed slight variations while individual stress wave and wavelength showed no clear changes and no periodic fluctuation. Although the manifested energy showed depletion over time, it remained concentrated. The peak value of the main positive stress wave (the first positive stress wave) decreased over propagation time and distance, from 60MPa at t=0.2s to about 39MPa at t=1.4s. These results indicated that the observed positive stress solitons were bright stress solitons while the negative solitons were dark stress solitons. Thus, if positive (bright) stress solitons represented compressive stress, then negative stress solitons represented tensile stress.

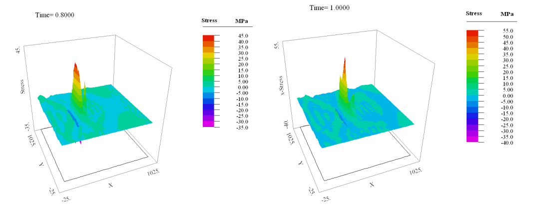
Based on the simulation results, the peak values of the soliton fluctuations often exceeded the ultimate stress value set in the simulated stress-strain curve (that is, the rupture stress value). This seemingly indicated that in the propagation of stress solitons through rock stratum, when stress value changed and approached rupture stress boundary, the rocks became unstable and easily ruptured. The alternating bright and dark stress solitons subjected the rocks to repeated compressive and tensile stress within a short period of time, rendering the rocks even more unstable and more easily ruptured.



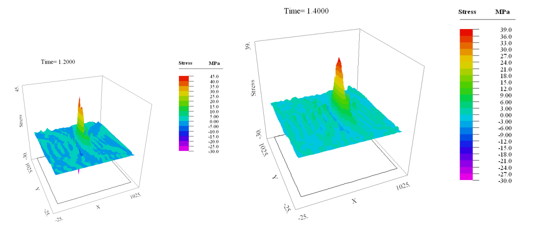
(a) (b)



(c) (d)

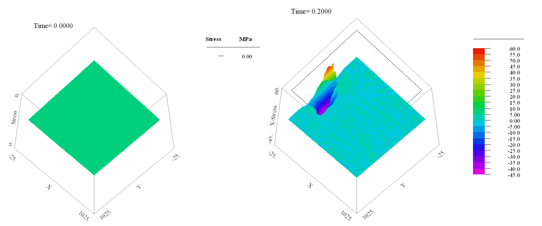


(e) (f)

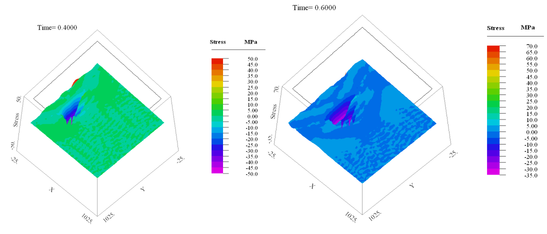


(g) (h)

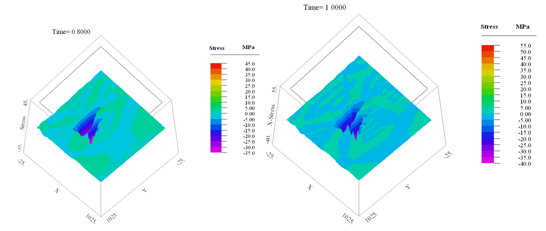
Fig. 6 The stress versus x and y positions for simulation lengths L=1000m and W=1000m, and was observed at chosen time from 0.0sec to 1.4sec step 0.2 sec. (a) t=0.0 sec (b) t=0.2sec (c) t=0.4sec (d) t=0.6sec (e) t=0.8sec (f) t=1.0sec (g) t=1.2sec (h) t=1.4sec were used to display the solitonary phenomena generated by impulse function of an external force



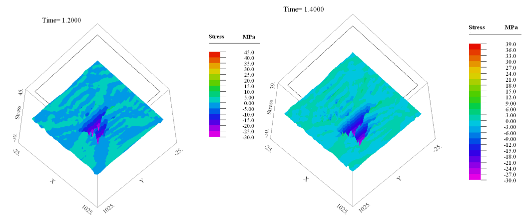
(a) (b)



(c) (d)



(e) (f)



(g) (h)

Fig. 7 The stress versus x and y positions for simulation lengths L=1000m and W=1000m, and was observed at another viewpoint. Chosen time from 0.0sec to 1.4sec step 0.2 sec, (a) t=0.0 sec (b) t=0.2sec (c) t=0.4sec (d) t=0.6sec (e) t=0.8sec (f) t=1.0sec (g) t=1.2sec (h) t=1.4sec were used to display the dark soliton by impulse unction of an external force in bottom viewpoint.

V. Conclusion

To illustrate the complete process of stress propagation with two dimensional nonlinear non-homogeneous wave equations, a computer program and computer-aided software are used. Using Non-linear analysis technology, a stress soliton with external forces was displayed. Based on this result, the process of stress propagation before rock rupture was obtained.

The proposed curve fitting method was used to develop the mathematical model analysis nonlinear wave equations should be helpful in geodynamical region. Finally, the impulse functions of the external forces were also to explore the soliton of stress propagation when a solid does not excess its elasticity limit.

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References

[1] D. H. Peregrine, “Water waves, nonlinear Schrodinger equations and their solutions,” *J. Austral. Math. Soc. Ser. B*, 25, 1983, p.16.

[2] M. J. Abowitz, P. A. Clarkson, Soliton, nonlinear evolution equation and Inverse scattering, London: Cambridge University Press, 1991.
[3] R. K. Dodd, J. C. Eilbeck, H. C. Morris, *Soliton and Nonlinear Equation Lodon:* Academic Press, 1984.
[4] E. Infeld, G. Rowlands, *Nonlinear Waves, Soliton and Chaos*, Cambridge, England: Cambridge University Press, 2000.
[5] C. H. Gu, B. L. Guo, Y.S. Li, C. W. Cao. C. Tian, G. Z. Tu, M. L. Ge. *Soliton Theory and Its Applicationa*, New York: Springer-Verlag, 1995.
[6] C. Rogers, W. K. Schief, Backlund and Darboux *Transformations gemetry and modem applications in soliton theory*, Cambridge University Press, Cambridge Texts in Applied Mathematics, 2002.
[7] F. Abdullaev, S. Darmanyan, P. Khabibullaev, *Optical Solitons*, Springer-Verlag, 1993.
[8] M.M. Oshhunov and S. Ozden, "The general stress and strain relationship in nonlinear materials", *International Journal of nonlinear mechanics*, 35, 2000, 763-767.
[9] R. Yoshinaka, T.V. Tran, and M. Osada, "Nonlinear stress- and strain-dependent behavior of soft rocks under cycle triaxial conditions", *Int. J. Rock Mech. Min. Sci*., Vol. 35, No. 7, pp. 941-955, 1998.
[10] Y.Y. Chang, and F.S. Jeng , *Study of discrepancies in mechanical behavior of dry and wet sandstones*, National Taiwan University, Master Thesis, June, 2007.