

# Nonlinear Equations with N-dimensional Telegraph Operator Iterated K-times

Jessada Tariboon

**Abstract**—In this article, using distribution kernel, we study the nonlinear equations with  $n$ -dimensional telegraph operator iterated  $k$ -times.

**Keywords**—Telegraph operator, Elementary solution, Distribution kernel.

## I. INTRODUCTION

THE telegraph equation arises in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. The interaction of convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physics, chemistry and biology. Further, the telegraph equation is more suitable than ordinary diffusion in modeling reaction-diffusion for such branches of applied sciences. We refer the reader to [1]-[4] and the references therein.

Kanantjai [5]-[6] has studied some properties and results of the distribution  $e^{\alpha x} \square^k \delta$  and solved the convolution equation

$$e^{\alpha x} \square^k \delta * u(x) = e^{\alpha x} \sum_{r=0}^m C_r \square^r \delta,$$

which is related to the ultra-hyperbolic equation, where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $\alpha x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$ ,  $C_r$  are given constants for  $r = 1, 2, \dots, m$ ,  $\square^k$  is the  $n$ -dimensional ultra-hyperbolic operator iterated  $k$  times defined by

$$\square^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

with  $p + q = n$  and  $\delta$  is the Dirac-delta distribution with  $\square^0 \delta = \delta$ ,  $\square^1 \delta = \square \delta$ .

In this work, by applying the distribution  $e^{\alpha x} \square^k \delta$ , we study the elementary solution of the following  $n$ -dimensional telegraph equation

$$\left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) := T^k u(x, t) = \delta(x, t), \quad (1)$$

where  $\Delta$  is the  $n$ -dimensional Laplacian operator iterated  $k$  times defined by

$$\Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^k,$$

Jessada Tariboon are with the Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.  
(e-mail: jessadat@kmutnb.ac.th).

and  $\beta$  is a positive constant. As an application, we solve the nonlinear equation with  $n$ -dimensional telegraph operator iterated  $k$ -times of the form

$$\left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) = f(x, t), \quad (2)$$

where  $f(t, x)$  is a generalized function.

## II. SOME DEFINITIONS AND LEMMAS

**Definition 1.** Let  $x = (x_1, x_2, \dots, x_n)$  be a point of  $\mathbb{R}^n$  and write

$$v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n.$$

Define by  $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$  designating the interior of forward cone and  $\bar{\Gamma}_+$  designating its closure.

For any complex number  $\gamma$ , we define the function

$$R_\gamma^H(v) = \begin{cases} \frac{v^{(\gamma-n)/2}}{K_n(\gamma)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \quad (3)$$

where the constant  $K_n(\alpha)$  is given by the formula

$$K_n(\gamma) = \frac{\pi^{(n-1)/2} \Gamma\left(\frac{2+\gamma-n}{2}\right) \Gamma\left(\frac{1-\gamma}{2}\right) \Gamma(\gamma)}{\Gamma\left(\frac{2+\gamma-p}{2}\right) \Gamma\left(\frac{p-\gamma}{2}\right)}. \quad (4)$$

Let  $\text{supp } R_\gamma^H(v) \subset \bar{\Gamma}_+$  where  $\text{supp } R_\gamma^H(v)$  denotes the support of  $R_\gamma^H(v)$ . The function  $R_\gamma^H$  is first introduced by Nozaki [7] and is called the ultra-hyperbolic kernel of Marcel Riesz. Moreover,  $R_\gamma^H(v)$  is an ordinary function if  $\text{Re}(\gamma) \geq n$  and is a distribution of  $\gamma$  if  $\text{Re}(\gamma) < n$ .

**Definition 2.** Let  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and write

$$s = x_1^2 + x_2^2 + \dots + x_n^2.$$

For any complex number  $\beta$ , define the function

$$R_\beta^e(s) = 2^{-\beta} \pi^{-n/2} \Gamma\left(\frac{n-\beta}{2}\right) \frac{s^{(\beta-n)/2}}{\Gamma\left(\frac{\beta}{2}\right)} \quad (5)$$

The function  $R_\beta^e(s)$  is called the elliptic kernel of Marcel Riesz and is ordinary function if  $\text{Re}(\beta) \geq n$  and is a distribution of  $\beta$  if  $\text{Re}(\beta) < n$ .

**Lemma 1.** [5] Let  $L$  be the partial differential operator defined by

$$L = \square - 2 \left( \sum_{i=1}^p \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} \right) + \left( \sum_{i=1}^p \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right). \quad (6)$$

Then

$$(e^{\alpha x} \square^k \delta) * u(x) = L^k u(x) = \delta \quad (7)$$

In addition, the unique elementary solution of (7) is given by  $u(x) = e^{\alpha x} R_{2k}^H(x)$ , where  $R_{2k}^H(x)$  is defined by (3) with  $\gamma = 2k$ .

**Lemma 2.** [8]  $e^{\alpha x} \delta^{(k)} = (D - \alpha)^k \delta$  where  $D \equiv \frac{d}{dx}$  and  $e^{\alpha x} \delta^{(k)}$  is a Tempered distribution of order  $k$  with support 0.

**Lemma 3.** [9] Let  $z$  be a complex number. Then

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z), \quad z \neq 0, -1, -2, \dots$$

### III. MAIN RESULTS

Now, we shall state and prove the following main results.

**Theorem 1.** Let  $T^k$  be the partial differential operator which iterated  $k$ -times defined by

$$T^k = \left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k, \quad (8)$$

where  $\Delta$  is the  $n$ -dimensional Laplacian operator and  $\beta$  is a given positive constant. Then  $u(x, t) = e^{-\beta t} M_{2k}(w)$  is a unique elementary solution of (1), where  $M_\eta(w)$  is defined by

$$M_\eta(w) = \begin{cases} \frac{w^{(\eta-n)/2}}{H_{n+1}(\eta)} & \text{if } t \in \Gamma_+, \\ 0 & \text{if } t \notin \Gamma_+, \end{cases} \quad (9)$$

where  $w = t^2 - x_1^2 - x_2^2 - \dots - x_n^2$ ,  $t$  is the time and

$$H_{n+1}(\eta) = \pi^{(n-1)/2} 2^{\eta-1} \Gamma\left(\frac{\eta-n+1}{2}\right) \Gamma\left(\frac{\eta}{2}\right). \quad (10)$$

**Proof.** Firstly, we define the  $n+1$ -dimensional ultra-hyperbolic operator as

$$\square_{n+1} = \left( \frac{\partial^2}{\partial t^2} - \Delta \right).$$

Setting  $\alpha_2 = \alpha_3 = \dots = \alpha_n = 0$ , we have

$$e^{\alpha(t,x)} \square_{n+1}^k \delta = e^{\alpha t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right)^k \delta(x, t).$$

Applying Lemma 3 for  $p = 1$ ,  $q = n$  and  $p + q = n + 1$ , (3) and (4) are reduced to (9) and (10), respectively.

Indeed, we have  $\delta(x, t) = \delta(x)\delta(t)$  and  $e^{\alpha t} \delta(x) = \delta(x)$ . Using Lemma 2, we get

$$\begin{aligned} e^{\alpha t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) \delta(x, t) &= e^{\alpha t} \frac{\partial^2}{\partial t^2} \delta(x, t) - e^{\alpha t} \Delta \delta(x, t) \\ &= \left( \frac{\partial}{\partial t} - \alpha_1 \right)^2 \delta(x, t) - \Delta e^{\alpha t} \delta(x, t) \\ &= \left( \frac{\partial^2}{\partial t^2} - 2\alpha_1 \frac{\partial}{\partial t} + \alpha_1^2 - \Delta \right) \delta(x, t). \end{aligned}$$

Substituting  $\alpha_1 = -\beta$ , it follows that

$$\begin{aligned} e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) \delta(x, t) &= \left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right) \delta(x, t) \\ &= T \delta(x, t) \end{aligned}$$

Convolving  $k$ -times for both sides of the above equation by  $e^{-\beta t} (\frac{\partial^2}{\partial t^2} - \Delta) \delta(x, t)$ , we have

$$\begin{aligned} e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) \delta(x, t) * \dots * e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) \delta(x, t) \\ = e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right)^k \delta(x, t) \\ = T \delta(x, t) * \dots * T \delta(x, t) \\ = T^k \delta(x, t). \end{aligned}$$

Then (1) can be written as

$$T^k u(x, t) = e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right)^k \delta(x, t) * u(x, t) = \delta(x, t).$$

Convolving both sides of the above equation by  $e^{-\beta t} M_{2k}(w)$  and Applying Lemma 1, we have

$$u(x, t) = e^{-\beta t} M_{2k}(w),$$

where  $M_{2k}(w)$  is defined by (9) with  $\eta = 2k$ .  $\square$

**Theorem 2.** Given the equation

$$\left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) = f(x, t), \quad (11)$$

where  $f(x, t)$  is a given generalized function and  $u(x, t)$  is an unknown function. Then,

$$u(x, t) = e^{-\beta t} M_{2k}(w) * f(x, t). \quad (12)$$

**Proof.** Convolving both sides of (11) by  $e^{-\beta t} M_{2k}(w)$  and applying the Theorem 1, we obtain (12) as required.  $\square$

**Remark 3.** By using the method of proving Theorem 1 together with suitable modifications, we have  $u(x, t) = e^{-\beta t} (-1)^k R_{2k}^e(s)$  is a unique elementary solution of the following equation

$$\left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 + \Delta \right)^k u(x, t) = \delta(x, t), \quad (13)$$

where  $R_{2k}^e(s)$  is defined by Definition 2 with  $\beta = 2k$ ,  $s = t^2 + x_1^2 + x_2^2 + \dots + x_n^2$  and a constant  $n$  in (5) is replaced by  $n + 1$ .

### ACKNOWLEDGMENT

This research is supported by King Mongkut's University of Technology North Bangkok, Thailand.

### REFERENCES

- [1] D.M. Pozar, Microwave Engineering, New York, Addison-Wesley, 1990.
- [2] A. Jeffrey, Advanced Engineering Mathematics, Harcourt Academic Press, 2002.
- [3] A. Jeffrey, Applied Partial Differential Equations, New York, Academic Press, 2002.
- [4] M. Dehghan, A. Ghesmati, Solution of second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation (DRBIE) method, Eng. Anal. Bound. Elem. 34 (2010), 51-59.
- [5] A. Kananthai, On the distribution related to the ultra-hyperbolic equations, J. Comput. Appl. Math. 84 (1997), 101-106.
- [6] A. Kananthai, On the convolution equation related to the N-dimensional ultra-hyperbolic operator, J. Comput. Appl. Math. 115 (2000), 301-308.

- [7] Y. Nozaki, On Riemann-Liouville integral of ultra-hyperbolic type, Kodai Math. Semin. Report, 6(2) (1964), 69-87.
- [8] A. Kananthai, On the distribution  $e^{\alpha t} \delta^{(k)}$  and its applications, Proc. Annual Conf. Math. KMUTL, Thailand, 1996.
- [9] Bateman, Manuscript Project, Higher Transcendental Functions, Vol. 1, Mc-Graw Hill, New York, 1953.