International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:8, No:5, 2014

Flow and Heat Transfer over a Shrinking Sheet: A Stability Analysis

Anuar Ishak

***Abstract*—**The characteristics of fluid flow and heat transfer over a permeable shrinking sheet is studied. The governing partial differential equations are transformed into a set of ordinary differential equations, which are then solved numerically using MATLAB routine boundary value problem solver bvp4c. Numerical results show that dual solutions are possible for a certain range of the suction parameter. A stability analysis is performed to determine which solution is linearly stable and physically realizable.

***Keywords*—**Dual solutions, heat transfer, shrinking sheet, stability analysis.

I. Introduction

THE flow and heat transfer over a linearly stretching sheet was first considered by Crane [1], who reported the solution in a closed analytical form. Gupta and Gupta [2] extended this problem to a permeable stretching sheet. Grubka and Bobba [3] realized that Crane's solution to the boundary layer equation also happens to be an exact solution to the Navier-Stokes equations. Since then, many authors have considered various aspects of this problem such as Chen and Char [4], Chen [5] and Ishak et al. [6]-[8], among others.

Different from the flow over a stretching sheet, the flow over a shrinking sheet only received the attention quite recently. Miklavčič and Wang [9] was the first who studied the properties of a viscous flow due to a shrinking sheet, and found that the solutions are non-unique. The flow is unlikely to exist unless adequate suction on the boundary is imposed, since vorticity of the shrinking sheet is not confined within a boundary layer. This problem was then extended by Fang and Zhang [10] to magnetohydrodynamic flow, and successfully obtained the closed form analytical solution. Moreover, the solution obtained by Fang and Zhang [10] is also an exact solution of the governing Navier-Stokes equations for that problem, and they reported greatly different solution behavior with multiple solution branches compared to the corresponding stretching sheet problem.

The present study investigates the stability of the non-unique solution for the flow over a shrinking sheet reported by Miklavčič and Wang [9] and Fang and Zhang [10].

II. Mathematical Formulation

Consider a steady boundary layer flow of a viscous fluid over a linearly shrinking sheet with suction at the boundary as shown in Fig. 1. It is assumed that the shrinking velocity is in the form $u_w(x) = cx$, where c is a constant and is maintained at a constant temperature T_w. It is also assumed that the mass flux velocity is v_0 with $v_0 < 0$ for suction and $v_0 > 0$ for injection. Under these assumptions, the steady governing continuity, momentum and energy boundary layer equations are [11]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where u and v are the velocity components along the x- and y-axes respectively, T is the fluid temperature, α is the thermal diffusivity and ν is the kinematic viscosity.

The equations are subjected to the boundary conditions

$$\begin{aligned} &u = \lambda u_w(x), \quad v = v_0, \quad T = T_w \quad \text{at} \quad y = 0 \\ &u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty \end{aligned} \tag{4}$$

where λ is the stretching/shrinking parameter with $\lambda = 1$ for stretching and $\lambda = -1$ for shrinking.

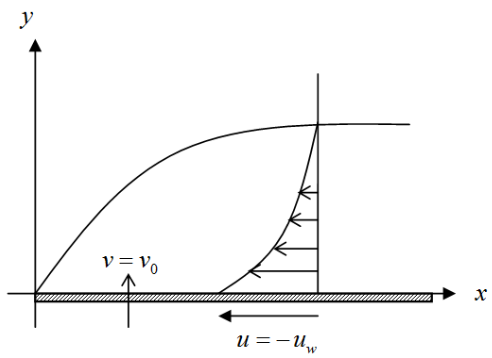

Fig. 1 Physical model and coordinate system

In order to solve (1) to (3) subject to the boundary conditions (4), we introduce the following similarity transformation:

A. Ishak is with the School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Malaysia (phone: +603-8921-5785; fax: +603-8925-4519; e-mail: anuarishak@yahoo.com).

$$\eta = \left(\frac{u_w}{\nu x}\right)^{1/2} y, \quad \psi = \left(\nu x u_w\right)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad (5)$$

where ψ is the stream function defined as $u = \partial\psi / \partial y$ and $v = -\partial\psi / \partial x$, which identically satisfies (1).

Substituting (5) into (2) and (3) we obtain the following ordinary differential (similarity) equations

$$f''' + f f'' - f'^2 = 0 \qquad (6)$$

$$\frac{1}{\Pr}\theta'' + f\theta' = 0 \qquad (7)$$

where prime denotes differentiation with respect to η and $\Pr = \nu / \alpha$ is the Prandtl number. The boundary conditions (4) become

$$\begin{aligned} & f(0) = S, \quad f'(0) = \lambda, \quad \theta(0) = 1 \\ & f'(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \end{aligned} \qquad (8)$$

where $S = -v_0 / (\nu c)^{1/2}$ is the constant mass transfer parameter with $S > 0$ for suction and $S < 0$ for injection.

The quantities of physical interest are the skin friction coefficient C_f, and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho u_w^2(x)}, \quad Nu_x = \frac{x q_w}{k(T_w - T_\infty)} \qquad (9)$$

where τ_w is the surface shear stress along the plate and q_w is the heat flux from the plate, which are defined as

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (10)$$

Using (5) we get

$$\mathrm{Re}_x^{1/2} C_f = f''(0), \quad \mathrm{Re}_x^{-1/2} Nu_x = -\theta'(0) \qquad (11)$$

where $\mathrm{Re}_x = u_w x / \nu$ is the local Reynolds number.

III. Stability of Solutions

In order to perform a stability analysis, we consider the unsteady problem. Equation (1) holds, while (2) and (3) are replaced by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (12)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (13)$$

where t denotes the time. Based on the variables (5), we introduce the following new dimensionless variables:

$$\begin{aligned} & \eta = \left(\frac{u_w}{\nu x}\right)^{1/2} y, \quad \psi = \left(\nu x u_w\right)^{1/2} f(\eta,\tau), \\ & \theta(\eta,\tau) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \tau = at \end{aligned} \qquad (14)$$

so that (2) and (3) can be written as

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} - \left(\frac{\partial f}{\partial \eta}\right)^2 - \frac{\partial^2 f}{\partial \eta\,\partial \tau} = 0 \qquad (15)$$

$$\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial \eta^2} + f\frac{\partial \theta}{\partial \eta} - \frac{\partial \theta}{\partial \tau} = 0 \qquad (16)$$

and are subjected to the boundary conditions

$$\begin{aligned} & f(0,\tau) = S, \quad \frac{\partial f}{\partial \eta}(0,\tau) = \lambda, \quad \theta(0,\tau) = 1 \\ & \frac{\partial f}{\partial \eta}(\eta,\tau) \to 0, \quad \theta(\eta,\tau) \to 0 \qquad \text{as} \quad \eta \to \infty \end{aligned} \qquad (17)$$

To test the stability of the steady flow solution $f(\eta) = f_0(\eta)$ and $\theta(\eta) = \theta_0(\eta)$ satisfying the boundary-value problem (1)-(4), we write (see [12]-[14]),

$$\begin{aligned} & f(\eta,\tau) = f_0(\eta) + e^{-\gamma\tau} F(\eta,\tau), \\ & \theta(\eta,\tau) = \theta_0(\eta) + e^{-\gamma\tau} G(\eta,\tau), \end{aligned} \qquad (18)$$

where γ is an unknown eigenvalue, and $F(\eta,\tau)$ and $G(\eta,\tau)$ are small relative to $f_0(\eta)$ and $\theta_0(\eta)$. Solutions of the eigenvalue problem (15)-(17) give an infinite set of eigenvalues $\gamma_1 < \gamma_2 < \cdots$; if the smallest eigenvalue is negative, there is an initial growth of disturbances and the flow is unstable; but when γ_1 is positive, there is an initial decay and the flow is stable. Introducing (18) into (15) and (16), we get the following linearized problem

$$\frac{\partial^3 F}{\partial \eta^3} + f_0\frac{\partial^2 F}{\partial \eta^2} + f_0'' F - \left(2f_0' - \gamma\right)\frac{\partial F}{\partial \eta} - \frac{\partial^2 F}{\partial \eta\,\partial \tau} = 0 \qquad (19)$$

$$\frac{1}{\Pr}\frac{\partial^2 G}{\partial \eta^2} + f_0\frac{\partial G}{\partial \eta} + \theta_0' F + \gamma G - \frac{\partial G}{\partial \tau} = 0 \qquad (20)$$

along with the boundary conditions

$$\begin{aligned} & F(0,\tau) = 0, \quad \frac{\partial F}{\partial \eta}(0,\tau) = 0, \quad G(0,\tau) = 0, \\ & \frac{\partial F}{\partial \eta}(\eta,\tau) \to 0, \quad G(\eta,\tau) \to 0 \qquad \text{as} \quad \eta \to \infty \end{aligned} \qquad (21)$$

The solutions $f(\eta)=f_0(\eta)$ and $\theta(\eta)=\theta_0(\eta)$ of the steady equations (6) and (7) are obtained by setting $\tau=0$. Hence $F=F_0(\eta)$ and $G=G_0(\eta)$ in (19) and (20) identify initial growth or decay of the solution (18). In this respect, we have to solve the linear eigenvalue problem

$$F_0''' + f_0 F_0'' + f_0'' F_0 - \left(2f_0' - \gamma\right)F_0' = 0 \tag{22}$$

$$\frac{1}{\Pr} G_0'' + f_0 G_0' + \theta_0' F_0 + \gamma G_0 = 0 \tag{23}$$

along with the boundary conditions

$$\begin{aligned} &F_0(0)=0, \quad F_0'(0)=0, \quad G_0(0)=0 \\ &F_0'(\eta)\to 0, \quad G_0(\eta)\to 0 \quad \text{as} \quad \eta\to\infty \end{aligned} \tag{24}$$

It should be stated that for particular values of Pr and γ, the stability of the corresponding steady flow solutions $f_0(\eta)$ and $\theta_0(\eta)$ are determined by the smallest eigenvalue γ. As it has been suggested by Harris et al. [15], the range of possible eigenvalues can be determined by relaxing a boundary condition on $F_0(\eta)$ or $G_0(\eta)$. For the present problem, we relax the condition that $F_0'(\eta)\to 0$ as $\eta\to\infty$ and for a fixed value of γ we solve the system (26, 27,28) along with the new boundary condition $F_0''(0)=1$.

IV. Results and Discussion

The system of equations (6)-(8) was solved numerically using the bvp4c solver in MATLAB software. In order to validate the numerical results obtained, we have compared our results with the stretching case, $\lambda=1$, which showed an excellent agreement.

For the stretching case, $\lambda=1$, the exact analytical solution for the flow field has been reported by Gupta and Gupta [2], while the solution for the thermal field in terms of Kummer's functions has been obtained by Chen and Char [4], and they are respectively given by

$$f(\eta)=\zeta-\frac{1}{\zeta}e^{-\zeta\eta}, \tag{25}$$

$$\theta(\eta)=\frac{F\,(\Pr-1,\,\Pr+1,\,-\Pr\cdot e^{-\zeta\eta}/\zeta^2)}{F(\Pr-1,\,\Pr+1,\,-\Pr/\zeta^2)}\,e^{-\zeta\eta\Pr}, \tag{26}$$

where $S=\zeta-1/\zeta$ and $\zeta>0$, with $\zeta>1$ for suction, $\zeta<1$ for injection, while $\zeta=1$ corresponds to the impermeable surface that has been considered by Crane [1]. In (26), $F(a,b,z)$ denotes the Kummer's function (see Abramowitz and Stegun [16]). By using (25) and (26), the skin friction coefficient $f''(0)$ and the local Nusselt number $-\theta'(0)$ can be shown to be given by

$$\begin{aligned} &f''(0)=-\zeta, \\ &\theta'(0)=-\zeta\Pr+\frac{\Pr-1}{\Pr+1}\frac{\Pr}{\zeta}\frac{M(\Pr,\,\Pr+2,\,-\Pr/\zeta^2)}{M(\Pr-1,\,\Pr+1,\,-\Pr/\zeta^2)}. \end{aligned} \tag{27}$$

Moreover, when $\Pr=1$, the solution $\theta(\eta)$ given in (26) can be expressed as $\theta(\eta)=f'(\eta)=e^{-\zeta\eta}$, which implies

$$-\theta'(0)=\zeta. \tag{28}$$

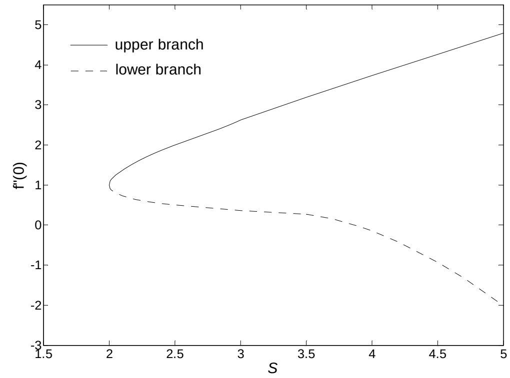

Fig. 2 Variation of the skin friction coefficient $f''(0)$ with S when $\lambda=-1$ (shrinking case)

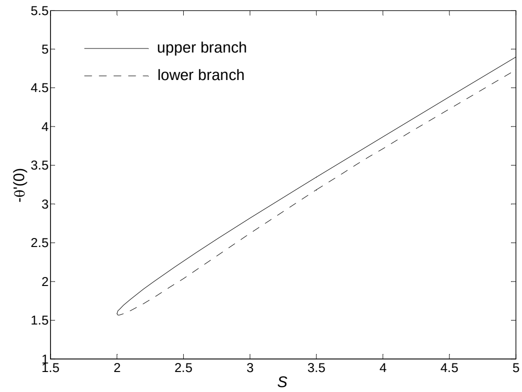

Fig. 3 Variation of the local Nusselt number $-\theta'(0)$ with S when $\Pr=1$ and $\lambda=-1$ (shrinking case)

Different from the stretching case, which shows unique solution, see (25) and (26); the solutions for the shrinking case as presented in Figs. 2 and 3 are non-unique. The validity of these numerical solutions is supported by the velocity and temperature profiles presented in Figs. 4 and 5. The existence of dual solutions for the shrinking sheet was first reported by Miklavčič and Wang [9]. As shown in Figs. 2 and 3, respectively for the skin friction coefficient and the heat transfer rate at the surface, two solutions were obtained for the same value of the suction parameter S for the range $S>2$. We term these solutions as upper and lower branch solutions, by how they appear in Figs. 2 and 3, i.e. the upper branch solution has higher values of $f''(0)$ and $-\theta'(0)$ than the lower branch solution. It is worth mentioning that the exact solution for the flow field was reported by Fang and Zhang [10] as

$$f(\eta) = S - \frac{2}{S \pm \sqrt{S^2 - 4}} + \frac{2}{S \pm \sqrt{S^2 - 4}} e^{-\frac{S \pm \sqrt{S^2-4}}{2}\eta} \quad (29)$$

which confirmed that there are two solutions for $S > 2$. The solution is unique at $S = 2$, i.e. $f''(0) = 1$, which is in agreement with the result presented in Fig. 2.

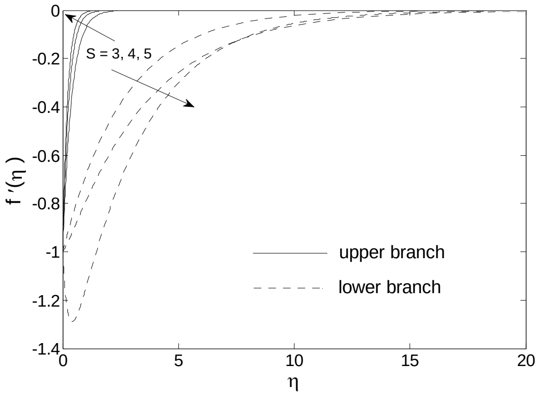

Fig. 4 Velocity profiles for different values of S when $\lambda = -1$ (shrinking case)

To test the stability of the solutions, we perform a stability analysis and find the eigenvalues γ in (18). If the smallest eigenvalue is negative, there is an initial growth of disturbances and the flow is unstable; while when the smallest eigenvalue is positive, there is an initial decay and the flow is stable. The smallest eigenvalues γ for selected values of S are presented in Table I which shows that γ is positive for the upper branch solution and negative for the lower branch solution. Thus, the upper branch solution is stable, while the lower branch solution is unstable. The transition from positive (stable) to negative (unstable) values of γ occurs at the turning points of the parametric solution curves ($S = 2$), which is shown in Fig. 2.

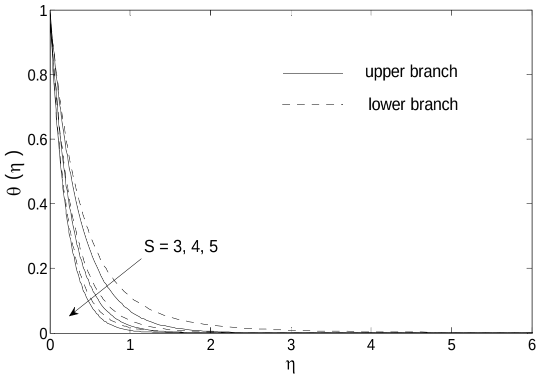

Fig. 5 Temperature profiles for different values of S when $\Pr = 1$ and $\lambda = -1$ (shrinking case)

TABLE I
SMALLEST EIGEN VALUES γ AT SEVERAL VALUES OF S

S	Upper Branch	Lower Branch
2.1	0.4040	-0.3355
2.3	0.7574	-0.5371
2.5	1.0527	-0.6511
3.0	1.7952	-0.8042
4.0	3.5817	-0.9514
5.0	5.7390	-1.4885

Since the upper branch solution is stable and likely physically realizable, our next discussion is about these solutions. Fig. 2 shows that the skin friction coefficient increases as S increases. This is due to the fact that increasing suction increases the friction at the solid-fluid interface, and thus increases the skin friction coefficient. As a result, the local Nusselt number, which represents the heat transfer rate at the surface, increases as presented in Fig. 3.

V. CONCLUSIONS

Different from the stretching case, numerical results showed that dual solutions, upper branch and lower branch, are possible for a certain range of the shrinking parameter. The stability analysis showed that there is an initial decay for the upper branch solution, while there is an initial growth of disturbances for the lower branch solution. Thus, the upper branch solution is linearly stable, while the lower branch solution is linearly unstable. Both the skin friction coefficient and the heat transfer rate at the surface increase as the magnitude of suction increases.

ACKNOWLEDGMENT

The financial support received from the Universiti Kebangsaan Malaysia (Project Code: DIP-2012-31) is gratefully acknowledged.

REFERENCES

[1] L.J. Crane, "Flow past a stretching plate," *Z. angew. Math. Phys.*, vol. 21, pp. 645-647, 1970.
[2] P.S. Gupta and A.S. Gupta, "Heat and mass transfer on a stretching sheet with suction or blowing," *Can. J. Chem. Eng.*, vol. 55, pp. 744-746, 1977.

[3] L.J. Grubka and K.M. Bobba, “Heat transfer characteristics of a continuous, stretching surface with variable temperature,” *ASME J. Heat Transfer*, vol. 107, pp. 248–250, 1985.
[4] C.-K. Chen and M.-I. Char, “Heat transfer of a continuous, stretching surface with suction or blowing,” *J. Math. Anal. Appl*., vol. 135, pp. 568-580, 1988
[5] C.H. Chen, “Laminar mixed convection adjacent to vertical, continuously stretching sheets,” *Heat Mass Transfer*, vol. 33, pp. 471–476, 1998.
[6] A. Ishak, R. Nazar and I. Pop, “Unsteady mixed convection boundary layer flow due to a stretching vertical surface,” *Arabian J. Sci. Eng*., vol. 31, pp. 165-182, 2006.
[7] A. Ishak, R. Nazar and I. Pop, “Boundary layer flow and heat transfer over an unsteady stretching vertical surface,” *Meccanica*, vol. 44, pp. 369–375, 2009.
[8] A. Ishak, K. Jafar, R. Nazar and I. Pop, “MHD stagnation point flow towards a stretching sheet,” *Physica A*, vol. 388, pp. 3377–3383, 2009.
[9] M. Miklavčič and C.Y. Wang, “Viscous flow due to a shrinking sheet,” *Quart. Appl. Math*. vol. 64, pp. 283-290, 2006.
[10] T. Fang and J. Zhang, “Closed-form exact solutions of MHD viscous flow over a shrinking sheet,” *Commun. Nonlinear Sci. Numer. Simulat*., vol. 14, pp. 2853–2857, 2009.
[11] F.M. White, *Viscous Fluid Flow*. Boston: McGraw Hill, 2006.
[12] P.D. Weidman, D.G. Kubitschek and A.M.J. Davis, “The effect of transpiration on self-similar boundary layer flow over moving surfaces,” *Int. J. Eng. Sci*., vol. 44, pp. 730-737, 2006.
[13] A. Postelnicu and I. Pop, “Falkner–Skan boundary layer flow of a power-law fluid past a stretching wedge,” *Appl. Math. Comput*. vol. 217, pp. 4359-4368, 2011.
[14] A.V. Roşca and I. Pop, “Flow and heat transfer over a vertical permeable stretching/ shrinking sheet with a second order slip,” *Int. J. Heat Mass Transfer*, vol. 60, 355-364, 2013.
[15] S.D. Harris, D.B. Ingham and I. Pop, “Mixed convection boundary-layer flow near the stagnation point on a vertical surface in a porous medium: Brinkman model with slip,” *Transport Porous Media*, vol. 77, pp. 267-285, 2009.
[16] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1965.