Some Results on Preconditioned Modified Accelerated Overrelaxation Method

Guangbin Wang, Deyu Sun, Fuping Tan

***Abstract*—**In this paper, we present new preconditioned modified accelerated overrelaxation (MAOR) method for solving linear systems. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned MAOR method converges faster than the MAOR method whenever the MAOR method is convergent. Finally, we give one numerical example to confirm our theoretical results.

***Keywords*—**Preconditioned, MAOR method, linear system, convergence, comparison.

I. Introduction

SOMETIMES, one has to solve a nonsingular linear system as

$$Ay = f\,, \tag{1}$$

where

$$A = \begin{pmatrix} I_1 - B & H \\ K & I_2 - C \end{pmatrix}$$
$$B = (b_{ij})_{p\times p}, C = (c_{ij})_{q\times q},$$
$$K = (k_{ij})_{q\times p}, H = (h_{ij})_{p\times q}, p + q = n\,.$$

Hadjidimos et al. [1] proposed a class of modified accelerated over-relaxation (MAOR) method when the coefficient matrix is a generalized consistently ordered matrix. In [2], Song established sufficient and/or necessary conditions for convergence of the MAOR method.

In order to solve the linear system (1) using the MAOR method, we split A as

$$A = I - L - U$$

with

$$I = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix}, U = \begin{pmatrix} B & -H \\ 0 & C \end{pmatrix}.$$

Then, the MAOR method can be defined by

$$y^{(k+1)} = M_{\Omega,\Gamma}\, y^{(k)} + c, k = 0,1,2,\cdots$$

with

$$M_{\Omega,\Gamma} = (I - \Gamma L)^{-1}(I - \Omega + (\Omega - \Gamma)L + \Omega U)$$
$$c = (I - \Gamma L)^{-1}\Omega f$$

and

$$\Omega = \begin{pmatrix} \omega_1 I_1 & 0 \\ 0 & \omega_2 I_2 \end{pmatrix}, \Gamma = \begin{pmatrix} \gamma_1 I_1 & 0 \\ 0 & \gamma_2 I_2 \end{pmatrix}$$

where $\omega_1, \omega_2, \gamma_1, \gamma_2$ are positive real numbers.

Then, the iteration matrix $M_{\Omega,\Gamma}$ can be rewritten by

$$M_{\Omega,\Gamma} = \begin{pmatrix} (1-\omega_1)I_1 + \omega_1 B & -\omega_1 H \\ (\omega_1\gamma_2 - \omega_2)K - \omega_1\gamma_2 KB & (1-\omega_2)I_2 + \omega_2 C + \omega_1\gamma_2 KH \end{pmatrix}.$$

In [3], authors presented three kinds of preconditioners for preconditioned modified accelerated overrelaxation method to solve systems of linear equations. They showed that the convergence rate of the preconditioned modified accelerated overrelaxation method is better than that of the original method, whenever the original method is convergent.

This paper is organized as follows. In Section II, we give some important definitions and the known results as the preliminaries of the paper. In Section III, we propose three preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned MAOR method converges faster than the MAOR method whenever the MAOR method is convergent. In Section IV, we give an example to confirm our theoretical results.

II. Preliminaries

Definition 1 Let $A = (a_{ij})_{n\times n}$, and $B = (b_{ij})_{n\times n}$. We say $A > B$ if $a_{ij} > b_{ij}$ for all $i, j = 1, 2, \cdots, n$.

Definition 2 Let $A = (a_{ij})_{n\times n}$, and $B = (b_{ij})_{n\times n}$. We say $A \ge B$ if $a_{ij} \ge b_{ij}$ for all $i, j = 1, 2, \cdots, n$.

In this paper, $\rho(\cdot)$ denotes the spectral radius of a matrix.

Lemma 1 [4,5] Let $A \in R^{n\times n}$ be nonnegative and irreducible. Then

* This work was supported by the National Natural Science Foundation of China (Grant No. 11001144) and the Natural Science Foundation of Shandong Province of China (ZR2012AL09).

Guangbin Wang and Deyu Sun are with the Department of Mathematics, Qingdao University of Science and Technology, Qingdao 266061, China (e-mail: wguangbin750828@sina.com).

Fuping Tan is with the Department of Mathematics, Shanghai University, Shanghai 200444, China (e-mail: tanfp@163.com).

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;

(ii) for $\rho(A)$, there corresponds an eigenvector $x > 0$.

(iii) if $0 \neq \alpha x \leq Ax \leq \beta x, \alpha x \neq Ax, Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

III. Comparison Results

We consider the preconditioned linear system

$$\widetilde{A} y = \tilde{f}, \tag{2}$$

where $\widetilde{A} = (I + \tilde{S})A$ and $\tilde{f} = (I + \tilde{S})f$ with

$$\tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

S is a $p \times p$ matrix with $1 < p < n$.

We take S as follows:

$$S_1 = \begin{pmatrix} 0 & b_{12} & \cdots & 0 & 0 \\ b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & b_{p-1,p} \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ 0 & b_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 \\ 0 & 0 & b_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{p-1,p} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now, we obtain three preconditioned linear systems with coefficient matrices

$$\widetilde{A}_i = \begin{pmatrix} I_1 - [B - S_i(I_1 - B)] & (I_1 + S_i)H \\ K & I_2 - C \end{pmatrix}, i = 1,2,3,$$

matrix $\widetilde{A}_i$ can be decomposed by three different cases:

$$\widetilde{A}_i = I - \widetilde{L}_i - \widetilde{U}_i, i = 1,2,3$$

where

$$\widetilde{L}_i = \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix}, \widetilde{U}_i = \begin{pmatrix} B - S_i(I_1 - B) & -(I_1 + S_i)H \\ 0 & C \end{pmatrix},$$

then the preconditioned MAOR methods for solving (2) are defined as follows.

$$y^{(k+1)} = \widetilde{M}^{(i)}_{\Omega,\Gamma} y^{(k)} + v^{(i)}, k = 0,1,2,\cdots$$

where for $i = 1,2,3$

$$\widetilde{M}^{(i)}_{\Omega,\Gamma} = (I - \Gamma \widetilde{L}_i)^{-1}(I - \Omega + (\Omega - \Gamma)\widetilde{L}_i + \Omega \widetilde{U}_i),$$

are iteration matrices and

$$v^{(i)} = (I - \Gamma \widetilde{L}_i)^{-1} \Omega \tilde{b}.$$

Then, the iteration matrix $\widetilde{M}^{(i)}_{\Omega,\Gamma}$ can be rewritten by

$$\begin{aligned} &\widetilde{M}^{(i)}_{\Omega,\Gamma} \\ &= \begin{pmatrix} I_1 & 0 \\ -\gamma_2 K & I_2 \end{pmatrix} \begin{pmatrix} (1-\omega_1)I_1 + \omega_1[B - S_i(I_1 - B)] & -\omega_1(I_1 + S_i)H \\ -(\omega_2 - \gamma_2)K & (1-\omega_2)I_2 + \omega_2 C \end{pmatrix} \\ &= \begin{pmatrix} (1-\omega_1)I_1 + \omega_1[B - S_i(I_1 - B)] & -\omega_1(I_1 + S_i)H \\ (\omega_1\gamma_2 - \omega_2)K - \omega_1\gamma_2 K[B - S_i(I_1 - B)] & (1-\omega_2)I_2 + \omega_2 C + \omega_1\gamma_2 K(I_1 + S_i)H \end{pmatrix}. \end{aligned}$$

Now, we consider new preconditioners P_i^*

$$P_i^* = I + \widetilde{S}, i = 1,2,3,$$

where $\tilde{S} = \begin{pmatrix} S_i & 0 \\ 0 & V_i \end{pmatrix}$, S_i are defined as above, and

$$V_1 = \begin{pmatrix} 0 & c_{12} & \cdots & 0 & 0 \\ c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & c_{q-1,q} \\ 0 & 0 & \cdots & c_{q,q-1} & 0 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{21} & 0 & \cdots & 0 & 0 \\ 0 & c_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{q,q-1} & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & c_{12} & 0 & \cdots & 0 \\ 0 & 0 & c_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{q-1,q} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$\widetilde{A}_i^* = P_i^* A = \begin{pmatrix} I_1 - \left[B - S_i\left(I_1 - B\right)\right] & \left(I_1 + S_i\right)H \\ (I_2 + V_i)K & I_2 - \left[C - V_i(I_2 - C)\right] \end{pmatrix},$$

matrix $\widetilde{A}_i^*$ can be decomposed by three different cases:

$$\widetilde{A}_i^* = I - \widetilde{L}_i^* - \widetilde{U}_i^*, i = 1,2,3$$

where

$$\widetilde{L}_i^* = \begin{pmatrix} 0 & 0 \\ -(I_2 + V_i)K & 0 \end{pmatrix},$$
$$\widetilde{U}_i^* = \begin{pmatrix} B - S_i\left(I_1 - B\right) & -\left(I_1 + S_i\right)H \\ 0 & C - V_i(I_2 - C) \end{pmatrix}.$$

Then the preconditioned MAOR methods for solving $P_i^* Hy = P_i^* f$ are defined as follows

$$y^{(k+1)} = \widetilde{M}_{\Omega,\Gamma}^{*(i)} y^{(k)} + v^{*(i)}, k = 0,1,2,\cdots$$

where for $i = 1,2,3$,

$$\widetilde{M}_{\Omega,\Gamma}^{*(i)} = (I - \Gamma\widetilde{L}_i^*)^{-1}(I - \Omega + (\Omega - \Gamma)\widetilde{L}_i^* + \Omega\widetilde{U}_i^*), i = 1,2$$

and $v^{*(i)} = (I - \Gamma\widetilde{L}_i)^{-1}\Omega\widetilde{b}^*$.

Then, the iteration matrix $\widetilde{M}_{\Omega,\Gamma}^{*(i)}$ can be rewritten by

$$\begin{aligned}
&\widetilde{M}_{\Omega,\Gamma}^{*(i)} \\
&= \begin{pmatrix} I_1 & 0 \\ \gamma_2(I_2 + V_i)K & I_2 \end{pmatrix}^{-1} \\
&\begin{pmatrix} \left(1-\omega_1\right)I_1 + \omega_1\left[B - S_i\left(I_1 - B\right)\right] & -\omega_1\left(I_1 + S_i\right)H \\ -(\omega_2 - \gamma_2)(I_2 + V_i)K & \left(1-\omega_2\right)I_2 + \omega_2[C - V_i(I_2 - C)] \end{pmatrix} \\
&= \begin{pmatrix} \left(1-\omega_1\right)I_1 + \omega_1\left[B - S_i\left(I_1 - B\right)\right] & -\omega_1\left(I_1 + S_i\right)H \\ (\omega_1\gamma_2 - \omega_2)(I_2 + V_i)K & \left(1-\omega_2\right)I_2 + \omega_2[C - V_i(I_2 - C)] \\ -\omega_1\gamma_2 K\left[B - S_i\left(I_1 - B\right)\right] & +\omega_1\gamma_2(I_2 + V_i)K\left(I_1 + S_i\right)H \end{pmatrix}.
\end{aligned}$$

Theorem 1 Let $M_{\Omega,\Gamma}, \widetilde{M}_{\Omega,\Gamma}^{*(i)}$ be the iteration matrices associated of the MAOR and preconditioned MAOR methods, respectively. If the matrix A is irreducible with $K \le 0, H \le 0, B \ge 0, C \ge 0, b_{i,i+1} > 0, b_{i+1,i} > 0$, $c_{i,i+1} > 0, c_{i+1,i} > 0$ for some $i \in \{1,2,\cdots,p-1\}$, then either

$$\rho\left(\widetilde{M}_{\Omega,\Gamma}^{*(1)}\right) < \rho\left(M_{\Omega,\Gamma}\right) < 1$$

or

$$\rho\left(\widetilde{M}_{\Omega,\Gamma}^{*(1)}\right) > \rho\left(M_{\Omega,\Gamma}\right) > 1.$$

Proof. Since $K \le 0, H \le 0, B \ge 0, C \ge 0$, it is easy to prove that both $\widetilde{M}_{\Omega,\Gamma}^{*(1)}$ and $M_{\Omega,\Gamma}$ are irreducible and non-negative. By Lemma 1, there is a positive vector x such that

$$M_{\Omega,\Gamma}x = \lambda x$$

where $\lambda = \rho(M_{\Omega,\Gamma})$.

Then $(I - \Omega + (\Omega - \Gamma)L + \Omega U)x = \lambda(I - \Gamma L)x$

$$\begin{aligned}
&\widetilde{M}_{\Omega,\Gamma}^{*(1)}x - \lambda x \\
&= (I - \Gamma\widetilde{L}_1^*)^{-1}[(I - \Omega + (\Omega - \Gamma)\widetilde{L}_1^* + \Omega\widetilde{U}_1^*)x - \lambda(I - \Gamma\widetilde{L}_1^*)x] \\
&= (I - \Gamma\widetilde{L}_1^*)^{-1}\left[(\Omega - \Gamma)(\widetilde{L}_1^* - L) + \Omega(\widetilde{U}_1^* - U) - \lambda\Gamma(\widetilde{L}_1^* - L)\right]x \\
&= (I - \Gamma\widetilde{L}_1^*)^{-1}\Omega[(\widetilde{L}_1^* - L) + (\widetilde{U}_1^* - U)]x + (\lambda - 1)(I - \Gamma\widetilde{L}_1^*)^{-1}\Gamma(\widetilde{L}_1^* - L)x \\
&= (I - \Gamma\widetilde{L}_1^*)^{-1}\begin{pmatrix} -\omega_1 S_1(I_1 - B) & -\omega_1 S_1 H \\ -\omega_2 V_1 K & -\omega_2 V_1(I_2 - C) \end{pmatrix}x + (\lambda - 1)(I - \Gamma\widetilde{L}_1^*)^{-1}\begin{pmatrix} 0 & 0 \\ -\gamma_2 V_1 K & 0 \end{pmatrix}x.
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} -\omega_1 S_1(I_1 - B) & -\omega_1 S_1 H \\ -\omega_2 V_1 K & -\omega_2 V_1(I_2 - C) \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} -\omega_1(I_1 - B) & -\omega_1 H \\ -\omega_2 K & -\omega_2(I_2 - C) \end{pmatrix} \\
&= \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\left(\lambda\begin{pmatrix} I_1 & 0 \\ \gamma_2 K & I_2 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ \gamma_2 K & I_2 \end{pmatrix}\right) = (\lambda - 1)\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} I_1 & 0 \\ \gamma_2 K & I_2 \end{pmatrix}
\end{aligned}$$

So

$$\begin{aligned}
&\widetilde{M}_{\Omega,\Gamma}^{*(1)}x - \lambda x \\
&= (\lambda - 1)(I - \Gamma\widetilde{L}_1^*)^{-1}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} I_1 & 0 \\ \gamma_2 K & I_2 \end{pmatrix}x + (\lambda - 1)(I - \Gamma\widetilde{L}_1^*)^{-1}\begin{pmatrix} 0 & 0 \\ -\gamma_2 V_1 K & 0 \end{pmatrix}x. \\
&= (\lambda - 1)(I - \Gamma\widetilde{L}_1^*)^{-1}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}x = (\lambda - 1)\begin{pmatrix} I_1 & 0 \\ -\gamma_2(I_2 + V_i)K & I_2 \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}x
\end{aligned}$$

Since $b_{i,i+1} > 0, b_{i+1,i} > 0$, $c_{i,i+1} > 0, c_{i+1,i} > 0$, then $S_1 > 0, V_1 > 0$. We have

$$\begin{pmatrix} I_1 & 0 \\ -\gamma_2(I_2 + V_i)K & I_2 \end{pmatrix} > 0, \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} > 0$$

Then

If $\lambda < 1$, then $\widetilde{M}_{\Omega,\Gamma}^{*(1)}x - \lambda x < 0$. By Lemma 1, we know that $\rho(\widetilde{M}_{\Omega,\Gamma}^{*(1)}) < \rho(M_{\Omega,\Gamma}) < 1$.

If $\lambda > 1$, then $\widetilde{M}_{\Omega,\Gamma}^{*(1)}x - \lambda x > 0$. By Lemma 1, we know that $\rho(\widetilde{M}_{\Omega,\Gamma}^{*(1)}) > \rho(M_{\Omega,\Gamma}) > 1$.

According to the proof of Theorem1, we can prove the following two theorems easily.

Theorem 2 Let $M_{\Omega,\Gamma}, \widetilde{M}_{\Omega,\Gamma}^{*(2)}$ be the iteration matrices of the MAOR and preconditioned MAOR methods, respectively. If the matrix A is irreducible with $K \le 0$, $H \le 0$, $B \ge 0$, $C \ge 0$, $b_{i+1,i} > 0$, $c_{i+1,i} > 0$ for some $i \in \{1,2,\cdots,p-1\}$, then either

$$\rho\left(\widetilde{M}_{\Omega,\Gamma}^{*(2)}\right) < \rho\left(M_{\Omega,\Gamma}\right) < 1$$

or

$$\rho\left(\widetilde{M}^{*(2)}_{\Omega,\Gamma}\right) > \rho\left(M_{\Omega,\Gamma}\right) > 1.$$

Theorem 3 Let $M_{\Omega,\Gamma}$, $\widetilde{M}^{*(3)}_{\Omega,\Gamma}$ be the iteration matrices of the MAOR and preconditioned MAOR methods, respectively. If the matrix A is irreducible with $K \le 0$, $H \le 0$, $B \ge 0$, $C \ge 0$, $b_{i,i+1} > 0$, $c_{i,i+1} > 0$ for some $i \in \{1,2,\cdots,p-1\}$, then either

$$\rho\left(\widetilde{M}^{*(3)}_{\Omega,\Gamma}\right) < \rho\left(M_{\Omega,\Gamma}\right) < 1$$

or

$$\rho\left(\widetilde{M}^{*(3)}_{\Omega,\Gamma}\right) > \rho\left(M_{\Omega,\Gamma}\right) > 1.$$

Theorem 4 Under the assumptions of Theorem 1, then

(1). $\rho\left(\widetilde{M}^{*(1)}_{\Omega,\Gamma}\right) < \rho\left(\widetilde{M}^{(1)}_{\Omega,\Gamma}\right) < 1$, if $\rho\left(M_{\Omega,\Gamma}\right) < 1$ or

(2). $\rho\left(\widetilde{M}^{*(1)}_{\Omega,\Gamma}\right) > \rho\left(\widetilde{M}^{(1)}_{\Omega,\Gamma}\right) > 1$, if $\rho\left(M_{\Omega,\Gamma}\right) > 1$.

Proof.

$$\begin{aligned}&\widetilde{M}^{*(1)}_{\Omega,\Gamma}x - \widetilde{M}^{(1)}_{\Omega,\Gamma}x \\ &= (\widetilde{M}^{*(1)}_{\Omega,\Gamma}x - \lambda x) - (\widetilde{M}^{(1)}_{\Omega,\Gamma}x - \lambda x) = (\lambda-1)(I-\Gamma\widetilde{L}_1^*)^{-1}\begin{pmatrix} S_1 & 0 \\ 0 & V_1\end{pmatrix}x \\ &-(\lambda-1)(I-\Gamma\widetilde{L}_1)^{-1}\begin{pmatrix} S_1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} I_1 & 0 \\ \gamma_2 K & I_2\end{pmatrix}x = (\lambda-1)\begin{pmatrix} 0 & 0 \\ -\gamma_2(I_2+V_1)KS_1 & V_1\end{pmatrix}x,\end{aligned}$$

where $\lambda = \rho(M_{\Omega,\Gamma})$.

By assumptions, $V_1 > 0, -\gamma_2(I_2+V_1)KS_1 > 0$ then

$$\begin{pmatrix} 0 & 0 \\ -\gamma_2(I_2+V_1)KS_1 & V_1\end{pmatrix} > 0.$$

If $\lambda < 1$, then $\widetilde{M}^{*(1)}_{\Omega,\Gamma}x - \widetilde{M}^{(1)}_{\Omega,\Gamma}x < 0$, that is $\rho(\widetilde{M}^{*(1)}_{\Omega,\Gamma}) < \rho(\widetilde{M}^{(1)}_{\Omega,\Gamma}) < 1$.

If $\lambda > 1$, then $\widetilde{M}^{*(1)}_{\Omega,\Gamma}x - \widetilde{M}^{(1)}_{\Omega,\Gamma}x > 0$, that is $\rho(\widetilde{M}^{*(1)}_{\Omega,\Gamma}) > \rho(\widetilde{M}^{(1)}_{\Omega,\Gamma}) > 1$.

According to the proof of Theorem 4, we can prove the following two theorems easily.

Theorem 5 Under the assumptions of Theorem 2, then

(1). $\rho\left(\widetilde{M}^{*(2)}_{\Omega,\Gamma}\right) < \rho\left(\widetilde{M}^{(2)}_{\Omega,\Gamma}\right) < 1$, if $\rho\left(M_{\Omega,\Gamma}\right) < 1$ or

(2). $\rho\left(\widetilde{M}^{*(2)}_{\Omega,\Gamma}\right) > \rho\left(\widetilde{M}^{(2)}_{\Omega,\Gamma}\right) > 1$, if $\rho\left(M_{\Omega,\Gamma}\right) > 1$.

Theorem 6 Under the assumptions of Theorem 3, then

(1). $\rho\left(\widetilde{M}^{*(3)}_{\Omega,\Gamma}\right) < \rho\left(\widetilde{M}^{(3)}_{\Omega,\Gamma}\right) < 1$, if $\rho\left(M_{\Omega,\Gamma}\right) < 1$ or

(2). $\rho\left(\widetilde{M}^{*(3)}_{\Omega,\Gamma}\right) > \rho\left(\widetilde{M}^{(3)}_{\Omega,\Gamma}\right) > 1$, if $\rho\left(M_{\Omega,\Gamma}\right) > 1$.

IV. NUMERICAL EXAMPLE

The coefficient matrix A in (1) is given by

$$A = \begin{pmatrix} I_1 - B & H \\ K & I_2 - C\end{pmatrix},$$

where $B = (b_{ij})_{p\times p}$, $C = (c_{ij})_{(n-p)\times(n-p)}$, $K = (k_{ij})_{(n-p)\times p}$, and $H = (h_{ij})_{p\times(n-p)}$ with

$$b_{ii} = \frac{1}{10\times(i+1)},\ i = 1,2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times j + i},\ i < j,\ i = 1,2,\cdots,p-1,\ j = 2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+i},\ i > j,\ i = 2,\cdots,p,$$

$$j = 1,2,\cdots,p-1,$$

$$c_{ii} = \frac{1}{10\times(p+i+1)},\ i = 1,2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(p+j)+p+i},\ i < j,$$

$$i = 1,2,\cdots,n-p+1,\ j = 2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+p+i},\ i > j,$$

$$i =,2,\cdots,n-p,\ j = 1,2,\cdots,n-p-1,$$

$$k_{ij} = \frac{1}{30\times(p+i-j+1)+p+i} - \frac{1}{30},$$

$$i = 1,2,\cdots,n-p,\ j = 1,2,\cdots,p,$$

$$h_{ij} = \frac{1}{30\times(p+j)+i} - \frac{1}{30},\ i = 1,2,\cdots,p,$$

$$j = 1,2,\cdots,n-p.$$

TABLE I
THE SPECTRAL RADII OF THE MAOR AND PRECONDITIONED MAOR ITERATION MATRICES

N	5	10	15	20	25	30	40	50
P	3	5	5	10	15	20	25	30
w_1	0.4	0.5	0.6	0.75	0.8	0.8	0.95	0.95
w_2	0.5	0.55	0.7	0.8	0.8	0.85	0.9	0.95
γ_2	0.6	0.65	0.7	0.7	0.8	0.85	0.9	0.95
ρ	0.6345	0.5955	0.5803	0.6214	0.738	0.8868	1.2653	1.7481
ρ_1	0.6284	0.5867	0.5738	0.6124	0.7298	0.8825	1.2751	1.7766
ρ_2	0.6309	0.5915	0.5773	0.6176	0.7347	0.8851	1.2692	1.7596
ρ_3	0.6305	0.5904	0.5765	0.6159	0.7331	0.8842	1.2712	1.7652
ρ^*_1	0.6282	0.5817	0.5628	0.6029	0.7237	0.8803	1.2827	1.8013
ρ^*_2	0.6308	0.5895	0.5731	0.6139	0.7323	0.8842	1.2723	1.7696
ρ^*_3	0.6303	0.5873	0.5699	0.6103	0.7294	0.8829	1.2756	1.7793

Here $\rho=\rho\left(M_{\Omega,\Gamma}\right), \rho_1=\rho\left(\widetilde{M}^{(1)}_{\Omega,\Gamma}\right), \rho_2=\rho\left(\widetilde{M}^{(2)}_{\Omega,\Gamma}\right), \rho_3=\rho\left(\widetilde{M}^{(3)}_{\Omega,\Gamma}\right),$
$\rho_1^*=(\widetilde{M}^{*(1)}_{\Omega,\Gamma}), \rho_2^*=(\widetilde{M}^{*(2)}_{\Omega,\Gamma}), \rho_3^*=(\widetilde{M}^{*(3)}_{\Omega,\Gamma})$.

REFERENCES

[1] A. Hadjidimos, A. Psimarni, A. K. Yeyios, "On the convergence of the modified accelerated overrelaxation (MAOR) method," Applied Numerical Mathematics, vol.10, pp. 115–127, 1992.

[2] Y. Song, "On the convergence of the MAOR method", *Journal of Computational and Applied Mathematics*, vol.79, pp. 299-317,1997.

[3] M. T. Darvishi, P. Hessari, B.C. Shin, "Preconditioned modified AOR method for systems of linear equations", *International Journal for Numerical Methods in Biomedical Engineering*, vol.27, pp. 758-769, 2011.

[4] R. S. Varga, *Matrix Iterative Analysis,* in: Springer Series in Computational Mathematics, vol. 27, Springer-Verlag, Berlin, 2000.

[5] A. Berman, R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM Press, Philadelphia, 1994.