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Isospectral Hulthén Potential

Anil Kumar

Abstract—Supersymmetric Quantum Mechanics is an interesting framework to analyze nonrelativistic quantal problems. Using these techniques, we construct a family of strictly isospectral Hulthén potentials. Isospectral wave functions are generated and plotted for different values of the deformation parameter.

Keywords—Hulthén potential, Isospectral Hamiltonian.

I. Introduction

THE exact analytical solution of the Schrödinger equation for its bound state energy levels and eigenfunction is fundamental in understanding the bound energy spectrum in nonrelativistic and relativistic quantum mechanics. The wave function contains all necessary information for the complete description of a quantum mechanical system. There are only a few potentials for which the Schrödinger equation can be solved explicitly. One of these exactly solvable potentials is the Hulthén potential. The Hulthén potential [1-5] is one of the important exponential potentials which is extensively used to describe the atomic interactions. It has many applications in atomic physics, nuclear and high energy physics, solid state physics and chemical physics [6,7].

We use the isospectral Hamiltonian approach to study the isospectral potential and their wave functions. Two Hamiltonians are said to be strictly isospectral, if they have exactly same energy eigenvalue spectrum and S-matrix [8-10], whereas the wave functions and their dependent quantities are different. Though the idea of generating isospectral Hamiltonians using the Gelfand-Levitan approach or the Darboux procedure were known for some time, the supersymmetric quantum mechanical techniques make the procedure look simpler [11-13]. When one deletes a bound state of a given potential $V(x)$ and re-introduce the state, it involves solving a first order differential equation. Thus, a set of one-dimensional family of potentials $\hat{V}(x,c)$ can be constructed which have the exactly same energy spectrum as that of $V(x)$. In general, for any one dimensional potential with n bound states, one can construct an n-parameter family of strictly isospectral potentials, i.e. potentials with eigenvalues, reflection and transmission coefficients identical to those for original potential. This aspect has been utilized profitably in many physical situations, which are of interest to various fields [14-20]. In soliton physics, the stability of the soliton/kink is ensured by the occurrence of a zero energy ground state of the stability equation when small oscillations around the soliton/kink are considered. The stability equation can be considered as an one dimensional Schrödinger equation with potential $V(x)$ and one can construct an isospectral partner for it. The partner stability equation will have the same energy spectrum as that of the original equation. Then, one can reconstruct the soliton solution and hence the potential, V(ϕ), which admits the soliton solution from the partner stability equation. This generalizes the class of Hamiltonians which admits soliton/kink solutions that share the same stability data [21-23]. The spectrum of a charged particle in uniform magnetic field consists of equally spaced Landau levels which are infinitely degenerate. Using isospectral deformation, it has been shown that equispaced spectrum can also be obtained for a wide class of non-uniform magnetic fields [24], [25]. In this paper, we consider the Hulthén potential and calculate the deformed potential and their eigenfunctions using isospectral Hamiltonian approach.

Anil Kumar is with the Department of Physics, JC DAV College, Dasuya 144 205 Punjab, India. (e-mail:anilkumarphys@gmail.com).

II. Isospectral Hamiltonian Approach

The connection between the bound state wave functions and the potential is one of the key ingredients in solving exactly for the spectrum of one dimensional potential problems. Once, we know the ground state wave function (ψ_0) and choose its energy to be zero, we can factorize the Hamiltonian as $H_1 = A^\dagger A$ where (in units $\hbar = 2m = 1$), $A = \frac{d}{dx} + W(x)$ and $A^\dagger = -\frac{d}{dx} + W(x)$ are the supersymmetric operators and $W(x) = -\frac{d}{dx}[\ln\psi_0(x)]$ is called the superpotential. We have

$$H_1\psi_n = A^\dagger A\psi_n = \epsilon_n\psi_n, \tag{1}$$

$$AA^\dagger(A\psi_n) = \epsilon_n(A\psi_n),$$

$$H_2(A\psi_n) = \epsilon_n(A\psi_n). \tag{2}$$

Here H_2 is the supersymmetric partner Hamiltonian of H_1, with eigenfunctions $\chi_n = A\psi_n$. It is obvious that H_2 has the same eigenvalue spectrum as that of H_1, but for the case $A\psi_0 = 0$, which is the case of supersymmetry broken. Explicitly, the relation between Hamiltonians reads,

$$E_n^{(2)} = E_{n+1}^{(1)}; \qquad E_0^{(1)} = 0,$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-\frac{1}{2}} A\psi_{n+1}^{(1)},$$

$$\psi_{n+1}^{(1)} = [E_n^{(2)}]^{-\frac{1}{2}} A^\dagger\psi_n^{(2)},$$

The superpotential relates the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ as

$$V_{1,2}(x) = W^2(x) \mp \frac{dW}{dx}. \tag{3}$$

It is well known that for the potential $V_2(x)$, the original potential $V_1(x)$ is not unique. The argument is as follows. Suppose H_2 has another factorization $BB^\dagger$, where $B = \frac{d}{dx} + \hat{W}(x)$, then, $H_2 = AA^\dagger = BB^\dagger$ but $H_1 = B^\dagger B$ is not $A^\dagger A$ rather it defines a certain new Hamiltonian. For superpotential $\hat{W}(x)$, the partner potential $V_2(x)$ is

$$V_2(x) = \hat{W}^2(x) + \hat{W}'(x). \tag{4}$$

Consider the most general solution as $\hat{W}(x) = W(x) + \phi(x)$, which demands that,

$$\phi^2(x) + 2W(x)\phi(x) + \phi^{'}(x) = 0. \tag{5}$$

The solution of the above equation is $\phi(x) = \frac{d}{dx}\ln[I(x)+c]$, where $I(x) = \int_{-\infty}^{x} \psi_0^2(x^{'})dx^{'}$ and c is a constant. Therefore, we obtain,

$$\hat{W}(x) = W(x) + \frac{d}{dx}\ln[I(x)+c]. \tag{6}$$

The corresponding one parameter family of potentials $\hat{V}_1(x,c)$ is given as

$$\hat{V}_1(x,c) = V_1(x) - 2\frac{d^2}{dx^2}(\ln(I(x)+c)). \tag{7}$$

The normalized ground state wave function corresponding to the potential $\hat{V}_1(x,c)$ reads

$$\hat{\psi}_0(x,c) = \frac{\sqrt{c(1+c)}\psi_0(x)}{I(x)+c}, \tag{8}$$

where $c \notin (0,-1)$. The excited state eigenfunctions for the potential $\hat{V}_1(x,\lambda)$ are given by

$$\hat{\psi}_{n+1}(x,c) = \psi_{n+1}(x) + \frac{1}{E_{n+1}}\left(\frac{I^{'}(x)}{I(x)+c}\right)\left(\frac{d}{dx}+W(x)\right)\psi_{n+1}(x). \tag{9}$$

The equations 7, 8 and 9 represent the one parameter family of isospectral potentials and the wave functions which shall be used to obtain the Hulthén Potential as a function of deformation parameter.

III. Isospectral Hulthén Potential

The Hulthén Potential is an interesting short range potential which is used in atomic physics, solid state physics, nuclear physics, particle physics and chemical physics. The potential is given as [5],

$$V(x) = -\frac{V_1 e^{-2ax}}{1-qe^{-2ax}} \tag{10}$$

The energy eigenvalues of the potential are obtained as

$$E_n = -a^2\left[n+1-\frac{V_1}{4qa^2(n+1)}\right]^2 \tag{11}$$

The normalized ground state eigenfunction reads

$$\psi_0(x) = \frac{e^{-2a\sqrt{\epsilon}x} - qe^{-2a(1+\sqrt{\epsilon})x}}{\left(\frac{1}{4a\sqrt{\epsilon}} - \frac{q}{a(1+2\sqrt{\epsilon})} + \frac{q^2}{4a(1+\sqrt{\epsilon})}\right)} \tag{12}$$

The excited state wave functions are

$$\psi_n(x) = N_n(e^{-2a\sqrt{\epsilon}x} - qe^{-2a(1+\sqrt{\epsilon})x})\, P[n, 2\sqrt{\epsilon}, 1, 1-2qe^{-2ax}] \tag{13}$$

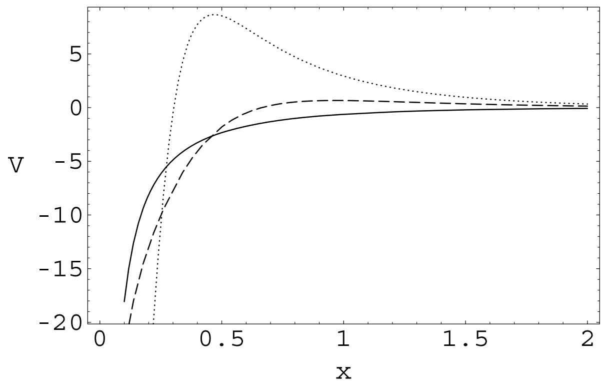

Fig. 1. The Hulthén potential for q=1, a=1, $V_1 = 4$ and deformation parameter $c = 2$ (dashed line) and $c = 1.1$ (dotted line). Solid line shows the undeformed potential.

where N_n is normalization constant and P is Jacobi's polynomial. The potential isospectral to the Hulthén potential is obtained after some calculations as

$$\hat{V}(x,c) = 2\left\{\left(\frac{e^{-4a\sqrt{\epsilon}x} - 2qJ_4 + q^2J_3}{J_1(c+J_2)}\right)^2 + \frac{4a\sqrt{\epsilon}J_3 - 4qa(1+2\sqrt{\epsilon})J_4 + 4q^2a(1+\sqrt{\epsilon})J_3}{J_1(c+J_2)}\right\} \tag{14}$$

where

$$J_1 = \left(\frac{1}{4a\sqrt{\epsilon}} - \frac{q}{a(1+2\sqrt{\epsilon})} + \frac{q^2}{4a(1+\sqrt{\epsilon})}\right)$$

$$J_2 = \frac{-\frac{e^{-4a\sqrt{\epsilon}x}}{4a\sqrt{\epsilon}} + \frac{qe^{-2a(1+2\epsilon)x}}{a(1+2\sqrt{\epsilon})} - \frac{q^2e^{-4a(1+\sqrt{\epsilon})x}}{4a(1+\sqrt{\epsilon})}}{\frac{1}{4a\sqrt{\epsilon}} - \frac{q}{a(1+2\sqrt{\epsilon})} + \frac{q^2}{4a(1+\sqrt{\epsilon})}}$$

$$J_3 = e^{-4a(1+\sqrt{\epsilon})x})$$

and

$$J_4 = e^{-2a(1+2\sqrt{\epsilon})x}$$

Using isospectral hamiltonian approach, the ground state wave function is calculated as

$$\hat{\psi}_0(x,c) = \frac{\sqrt{c(1+c)}(e^{-2a\sqrt{\epsilon}x} - qe^{-2a(1+\sqrt{\epsilon})x})}{\sqrt{J_1}(c+J_2)} \tag{15}$$

The excited state eigenfunction is obtained after some calculations as

$$\hat{\psi}_n(x,c) = \frac{2q(n+3+2\sqrt{\epsilon})e^{-2ax}J_5^3P[n,1+2\sqrt{\epsilon},2,\mu]}{J_6 - J_5P[n+1,2\sqrt{\epsilon},1,\mu]} \tag{16}$$

where

$$J_5 = qe^{-2a(1+\sqrt{\epsilon})x} - e^{-2a\sqrt{\epsilon}x}$$

$$J_6 = aJ_1(c+J_2)(n+2-\frac{V_1}{4qa^2(n+2)})^2$$

and

$$\mu = 1 - 2qe^{-2ax}$$

The potential is plotted for different values of the deformation

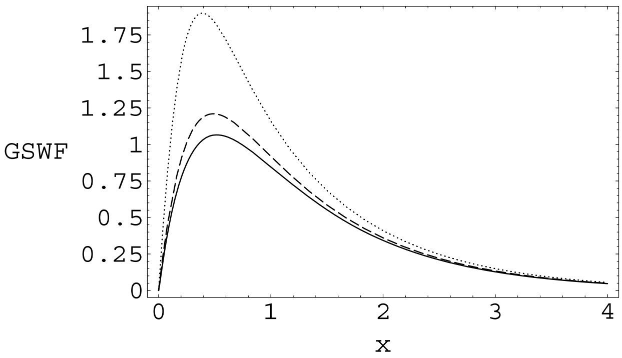

Fig. 2. Ground state wave function of Hulthén potential for q=1, a=1 and deformation parameter $c = 10$ (solid line), $c = 5$ (dashed line) and $c = 2$ (dotted line).

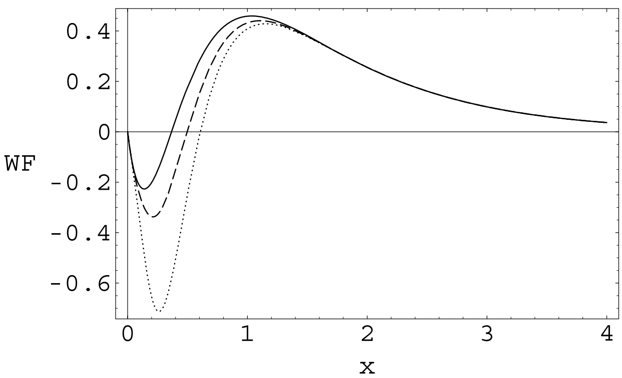

Fig. 3. First excited state wave function of Hulthén potential for q=1, a=1 and deformation parameter $c = 10$ (solid line), $c = 2$ (dashed line) and $c = 1.3$ (dotted line).

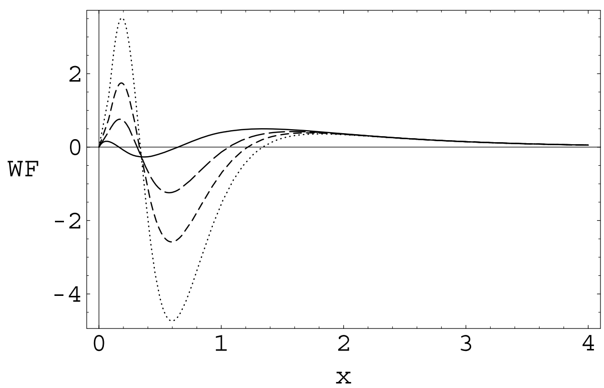

Fig. 4. Second excited state state wave function of Hulthén potential for q=1, a=1 and deformation parameter $c = 10$ (dashed line), $c = 5$ (small dashed line), $c = 3$ (dotted line) and solid line is for undeformed case.

parameter in Fig.1. It is found that with the decrease in value of the parameter, the deformation in the potential increases. The wave function corresponding to ground state, first excited state and second excited state are plotted in Fig.2, Fig.3 and Fig.4 for different values of deformation parameter. It is noted that for large values of deformation parameter, the wave function approaches towards the undeformed wave functions.

IV. Conclusion

We have presented the calculations for Hulthén potential using isospectral Hamiltonian approach, which deals with first order differential equation. A class of Hulthén potential and their eigenfunctions is obtained having same eigenvalue spectrum. The deformed potential and the eigenfunctions approaches to their undeformed case for large values of the deformation parameter.

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