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Relative Injective Modules and Relative Flat Modules

Jianmin Xing, Rufeng Xing

Abstract—Let R be a ring, n a fixed nonnegative integer. The concepts of $(n,0)$-FI-injective and $(n,0)$-FI-flat modules, and then give some characterizations of these modules over left n-coherent rings are introduced . In addition, we investigate the left and right n-$\mathcal{FI}$-resolutions of R-modules by left (right) derived functors $\mathrm{Ext}_n(-,-)$ ($\mathrm{Tor}^n(-,-)$) over a left n-coherent ring, where n-$\mathcal{FI}$ stands for the categories of all $(n,0)$- injective left R-modules. These modules together with the left or right derived functors are used to study the $(n,0)$-injective dimensions of modules and rings.

Keywords—$(n,0)$-injective module, $(n,0)$-injective dimension, $(n,0)$-FI-injective(flat) module, (Pre)cover, (Pre)envelope.

I. Introduction

THROUGHOUT this paper, n is a positive integer unless a special note. R denotes an associative ring with identity and all modules considered are unitary. $M_R({}_RM)$ denotes a right(left) R-module. For an R-module M, $E(M)$ stands for the injective envelope of M, the character module $\mathrm{Hom}_Z(M,Q/Z)$ is denoted by M^+, and $\mathrm{id}(M)$($\mathrm{fd}(M)$) is the injective(flat) dimension of M.

B. Stenstr$\ddot{o}$m [11] defined and studied FP-injective modules. FP-injective modules are also called absolutely pure modules[9], these modules have been studied by many authors. In the paper [11], right Noetherian rings, right coherent rings, right semihereditary rings and regular rings are characterized by FP-injective right R-modules. It has been recently proven that every left R-module has an FP-injective cover over a left coherent ring R in the paper [9].On the other hand, every left R-module M has an FP-injective preenvelope over any ring in the paper [6]. In the paper [7], L.X.Mao and N.Q.Ding introduced the definitions of FI-injective and FI-flat modules and give some characterizations of these modules over left coherent rings. FI-injective and FI-flat modules together with the left derived functors of Hom are used to study the FP-injective dimensions of modules and rings.

As generalizations of the paper [7], we introduce the definitions of $(n,0)$-FI-injective and $(n,0)$-FI-flat modules and give some characterizations of these modules over left n-coherent rings. In addition, we investigate the left and right n-$\mathcal{FI}$-resolutions of R-modules by left (right) derived functors $\mathrm{Ext}_n(-,-)$ ($\mathrm{Tor}^n(-,-)$) over a left n-coherent ring, where n-$\mathcal{FI}$ stands for the categories of all $(n.0)$-injective left R-modules. These modules together with the left or right derived functors are used to study the $(n,0)$-injective dimensions of modules and rings.

We recall some known notions and facts needed in the sequel.

Let R be a ring and n be a non-negative integer. A left R-module M is called n-presented in case there is an exact sequence of left R-modules $F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ in which every F_i is a finitely generated free [3], equivalently projective left R-module. Let n,d be non-negative integers. According to [13] , a left R-module M is called (n,d)-injective(respectively (n,d)-flat) if $\mathrm{Ext}^{d+1}(N,M)=0$(respectively $\mathrm{Tor}_{d+1}(N,M)=0$) for all n-presented left (respectively right) R-modules N. The $(n,0)$-injective($(n,0)$-flat) dimension of M[14], denoted by $(n,0)$-id(M)($(n,0)$-fd(M)), is defined to be the smallest nonnegative integer m such that $\mathrm{Ext}^{m+1}(F,M)=0$($\mathrm{Tor}_{m+1}(F,M)=0$) for every n-presented left R-module F (if no such m exists, set $(n,0)$- id(M)($(n,0)$-fd(M)) $=\infty$), and l.$(n,0)$-dim(R) (l.$(n,0)$-wdim(R)) is defined as $\sup\{(n,0)\text{-id}(M)((\text{n,0})-fd(\text{M})) : M \text{ is a left } R\text{-module}\}$.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [5], we say that a homomorphism $\varphi : M \longrightarrow C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\varphi, C') : \mathrm{Hom}\ (C,C') \longrightarrow \mathrm{Hom}(M,C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\varphi : M \longrightarrow C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g : C \longrightarrow C$ such that $g\varphi = \varphi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers)may not exist in general, but if they exist, they are unique up to isomorphism. A homomorphism $\varphi : M \longrightarrow C$ with $C \in \mathcal{C}$ is said to a $\mathcal{C}$-envelope with the unique mapping property [5] if for any homomorphism $f : M \longrightarrow C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C \longrightarrow C'$ such that $g\varphi = f$. Dually we have the definition of a $\mathcal{C}$-cover with the unique mapping property.

In what follows, we write ${}_R\mathcal{M}$ and n-$\mathcal{FI}$ for the categories of all left R-modules and all $(n,0)$- injective left R-modules, respectively. According to Costa[7],a ring R is called a left n-coherent ring in case every n-presented left R-module is $(n+1)$-presented. It is easy to see that R is left 0-coherent(resp.1-coherent)if and only if it is left noetherian(resp. coherent), and every n-coherent ring is m-coherent for $m \geq n$. n-coherent rings have been investigated by many authors(see Chen and Ding[1,4],Costa[3]). For $n \geq 1$, it has been proven that every left R-module M has an $(n,0)$-injective preenvelope over any ring in [8]. So M has a right n-$\mathcal{FI}$-resolution, that is, there is a $\mathrm{Hom}(-, n\text{-}\mathcal{FI})$ exact complex $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow$

Jianmin Xing is with the College of Mathematics and Physics, Qingdao University of Science and Technology , Qingdao, 266061, P.R.China (e-mail: xingjianmin1978@163.com).
Rufeng Xing is with the College of Physical and Environmental Oceanography, Ocean University of China , Qingdao, 266100, P.R.China (e-mail: 100season@sina.com).

$\cdots$ with each $F^i(n,0)$-injective. Obviously, the complex is exact. Let

$$L^0 = M, L^1 = \text{coker}(M \longrightarrow F_0),$$
$$L^i = \text{coker}(F^{i-2} \longrightarrow F^{i-1}) \quad \text{for } i \geq 2$$

The nth cokernel $L_n (n \geq 0)$ is called the nth n-$\mathcal{FI}$-cosyzygy of M .

On the other hand, for $n \geq 1$, it has been proven that every left R-module has an $(n,0)$-injective cover over a left n-coherent ring R [8]. So every left R-module M has a left n-$\mathcal{FI}$-resolution, that is, there is a Hom (n-$\mathcal{FI}, -)$ exact complex $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$(not necessarily exact) with each $F_i(n,0)$-injective. Write

$$K_0 = M, K_1 = \ker(F_0 \longrightarrow M),$$
$$K_i = \ker(F_{i-1} \longrightarrow F_{i-2}) \quad \text{for } i \geq 2.$$

The nth kernel $K_n(n \geq 0)$ is called the nth n-$\mathcal{FI}$-syzygy of M.

Note that $\text{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times_R \mathcal{M}$ by n-$\mathcal{FI} \times n$-$\mathcal{FI}$ for a left n-coherent ring R (see[6, Definition 8.2.13]). Thus the nth left derived functor of $\text{Hom}(-,-)$, which is denoted by $\text{Ext}_n(-,-)$, can be computed using a right n-$\mathcal{FI}$-resolution of the first variable or a left n-$\mathcal{FI}$-resolution of the second variable. Following [6,Definition 8.4.1], the left n-$\mathcal{FI}$-dimension of a left R-module M, denoted by left n-$\mathcal{FI}$-dim$M$, is defined as $\inf\{m$: there is a left n-$\mathcal{FI}$-resolution of the form $0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ of $M\}$. If there is no such m, set left n-$\mathcal{FI}$-dim$(M) = \infty$. The global left n-$\mathcal{FI}$ dimension of ${}_R\mathcal{M}$, denoted by gl left n-$\mathcal{FI}$-dim$\mathcal{M}$, is defined to be $\sup\{$ left n-$\mathcal{FI}$-dim$(M) : M \in_R \mathcal{M}\}$ and is infinite otherwise. The right versions can be defined similarly.

Recall that a left R-module M is called reduced [6] if M has no nonzero injective submodules.

In Section II of this paper, we introduce the concepts of $(n,0)$-FI-injective and $(n,0)$-FI-flat modules. It is shown that a left R-module M is $(n,0)$-FI-injective if and only if M is a kernel of an $(n,0)$-injective precover $A \longrightarrow B$ with A injective. For a left n-coherent ring R, we prove that a left R-module M is $(n,0)$-FI-injective if and only if M is a direct sum of an injective left R-module and a reduced $(n,0)$-FI-injective left R-module; an n-presented right R-module M is $(n,0)$-FI-flat if and only if M is a cokernel of an $(n,0)$-flat preenvelope of a right R-module.

In Section III, we investigate the $(n,0)$-injective dimensions of modules and rings in terms of $(n,0)$-FI-injective and $(n,0)$-FI-flat modules and the left derived functors $\text{Ext}_n(-,-)$. Let R be a left n-coherent ring. We first give some characterizations of left n-hereditary rings. It is proven that R is left n-hereditary(i.e., l.$(n.0)$-dim$(R) \leq 1$) if and only if the canonical map σ :$\text{Ext}_0(M,N) \longrightarrow \text{Hom}(M,N)$ is a monomorphism for all left R-modules M and N if and only if every $(n,0)$-FI-injective left R-module is injective if and only if every$(n,0)$-FI-flat right R-module is flat. Then it is shown that l.$(n,0)$-dim$(R) \leq m (m \geq 2)$ if and only if $\text{Ext}_{m+k}(M,N) = 0$ for all left R-modules M, N and all $k \geq -1$.

In Section IV, we first investigate that the $- \otimes -$ on $\mathcal{M}_R \times_R \mathcal{M}$ is right balanced by n-$\mathcal{F}$ $\times n$-$\mathcal{FI}$ in the n-coherent ring, where n-$\mathcal{F}$ stands for the class of all $(n,0)$-flat modules. Then we introduce the right derived functors $\text{Tor}^n(-,-)$ and give some characteristic of right n-$\mathcal{F}$-dim M and n-$\mathcal{FI}$-dim M for any R-module M in the n-coherent ring R .

Let M and N be R-modules. $\text{Hom}(M,N)$ (respectively $\text{Ext}^n(M,N)$) means $\text{Hom}_R(M,N)$ (respectively $\text{Ext}^n_R(M,N)$), and similarly $M \otimes N$(respectively $\text{Tor}_n(M,N)$) denotes $M \otimes_R N$ (respectively $\text{Tor}^R_n(M,N)$) for an integer $n \geq 1$ throughout this paper. For unexplained concepts and notations, we refer the reader to [6,10,12].

II. $(n,0)$-FI-Injective Modules and $(n,0)$-FI-Flat Modules

Definition 1 A left R-module M is called $(n,0)$-FI-injective if $\text{Ext}^1(G,M) = 0$ for any $(n,0)$-injective left R-module G.

A right R-module N is said to be $(n,0)$-FI-flat if $\text{Tor}_1(N,G) = 0$ for any $(n,0)$-injective left R-module G.

Remark 1 (1) A right R-module M is $(n,0)$-FI-flat if and only if M^+ is $(n,0)$-FI-injective by the standard isomorphism: $\text{Ext}^1(N,M^+) \simeq \text{Tor}_1(M,N)^+$ for any left R-module N.

(2) We note that by the above definitions that $(1,0)$-FI-injective (flat) modules are FI-injective (flat)module in [7] and any FI-injective (flat) module is $(n,0)$-FI-injective (flat) for any $n \geq 1$.

Proposition 1 Let $\{M_i\}_I$ be family of right R-module

(1) $\oplus_I M_i$ is $(n,0)$-FI-flat if and only if each M_i is $(n,0)$-FI-flat;

(2) $\prod_I M_i$ is $(n,0)$-FI-injective if and only if each M_i is $(n,0)$-FI-injective.

Proof (1) By $\text{Tor}_1(G, \oplus_I M_i) \simeq \oplus_I \text{Tor}_1(G,M_i)$;

(2)By $\text{Ext}^1(G, \prod_I M_i) \simeq \prod_I \text{Ext}^1(G,M_i)$.

Definition 2 A ring R is said to be $(n,0)$-IP-ring if every $(n,0)$-injective R-module is projective ; R is said to be $(n,0)$-IF-ring if every $(n,0)$-injective R-module is flat. It is trivial to show that if $n \geq n'$, then every $(n,0)$-IP(IF) ring is an $(n',0)$-IP(IF) ring and every $(0,0)$-IP-ring is an quasi-Frobenius ring and every $(0,0)$-IF-ring is an IF ring.

Next,we shall see that the class of right $(n,0)$-IP(IF) -rings contains several important known rings.

Proposition 2 Let R be a ring.

(1) R is a right $(n,0)$-IP-ring if and only if every right module is $(n,0)$-FI-injective.

(2) R is a right $(n,0)$-IF-ring if and only if every left module is $(n,0)$-FI-flat.

(3) If R is a right $(n,0)$-IP-ring,then R is a right $(n,0)$-IF-ring.

Proof Directly by the definitions.

Corollary 1 Let R be a ring.

(1) R is a right quasi-Frobenius if and only if every right module is FI-injective.

(2) R is a right IF-ring if and only if every left module is FI-flat.

(3) If R is a right quasi-Frobenius, then R is a right IF -ring.

Proposition 3 The following hold for a left n-coherent ring R:

(1) A left R-module M is injective if and only if M is $(n,0)$-FI-injective and $(n,0)$-id$(M) \leq 1$.

(2) A right R-module N is flat if and only if N is $(n,0)$-FI-flat and $(n,0)$-fd$(N) \leq 1$.

Proof (1) "Only if" part is trivial.

"If" part. Let M be an $(n,0)$-FI-injective left R-module and $(n,0)$-id$(M) \leq 1$. Then there is an exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$ with E injective. Note that L is $(n,0)$-injective by[14, Theorem 2.12] since R is a left n-coherent ring. So the exact sequence is split, and hence M is injective.

(2)"Only if" part is trivial.

"If" part. For any $(n,0)$-FI-flat right R-module N with $(n,0)$-fd$(N) \leq 1$, we have N^+ is $(n,0)$-FI-injective by Remark 2.2 Thus N^+ is injective by (1) since $(n,0)$-id$(N^+) \leq 1$ by [14, Theorem 2.15]. So N is flat.

Proposition 4 The following are equivalent for a left R-module M:

(1) M is $(n,0)$-FI-injective.

(2) For every exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$, where E is $(n,0)$-injective, $E \longrightarrow L$ is an $(n,0)$-injective precover of L.

(3) M is a kernel of an $(n,0)$-injective precover $f : A \longrightarrow B$ with A injective.

(4) M is injective with respect to every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where C is $(n,0)$-injective.

Proof (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear by definitions.

(2) $\Rightarrow$ (3) is obvious since there exists a short exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$.

(3) $\Rightarrow$ (1) Let M be a kernel of an $(n,0)$-injective precover $f : A \longrightarrow B$ with A injective. Then we have an exact sequence $0 \longrightarrow M \longrightarrow A \longrightarrow A/M \longrightarrow 0$. So, for any $(n,0)$-injective left R-module N, the sequence $\mathrm{Hom}(N,A) \longrightarrow \mathrm{Hom}(N,A/M) \longrightarrow \mathrm{Ext}^1(N,M) \longrightarrow 0$ is exact. It is easy to verify that $\mathrm{Hom}(N,A) \longrightarrow \mathrm{Hom}(N,A/M) \longrightarrow 0$ is exact by (3). Thus $\mathrm{Ext}^1(N,M) = 0$, and so (1) follows.

(4) $\Rightarrow$ (1). For each $(n,0)$-injective left R-module N, there exists a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective, which induces an exact sequence $\mathrm{Hom}(P,M) \longrightarrow \mathrm{Hom}(K,M) \longrightarrow \mathrm{Ext}\ ^1(N,M) \longrightarrow 0$. Note that $\mathrm{Hom}(P,M) \longrightarrow \mathrm{Hom}(K,M) \longrightarrow 0$ is exact by (4). Hence $\mathrm{Ext}^1(N,M) = 0$, as desired.

Proposition 5 Let R be a left n-coherent ring. Then the following are equivalent for a left R- module M:

(1)M is a reduced $(n,0)$-FI-injective left R-module.

(2)M is a kernel of an $(n,0)$-injective cover $f : A \longrightarrow B$ with A injective.

Proof (1) $\Longrightarrow$ (2) By Proposition 4, the natural map $\pi : E(M) \longrightarrow E(M)/M$ is an $(n,0)$-injective precover. Note that $E(M)/M$ has an $(n,0)$-injective cover, and $E(M)$ has no nonzero direct summand K contained in M since M is reduced. It follows that $\pi : E(M) \longrightarrow E(M)/M$ is an $(n,0)$-injective cover by [12,Corollary 1.2.8], and hence (2) follows.

(2) $\Longrightarrow$ (1) Let M be a kernel of an $(n,0)$-injective cover $\alpha : A \longrightarrow B$ with$A$ injective. By Proposition 4, M is $(n,0)$-FI-injective. Now let K be an injective submodule of M. Suppose $A = K \oplus L, p : A \longrightarrow L$ is the projection and $i : L \longrightarrow A$ is the inclusion . It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism since α is a cover. Thus i is epic, and hence $A = L, K = 0$. So M is reduced.

Theorem 1 Let R be a left n-coherent ring. Then a left R-module M is $(n,0)$-FI-injective if and only if M is a direct sum of an injective left R-module and a reduced $(n,0)$-FI-injective left R-module.

Proof"If" part is clear.

"Only if" part. Let M be an $(n,0)$-FI-injective left R-module. Consider the exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$. Note that $E(M) \longrightarrow E(M)/M$ is an $(n,0)$-injective precover of $E(M)/M$ by Proposition 2.8. But $E(M)/M$ has an $(n,0)$-injective cover $L \longrightarrow E(M)/M$, so we have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{f} & L & \to & E(M)/M & \to & 0 \\
 & & \downarrow^{\varphi} & & \downarrow^{\gamma} & & \| & & \\
0 & \to & M & \xrightarrow{\alpha} & E(M) & \to & E(M)/M & \to & 0 \\
 & & \downarrow^{\sigma} & & \downarrow^{\beta} & & \| & & \\
0 & \to & K & \xrightarrow{f} & L & \to & E(M)/M & \to & 0
\end{array}$$

Note that $\beta\gamma$ is an isomorphism, and so $E(M) = \ker(\beta) \oplus \mathrm{im}(\gamma)$. Thus L and $\ker(\beta)$ are injective (for $\mathrm{im}(\gamma) \simeq L$). Therefore K is a reduced $(n,0)$-FI-injective module by Proposition 9. Since $\sigma\varphi$ is an isomorphism by the Five Lemma, we have $M = \ker(\sigma) \oplus \mathrm{im}(\varphi)$,where $\mathrm{im}(\varphi) \simeq K$.In addition, we get the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker(\sigma) & \to & \ker(\beta) & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \xrightarrow{\alpha} & E(M) & \to & E(M)/M & \to & 0 \\
 & & \downarrow^{\sigma} & & \downarrow^{\beta} & & \| & & \\
0 & \to & K & \xrightarrow{f} & L & \to & E(M)/M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Hence $\ker(\sigma) \simeq \ker(\beta)$ by the 3×3 Lemma [10,Exercise6.16,p.175]. This completes the proof.

It is well known that if R is a left n-coherent ring , then every right R-module has a $(n,0)$-flat preenvelope (see[13]). Here we have

Proposition 6 Let R be a left n-coherent ring.

(1) If L is a cokernel of a $(n,0)$-flat preenvelope $f : K \longrightarrow F$ of a right R-module K,where F is flat, then L is $(n,0)$-FI-flat.

(2) If M is an n-presented $(n,0)$-FI-flat right R-module, then M is a cokernel of an $(n,0)$-flat preenvelope.

Proof (1) There is an exact sequence $0 \longrightarrow \mathrm{im}(f) \xrightarrow{i} F \longrightarrow L \longrightarrow 0$. It is clear that i :$\mathrm{im}(f) \longrightarrow F$ is an $(n,0)$-flat preenvelope. For any $(n,0)$-injective left R-module N, N^+ is $(n,0)$-flat by [14,Theorem 2.15]. Thus we obtain an exact sequence

$$\mathrm{Hom}(F, N^+) \longrightarrow \mathrm{Hom}(\mathrm{im}(f), N^+) \longrightarrow 0,$$

which yields the exactness of $(F \otimes N)^+ \longrightarrow (\mathrm{im}(f) \otimes N)^+ \longrightarrow 0$. So the sequence $0 \longrightarrow \mathrm{im}(f) \otimes N \longrightarrow F \otimes N$ is exact. But the flatness of F implies the exactness of $0 = \mathrm{Tor}_1(F, N) \longrightarrow \mathrm{Tor}_1(L, N) \longrightarrow \mathrm{im}(f) \otimes N \longrightarrow F \otimes N$, and hence $\mathrm{Tor}_1(L, N) = 0$.

(2) Let M be an n-presented $(n, 0)$-FI-flat right R-module.There is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P finitely generated projective and K is $(n-1)$-presented. We claim that $K \longrightarrow P$ is an $(n, 0)$-flat preenvelope. In fact, for any $(n, 0)$-flat right R-module F, we have $\mathrm{Tor}_1(M, F^+) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{array}{ccccc} 0 & \longrightarrow & K \otimes F^+ & \xrightarrow{\alpha} & P \otimes F^+ \\ & & \downarrow^{\tau_{K,F}} & & \downarrow^{\tau_{P,F}} \\ & & \mathrm{Hom}(K, F)^+ & \xrightarrow{\theta} & \mathrm{Hom}(P, F)^+. \end{array}$$

Note that $\tau_{K,F}$ is an epimorphism and $\tau_{P,F}$ is an isomorphism by [2, Lemma 2]. Thus θ is a monomorphism, and hence $\mathrm{Hom}(P, F) \longrightarrow \mathrm{Hom}(K, F)$ is epic, as required.

We shall say that a right R-module M is strongly $(n, 0)$ -FI-flat if $\mathrm{Tor}_i(M, G) = 0$ for all $(n, 0)$-injective left R-modules G and all $i \geq 1$. Similarly, a left R-module N will be called strongly $(n, 0)$ - FI-injective if $\mathrm{Ext}^i(G, N) = 0$ for all $(n, 0)$-injective left R-modules G and all $i \geq 1$.

Theorem 2 Let R be a left and right n-coherent ring. Consider the following conditions:

(1) $(n, 0)$-id$({}_RR) \leq 1$.

(2) Every submodule of an $(n, 0)$-FI-flat right R-module, which factor module is n-presented, is $(n, 0)$-FI-flat.

(3) Every n-presented $(n, 0)$-FI-flat right R-module is strongly $(n, 0)$-FI-flat.

(4) Every $(n, 0)$-FI-injective left R-module is strongly $(n, 0)$-FI-injective.

(5) Every quotient of an $(n, 0)$-FI-injective left R-module is $(n, 0)$-FI-injective.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftarrow (4) \Leftarrow (5)$.

Proof (1) $\Rightarrow$ (2) Let A be a submodule of an $(n, 0)$-FI-flat right R-module B such that B/A is n-presented and M an $(n, 0)$- injective left R-module. Then one gets an exact sequence $\mathrm{Tor}_2(B/A, M) \longrightarrow \mathrm{Tor}_1(A, M) \longrightarrow \mathrm{Tor}_1(B, M) = 0$. On the other hand, there is a pure exact sequence $0 \longrightarrow M \longrightarrow \prod(R_R)^+$ since $(R_R)^+$ is a cogenerator in $R-$Mod. Thus we get a split exact sequence $(\prod(R_R)^+)^+ \longrightarrow M^+ \longrightarrow 0$. Note that $(n, 0)$-fd$((R_R)^+) = (n, 0)$-id$(R_R) \leq 1$ by [14,Theorem 2.15], and so $(n, 0)$-fd$(\prod(R_R)^+) \leq 1$ since R is right n-coherent. It follows that $(n, 0)$-id$((\prod(R_R)^+)^+) = (n, 0)$-fd$(\prod(R_R)^+) \leq 1$ by [14,Theorem 2.15]. Hence $(n, 0)$-fd$(M) = (n, 0)$-id$(M^+) \leq 1$. Thus $\mathrm{Tor}_2(B/A, M) = 0$ by the condition, and so $\mathrm{Tor}_1(A, M) = 0$. Therefore, A is $(n, 0)$-FI-flat.

(2) $\Rightarrow$ (3) Let M be an n-presented $(n, 0)$-FI-flat right R-module. Then there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective. So K is $(n, 0)$-FI-flat by (2). Thus M is strongly $(n, 0)$-FI-flat by induction.

(5) $\Rightarrow$ (4) Let M be an $(n, 0)$-FI-injective left R-module.Then there is an exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$ with E injective. So L is $(n, 0)$-FI-injective by(5). It is easy to check that M is strongly $(n, 0)$-FI-injective by induction.

(4) $\Rightarrow$ (3) holds by Remark 2 and the standard isomorphism: $\mathrm{Ext}^n(N, M^+) \simeq \mathrm{Tor}_n(M, N)^+$ for any right R-module M, any left R-module N and any $n \geq 1$ (see[10, p.360]).

Recall that a short exact sequence of right R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called n-pure if every n-presented right R-module is projective with respect to this sequence[14]. In this case , A is said to be an n-pure submodule of B .It is easy to see that the pure exact sequence is 1-pure exact in this definition, and the pure exact sequence must be n-pure . Let A be a pure submodule of the right R-module B , A must be an n-pure submodule of B.

Proposition 7 A left $(n, 0)$-FI-injective R-module N is $(n, 0)$-injective if and only if, for every n-presented left R-module M, every homomorphism $f : M \longrightarrow L$ factors through an injective left R-module,where L is a cokernel of injective envelope of N.

Proof "Only if" part. There is an exact sequence $0 \longrightarrow N \longrightarrow E(N) \xrightarrow{\pi} L \longrightarrow 0$ with E injective. Since the exact sequence is n-pure, there exists $g : M \longrightarrow E$ such that $\pi g = f$, as required.

"If" part. It is enough to show that the exacts equence $0 \longrightarrow N \xrightarrow{i} E(N) \xrightarrow{\pi} L \longrightarrow 0$ is n-pure by [14,Theorem 2.2]. Let M be any n-presented right R-module. For any $f : M \longrightarrow L$, there exist an injective left R-module Q and $g : M \longrightarrow Q$ and $h : Q \longrightarrow L$ such that $f = hg$ by hypothesis. Note that $E(N) \xrightarrow{\pi} L$ is a precover of L, since N is FI-injective by Proposition 4. Thus there exists $\alpha : Q \longrightarrow E(N)$ such that $h = \pi\alpha$,and so $f = \pi\alpha g$. Therefore we get an exact sequence $\mathrm{Hom}(M, E(N)) \longrightarrow \mathrm{Hom}(M, L) \longrightarrow 0$. So N is $(n, 0)$-injective.

III. $(n, 0)$-Injective Dimensions and the Left Derived Functors of Hom

As is mentioned in the introduction, if R is a left n-coherent ring, then $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times_R \mathcal{M}$ by n-$\mathcal{FI} \times n$-$\mathcal{FI}$. Let $\mathrm{Ext}_n(-,-)$ denote the nth left derived functor of $\mathrm{Hom}(-,-)$ with respect to the pair n-$\mathcal{FI} \times n$-$\mathcal{FI}$. Then, for two left R-modules M and N, $\mathrm{Ext}_n(M, N)$ can be computed using a right n-$\mathcal{FI}$-resolution of M or a left n-$\mathcal{FI}$-resolution of N.

Let $0 \longrightarrow M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \longrightarrow \cdots$ be a right n-$\mathcal{FI}$-resolution of M. Applying $\mathrm{Hom}(-, N)$,we obtain the deleted complex $\cdots \longrightarrow \mathrm{Hom}(F^1, N) \xrightarrow{f^*} \mathrm{Hom}(F^0, N) \longrightarrow 0$. Then $\mathrm{Ext}_n(M, N)$ exactly the nth homology of the complex above. There is a canonical map σ :

$$\mathrm{Ext}_0(M, N) = \mathrm{Hom}(F^0, N)/\mathrm{im}(f^*) \to \mathrm{Hom}(M, N)$$

defined by $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g$ for $\alpha \in \mathrm{Hom}(F^0, N)$.

Proposition 8 Let R be a left n-coherent ring. The following are equivalent for a left R-module M:

(1) M is $(n, 0)$-injective.

(2) The canonical map σ : $\mathrm{Ext}_0(M,N) \longrightarrow \mathrm{Hom}(M,N)$ is an epimorphism for any left R- module N.

(3) The canonical map σ : $\mathrm{Ext}_0(M,M) \longrightarrow \mathrm{Hom}(M,M)$ is an epimorphism.

Proof $(1) \Rightarrow (2)$ is obvious by letting $F^0 = M$.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (1)$. By (3), there exists $\alpha \in \mathrm{Hom}(F^0, M)$ such that $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g = 1_M$. Thus M is isomorphic to a direct summand of F^0, and hence it is $(n,0)$-injective.

Corollary 2 The following are equivalent for a left n-coherent ring R .

(1) ${}_RR$ is $(n,0)$-injective.

(2) The canonical map σ : $\mathrm{Ext}_0({}_RR,N) \longrightarrow \mathrm{Hom}({}_RR,N)$ is an epimorphism for any left R- module N.

(3) The canonical map σ :$\mathrm{Ext}_0({}_RR,{}_RR) \longrightarrow \mathrm{Hom}({}_RR,{}_RR)$ is an epimorphism.

(4) Every (n-presented) left R-module has an epic $(n,0)$-injective cover.

(5) Every (n-presented) right R-module has a monic $(n,0)$-flat preenvelope.

(6) Every (n-presented) right R-module is a submodule of a $(n,0)$-flat right R-module.

Proof $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follow from Proposition 8.

$(1) \Rightarrow (4)$. Let M be a left R-module, then M has an $(n,0)$-injective cover g. On the other hand, there is an exact sequence $F \longrightarrow M \longrightarrow 0$ with F free. Since F is $(n,0)$–injective by (1), g is an epimorphism.

$(4) \Rightarrow (1)$.Let $f : N \longrightarrow_R R$ be an epic $(n,0)$-injective cover. Then ${}_RR$ is isomorphic to a direct summand of N, and so ${}_RR$ is $(n,0)$-injective.

$(1) \Leftrightarrow (5)$. by [13, Theorem 4.5]

$(5) \Rightarrow (6)$ is obvious.

$(6) \Rightarrow (5)$ follows since R is a left n-coherent ring and by [13, Proposition 4.1].

Proposition 9 Let R be a left n-coherent ring. Then the following are equivalent for a left R- module M:

(1) right n-$\mathcal{FI}$-dim $M \leq 1$.

(2) The canonical map σ : $\mathrm{Ext}_0(M,N) \longrightarrow \mathrm{Hom}(M,N)$ is a monomorphism for any left R- module N.

Proof $(1) \Rightarrow (2)$.By (1), M has a right n-$\mathcal{FI}$-resolution $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow 0$. Thus we get an exact sequence $0 \longrightarrow \mathrm{Hom}(F^1,N) \longrightarrow \mathrm{Hom}(F^0,N) \longrightarrow \mathrm{Hom}(M,N)$ for any left R-module N. Hence σ is a monomorphism.

$(2) \Rightarrow (1)$. Consider the exact sequence $0 \longrightarrow M \longrightarrow F^0 \longrightarrow L^1 \longrightarrow 0$, where $M \longrightarrow F^0$ is an $(n,0)$-injective preenvelope. We only need to show that L^1 is $(n,0)$-injective. By [6,Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccc}
 & & \mathrm{Ext}_0(L^1,L^1) & \longrightarrow & \mathrm{Ext}_0(F^0,L^1) \\
 & & \downarrow^{\sigma_1} & & \downarrow^{\sigma_2} \\
0 & \longrightarrow & \mathrm{Hom}(L^1,L^1) & \longrightarrow & \mathrm{Hom}(F^0,L^1)
\end{array}$$

$$\begin{array}{cccc}
\longrightarrow & \mathrm{Ext}_0(M,L^1) & \longrightarrow & 0 \\
 & \downarrow^{\sigma_3} & & \\
\longrightarrow & \mathrm{Hom}(M,L^1) & &
\end{array}$$

Note that σ_2 is an epimorphism by Proposition 8 and σ_3 is a monomorphism by (2).Hence σ_1 is an epimorphism by the Snake Lemma[10, Theorem 6.5].Thus L^1 is $(n,0)$-injective by Proposition 8, and so (1)follows.

Lemma 1 Let R be a left n-coherent ring. Then

(1) right n-$\mathcal{FI}$- dim $(M) = (n,0)$-id(M) for any left R-module M;

(2) $(n,0)$-wdim(R) = l.$(n,0)$-dim(R) = gl right n-$\mathcal{FI}$-dim $\mathcal{M}$.

Proof (1) It is clear that $(n,0)$-id$(M) \leq$ right n-$\mathcal{FI}$-dim M. Conversely, we may assume that $(n,0)$-id$(M) = m < \infty$. Let $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^{m-1}$ be a partial right n-$\mathcal{FI}$-resolution of M. Then we get an exact sequence $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^{m-1} \longrightarrow L \longrightarrow 0$ Therefore, L is $(n,0)$-injective by [14, Theorem 2.12], and so right right n-$\mathcal{FI}$-dim $M \leq m$,as desired.

(2) follows from [14, Theorem 2.15] and (1).

Lemma 2 ([7]) Let $\mathcal{C}$ be a class of R-modules and M an R-module.

(1) If $F \longrightarrow M$ and $G \longrightarrow M$ are $\mathcal{C}$-precovers with kernels K and L,respectively, then $K \oplus G \simeq L \oplus F$.

(2) If $M \longrightarrow F$ and $M \longrightarrow G$ are $\mathcal{C}$-preenvelopes with cokernels K and L,respectively, then $K \oplus G \simeq L \oplus F$.

Recall that a left R is called left n-hereditary[14] if every $(n-1)$- presented submodule of projective left R-module is projective.

Clearly, a ring R is left semihereditary if and only if it is right 1- hereditary. Left n-hereditary ring is left $(n+1)$-hereditary.

Lemma 3([14]) The following statements are equivalent for a ring R:

(1)R is left n-hereditary.

(2)R is left n-coherent and $l.(n,0)$-dim$(R) \leq 1$.

(3)Factor module of $(n,0)$-injective left R-module is $(n,0)$-injective.

(4)Factor module of injective left R-module is $(n,0)$-injective.

(5)R is a right $(n,1)$-ring.

Theorem 3 The following are equivalent for a left n-coherent ring R:

(1) R is a left n-hereditary ring (i.e. $l.(n,0)$-dim$(R) \leq 1$).

(2) The canonical map σ : $Ext_0(M,N) \longrightarrow Hom(M,N)$ is is monic for all left R-modules M and N.

(3) Every left R-module has a monic $(n,0)$-injective cover.

(4) Every $(n,0)$-FI-injective left R-module is injective.

(5) Every $(n,0)$-FI-injective left R-module is $(n,0)$-injective.

(6) Every (n-presented) $(n,0)$-FI-flat right R-module is flat.

(7) The kernel of any $(n,0)$-injective (pre)cover of a left R-module is $(n,0)$-injective.

(8) The cokernel of any $(n,0)$-injective preenvelope of a left R-module is $(n,0)$-injective.

(9) The kernel of any $(n,0)$-flat (pre)cover of a right R-module is flat.

Proof $(1) \Leftrightarrow (2)$ holds by Proposition 9 and Lemma 1.

$(1) \Rightarrow (4)$ follows from Proposition 3 and Lemma 1.

$(4) \Rightarrow (5)$ is trivial.

$(5) \Rightarrow (6)$.Let M be an $(n,0)$-FI-flat right R-module. Then M^+ is $(n,0)$-FI-injective by Remark 1, and hence M^+ is

$(n,0)$-injective by (5). So M is $(n,0)$-flat by [14, Theorem 2.15].

(1) $\Rightarrow$ (3) follows from Lemma 3 and [13, Proposition 4.9].

(3) $\Rightarrow$ (7). Let $f: F \longrightarrow M$ be an $(n,0)$-injective precover of a left R-module M and K =ker(f). Since there exists a monic $(n,0)$-injective cover $g: G \longrightarrow M$ by (3), we have $K \oplus G \simeq F$ by Lemma 2(1). So K is $(n,0)$-injective.

(7) $\Rightarrow$ (1).It is enough to show that any quotient of an $(n,0)$-injective left R-module is $(n,0)$- injective. But it is clear by Lemma 2.

(1) $\Leftrightarrow$ (8) follows from Lemma 1.

(1) $\Leftrightarrow$ (9) is obvious.

Theorem 4 Let R be a left n-coherent ring and an integer $m \geq 2$.The following are equivalent for a left R-module M:

(1) right n-$\mathcal{FI}$-dim $M \leq m$.

(2) $\mathrm{Ext}_{m+k}(M,N) = 0$ for all left R-modules N and all $k \geq -1$.

(3)$\mathrm{Ext}_{m-1}(M,N) = 0$ for all left R-modules N.

Proof (1) $\Rightarrow$ (2). Let $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^m \longrightarrow 0$ be a right n-$\mathcal{FI}$- resolution of M, which induces an exact sequence $0 \to \mathrm{Hom}(F^m,N) \to \mathrm{Hom}(F^{m-1},N) \to \mathrm{Hom}(F^{m-2},N)$ for any left R-module N. Hence $\mathrm{Ext}_m(M,N)$ =$\mathrm{Ext}_{m-1}(M,N) = 0$. Note that it is clear that $\mathrm{Ext}_{m+k}(M,N) = 0$ for all $k \geq 1$.Then (2)holds.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $0 \longrightarrow M \longrightarrow F^0 \longrightarrow \cdots \longrightarrow F^{m-2} \xrightarrow{f} F^{m-1} \xrightarrow{g} F^m \longrightarrow \cdots$ be a right n-$\mathcal{FI}$- resolution of M, with $L^m = \mathrm{coker}(F_{m-2} \longrightarrow F_{m-1})$. We only need to show that L^m is $(n,0)$-injective. In fact, we have the exact sequence $F^{m-1} \xrightarrow{\pi} L^m \longrightarrow 0$ and $0 \longrightarrow L^m \xrightarrow{\lambda} F^{m-1}$ such that $g = \lambda\pi$ by (3), Ext $_{m-1}(M,L^m) = 0$. Thus the sequence $0 \longrightarrow \mathrm{Hom}(F^m,L^m) \xrightarrow{g^*} \mathrm{Hom}(F_{m-1},L^m) \xrightarrow{f^*} \mathrm{Hom}(F^{m-2},L^m)$ is exact. Since $f^*(\pi) = \pi f = 0, \pi \in \ker(f^*) = \mathrm{im}(g^*)$. Thus there exists $h \in \mathrm{Hom}(F^m,L^m)$ such that $\pi = g^*(h) = hg = h\lambda\pi$, and hence $h\lambda = 1$ since π is epic. Therefore L^m is $(n,0)$-injective.

Corollary 3 The following are equivalent for a left n-coherent ring R and an integer $m \geq 2$:

(1) $l.(n,0)$-dim$(R) \leq m$.

(2) $\mathrm{Ext}_{m+k}(M,N) = 0$ for all left R-modules M and N, and all $k \geq -1$.

(3) $\mathrm{Ext}_{m-1}(M,N) = 0$ for all left R-modules M and N.

Proof It follows from Lemma 1 and Theorem 4.

It has been proven that R is a left coherent ring and l.FP-dim$(R) \leq 2$ if and only if every right R-module has an FP-injective cover with the unique mapping property . Now we have

Theorem 5 The following are equivalent for a ring R:

(1) R is left n-coherent and $l.(n,0)$-dim$(R) \leq 2$.

(2) Every left R-module has an $(n,0)$-injective cover with the unique mapping property.

(3) R is left n-coherent and $\mathrm{Ext}_1(M,N) = 0$ for all left R-modules M and N.

(4)R is left n-coherent and $\mathrm{Ext}_k(M,N) = 0$ for all left R-modules M, N and all $k \geq 1$.

Proof(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Corollary 3.

(1) $\Rightarrow$ (2).Let M be any left R-module. Then M has an $(n,0)$-injective cover $f: F \longrightarrow M$.It is enough to show that, for any $(n,0)$-injective left R-module G and any homomorphism $g: G \longrightarrow F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta: F/\mathrm{im}(g) \longrightarrow M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi: F \longrightarrow F/\mathrm{im}(g)$ is the natural map. Since $l.(n,0)$-dim$(R) \leq 2$, $F/\mathrm{im}(g)$ is $(n,0)$-injective. Thus there exists $\alpha: F/\mathrm{im}(g) \longrightarrow F$ such that $\beta = f\alpha$, and so we get the commutative diagram with an exact row:

$$\begin{array}{ccccccc} G & \xrightarrow{g} & F & \overset{\pi}{\Longleftrightarrow}_{\alpha} & F/\mathrm{im}(g) & \longrightarrow & 0 \\ & \searrow^{0} & \downarrow^{f} & \swarrow^{\beta} & & & \\ & & M & & & & \end{array}$$

Thus $f\alpha\pi = f$,and hence $\alpha\pi$ is an isomorphism. Therefore, π is monic, and so $g = 0$.

(2) $\Rightarrow$ (1). We first prove that R is a left n-coherent ring. Let $\{C_i, \varphi_j^i\}$ be a direct system with each C_i $(n,0)$-injective. By hypothesis, $\varinjlim C_i$ has an $(n,0)$-injective cover $\alpha: E \longrightarrow \varinjlim C_i$ with the unique mapping property. Let $\alpha_i: C_i \longrightarrow \varinjlim C_i$ satisfy $\alpha_i = \alpha_j\varphi_j^i$ whenever $i \leq j$. Then there exists$f_i: C_i \longrightarrow E$ such that $\alpha_i = \alpha f_i$ for any i. It follows that $\alpha f_i = \alpha f_j \varphi_j^i$, and so $f_i = f_j\varphi_j^i$ whenever $i \leq j$. Therefore, by the definition of direct limits, there exists $\beta: \varinjlim C_i \longrightarrow E$ such that $f_i = \beta\alpha_i$ and $f_j = \beta\alpha_j$. Thus $(\alpha\beta)\alpha_i = \alpha(\beta\alpha_i) = \alpha f_i = \alpha_i$ for any i. Therefore $\alpha\beta = 1_{\varinjlim C_i}$, by the definition of direct limits, and hence $\varinjlim C_i$ is a direct summand of E. So $\varinjlim C_i$ is $(n,0)$-injective. Thus R is a left n-coherent ring by [1].

Next we prove that $l.(n,0)$-dim$(R) \leq 2$. Let M be any left R-module. Then M has an $(n,0)$- injective cover $f: F \longrightarrow M$ with the unique mapping property. So $0 \longrightarrow F \longrightarrow M \longrightarrow 0$ is a left n-$\mathcal{FI}$-resolution. Thus gl left n-$\mathcal{FI}$-dim $_R\mathcal{M} = 0$, and hence $l.(n,0)$-dim$(R) \leq 2$ by Corollary 3.

Proposition 10 Let R be a left n-coherent ring. If M is an n-pure-injective left R-module , then $(n,0)$-id$(M) \leq m (m \geq 0)$ if and only if for the minimal left n-$\mathcal{FI}$-resolution $\cdots \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ of all n-pure-injective left R-module N, $\mathrm{Hom}(M,F_m) \longrightarrow \mathrm{Hom}(M,K_m)$ is an epimorphism.

Proof The proof is modeled on that of [6, Lemma 8.4.34].

We will proceed by induction on m. Let $m = 0$. If M is $(n,0)$- injective , it is clear that $\mathrm{Hom}(M,F_0) \longrightarrow \mathrm{Hom}(M,K_0)$ is an epimorphism, since $F_0 \longrightarrow N$ is an $(n,0)$-injective cover of N. Conversely, put $N = M$. Then $\mathrm{Hom}(M,F_0) \longrightarrow \mathrm{Hom}(M,M)$ is an epimorphism, and so M is $(n,0)$-injective.

Let $m \geq 1$. There is an exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$ with E injective. Then we have the following exact commutative diagrams:

$$\begin{array}{ccccc} \mathrm{Hom}(E,F_n) & \longrightarrow & \mathrm{Hom}(E,K_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}(M,F_n) & \longrightarrow & \mathrm{Hom}(M,K_n) & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(L, K_m) & \longrightarrow & \mathrm{Hom}(L, F_{m-1}) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(E, K_m) & \longrightarrow & \mathrm{Hom}(E, F_{m-1}) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(M, K_m) & \longrightarrow & \mathrm{Hom}(M, F_{m-1}) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}$$

$$\begin{array}{cccc}
 & 0 & & \\
 & \downarrow & & \\
\longrightarrow & \mathrm{Hom}(L, K_{m-1}) & & \\
 & \downarrow & & \\
\longrightarrow & \mathrm{Hom}(E, K_{m-1}) & \longrightarrow & 0 \\
 & \downarrow & & \\
\longrightarrow & \mathrm{Hom}(M, K_{m-1}) & &
\end{array}$$

Thus $(n, 0)$-id$(M) \leq m$ if and only if $(n, 0)$-id$(L) \leq m - 1$ by [14, Theorem 2.12.], if and only if $\mathrm{Hom}(L, F_{m-1}) \longrightarrow \mathrm{Hom}(L, K_{m-1})$ is an epimorphism by induction if and only if $\mathrm{Hom}(E, K_m) \longrightarrow \mathrm{Hom}(M, K_m)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, F_m) \longrightarrow \mathrm{Hom}(M, K_m)$ is an epimorphism by the first diagram.

IV. $(n, 0)$-Injective Dimensions and the Right Derived Functors of Tor

In this section,we introduce the right derived functors of Tor. If R is n-coherent, the $- \otimes -$ on $\mathcal{M}_R \times_R \mathcal{M}$ is right balanced by n-$\mathcal{F}$ $\times n$-$\mathcal{FI}$, where n-$\mathcal{F}$ stands for the class of all $(n, 0)$-flat modules. In fact, we need to show that if $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ is a right n-$\mathcal{F}$-resolution, which exists by [13, Lemma 4.1], and G is an $(n, 0)$-injective left R-module, then $0 \longrightarrow M \otimes G \longrightarrow F^0 \otimes G \longrightarrow F^1 \otimes G \longrightarrow \cdots$ is exact. Applying the functor $\mathrm{Hom}_Z(-, Q/Z)$ and using a standard identity we see the sequence $0 \longleftarrow \mathrm{Hom}(M, G^+) \longleftarrow \mathrm{Hom}(F^0, G^+) \longleftarrow \mathrm{Hom}(F^1, G^+) \longleftarrow \cdots$. But G^+ is $(n, 0)$-flat by [14,Theorem 2.15] and so this sequence is exact. This means the desired sequence is exact. Since right n-$\mathcal{FI}$-resolutions are exact , let $0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$ of a left R-module N, then $\cdots \longrightarrow G^{1+} \longrightarrow G^{0+} \longrightarrow N^+ \longrightarrow 0$ is a left n-$\mathcal{F}$-resolution . So applying the functor $\mathrm{Hom}(F, -)$ to above sequence, we get the exact sequence $\cdots \longrightarrow \mathrm{Hom}(F, G^{1+}) \longrightarrow \mathrm{Hom}(F, G^{0+}) \longrightarrow \mathrm{Hom}(F, N^+) \longrightarrow 0$ for F $\in n$-$\mathcal{F}$. Using a standard identity we get the exact sequence $0 \longrightarrow F \otimes N \longrightarrow F \otimes G^0 \longrightarrow F \otimes G^1 \longrightarrow \cdots$.

Let $\mathrm{Tor}^n(-, -)$ denote the nth right derived functor of $- \otimes -$ with respect to the pair n-$\mathcal{F} \times n$-$\mathcal{FI}$. Then, for two left R-modules M and N, $\mathrm{Tor}^n(M, N)$ can be computed using a right n-$\mathcal{F}$-resolution of M or a right n-$\mathcal{FI}$-resolution of N.

Lemma 4 If $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_4$ is an exact sequence of left R-moudules such that for every n-presented right R-module $P, P \otimes M_1 \longrightarrow P \otimes M_2 \longrightarrow P \otimes M_3 \longrightarrow P \otimes M_4$ is exact , then K =ker$(M_3 \longrightarrow M_4)$ is an n-pure submodule of M_3.

Proof $P \otimes M_1 \longrightarrow P \otimes M_2 \longrightarrow P \otimes M_3 \longrightarrow P \otimes M_4$ is exact and $P \otimes K \longrightarrow P \otimes M_3 \longrightarrow P \otimes M_4$ is a complex. Thus exactness of the first sequence means $0 \longrightarrow P \otimes K \longrightarrow P \otimes M_3$ is exact. This means K is an n-pure submodule of M_3

Theorem 6 Let R be a left n-coherent ring and an integer $m \geq 2$.The following are equivalent for a left R-module N:

(1) right n-$\mathcal{FI}$-dim $N \leq m$.

(2) $\mathrm{Tor}^{m+k}(M, N) = 0$ for all right R-modules M and all $k \geq -1$.

(3)$\mathrm{Tor}^m(M, N)$ $=\mathrm{Tor}^{m-1}(M, N) = 0$ for all right R-modules M.

(4)$\mathrm{Tor}^m(M, N)$ $=\mathrm{Tor}^{m-1}(M, N) = 0$ for all right n-presented R-modules M.

Proof (1) $\Rightarrow$ (2) Let $0 \longrightarrow N \longrightarrow A^0 \longrightarrow \cdots \longrightarrow A^n \longrightarrow 0$ be a right n-$\mathcal{FI}$-resolution of N. Then $M \otimes A^{n-2} \longrightarrow M \otimes A^{n-1} \longrightarrow M \otimes A^n \longrightarrow 0$ is exact and so $\mathrm{Tor}^{m-1}(M, N) = \mathrm{Tor}^m(M, N) = 0$. But clearly $\mathrm{Tor}^{m+k}(M, N) = 0$ for $k \geq -1$. Hence (2) holds.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). Let $0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$ be a right n-$\mathcal{FI}$-resolution of N. Then for any n-presented R-module M,$M \otimes A^{n-2} \longrightarrow M \otimes A^{n-1} \longrightarrow M \otimes A^n \longrightarrow M \otimes A^{n+1}$ is exact. So by Lemma 4, K =ker$(A^n \longrightarrow A^{n+1})$ is n-pure in A^n. But an n-pure submodule of $(n, 0)$-injective module is $(n, 0)$-injective by [14, Proposition 2.2]. But then $0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots A^{n-1} \longrightarrow K \longrightarrow 0$ is a right n-$\mathcal{FI}$-resolution of N and (1) holds.

Theorem 7 Let R be a left n-coherent ring and an integer $m \geq 2$.The following are equivalent for a left R-module N:

(1) right n-$\mathcal{F}$-dim $M \leq m$.

(2) $\mathrm{Tor}^{m+k}(M, N) = 0$ for all right R-modules N and all $k \geq -1$.

(3)$\mathrm{Tor}^m(M, N)$ $=\mathrm{Tor}^{m-1}(M, N) = 0$ for all right R-modules N.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ be a right n-$\mathcal{F}$-resolution of N. Then for any R-module N, $F^{n-2} \otimes N \longrightarrow F^{n-1} \otimes N \longrightarrow F^n \otimes N \longrightarrow F^{n+1} \otimes N$ is exact. So by Lemma 4, K =ker$(F^n \longrightarrow F^{n+1})$ is n-pure in F^n and so is $(n, 0)$-flat. But $F^{n-2} \longrightarrow F^{n-1} \longrightarrow K \longrightarrow 0$ is exact. Therefore, L =ker$(F^{n-2} \longrightarrow F^{n-1})$ is n-pure in F^{n-2} and so is $(n, 0)$-flat by [14, Corollary 2.20]. But then $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^{n-3} \longrightarrow L \longrightarrow 0$ is a right n-$\mathcal{F}$-resolution of M and so (1) holds.

Theorem 8 Let R be a left n-coherent ring and an integer $m \geq 0$. The following are equivalent

(1) For every $(n, 0)$-flat left R-module F, there is an exact sequence $0 \longrightarrow F \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^m \longrightarrow 0$ with each E^i is $(n, 0)$-injective.

(2) If $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ is a right n-$\mathcal{F}$-resolution of M, then the sequence is exact at F^k for $k \geq m - 1$, where $F^{-1} = M$.

(3) There is an exact sequence $0 \longrightarrow R \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^m \longrightarrow 0$ of left R-module with each E^i is $(n, 0)$-injective.

Proof (1) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (2) We recall that $-\otimes-$ is right balanced on $\mathcal{M}_R \times_R \mathcal{M}$ by n-$\mathcal{F} \times n$-$\mathcal{FI}$ with right derived functors Tor$^k(-,-)$.

If $m \geq 2$, using the exact sequence $0 \longrightarrow R \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^m \longrightarrow 0$, we get Tor$^k(M, R) = 0$ for $k \geq m-1$. Computing using $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ as in (2), we see that Tor$^k(M, R)$ is just the kth homology group of this complex, giving the desired result.

For $m = 1, 0 \longrightarrow R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow 0$ exact sequence gives Tor$^1(M, R) = 0$ so that , as above, $F^0 \longrightarrow F^1 \longrightarrow F^2$ is exact and $M \otimes R \longrightarrow$ Tor$^0(M, R)$ is onto. computing the latter morphism using $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1$ is exact.

If $m = 0$ then (3) means R is $(n, 0)$-injective as a left R-module. But the balance of $- \otimes -$ then gives $0 \longrightarrow M \otimes R \longrightarrow F^0 \otimes R \longrightarrow F^1 \otimes R \longrightarrow \cdots$ is exact . That is $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ is exact.

(2) $\Rightarrow$ (1). Assume (2) with $m \geq 2$. Let $0 \longrightarrow F \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^m \longrightarrow 0$ with each E^i is $(n, 0)$-injective. Then by (2), we get Tor$^k(M, F) = 0$ for $k \geq m-1$ since F is $(n, 0)$-flat. Computing using $0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ and using the Lemma 4 , we get $K =$ker$(E^m \longrightarrow E^{m+1})$ is n-pure in A^m and so K is also $(n, 0)$-injective. Hence $0 \longrightarrow F \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{m-1} \longrightarrow K \longrightarrow 0$ gives the desired exact sequence.

Now let $m = 1$.Then (2) says $M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ is exact. So Tor$^k(M, F) = 0$ for $k = 0$ and $M \otimes F \longrightarrow$ Tor$^0(M, F)$ is onto. Hence if $0 \longrightarrow F \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots 0$ is exact, $M \otimes F \longrightarrow M \otimes E^0 \longrightarrow M \otimes E^1 \longrightarrow M \otimes E^2$ is exact for all n-presented M. By Lemma 25, we again get the desired exact sequence $0 \longrightarrow F \longrightarrow E^0 \longrightarrow K \longrightarrow 0$ with $K =$ker$(E^1 \longrightarrow E^2)$.

If $m = 0$ then $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ exact means Tor$^k(M, F) = 0$ for $k > 0$ and $M \otimes F \longrightarrow$ Tor$^0(M, F)$ is isomorphism. This gives that $0 \longrightarrow M \otimes F \longrightarrow M \otimes E^0 \longrightarrow M \otimes E^1$ is exact for all M which implies F is an n-pure submodule of E^0 and so is $(n, 0)$-injective.

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