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Gorenstein Projective, Injective and Flat Modules Relative to Semidualizing Modules

Jianmin Xing, Rufeng Xing

Abstract—In this paper we study some properties of G_C-projective, injective and flat modules, where C is a semidualizing module and we discuss some connections between G_C-projective, injective and flat modules , and we consider these properties under change of rings such that completions of rings, Morita equivalences and the localizations.

Keywords—Semidualizing module, C-projective(injective,flat), G_C-projective (injective,flat), Commutative ring; Localizations .

I. Introduction

UNLESS stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary modules. Let R be a ring, we denote by R-Mod(Mod-R)the category of left(right) R-modules respectively. For any R-module M, we denote by $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ the projective dimension, injective dimension and flat dimension of M respectively, and denote by $M^+ = \mathrm{Hom}_Z(M, Q/Z)$ the characteristic module of M.For unexplained concepts and notations, we refer the reader to [2, 3, 8].

When R is two-sided Noetherian, Auslander and Bridger [1] introduced the G-dimension, G-$\dim_R(M)$ for every finitely generated R-module M. Several decades later, Enochs and Jenda [6,7] extended the ideas of Auslander and Bridger and introduced the Gorenstein projective, injective and flat dimensions. The homological properties of the Gorenstein projective dimension and some generalized versions of such a dimension have been studied by many authors,see [4,5, 6, 7, 9, 10, 11]. Foxby [9] and Golod [10] independently initiated the study of semidualizing modules (under different names). Examples include the rank 1 free module and a dualizing (canonical) module, when one exists. Golod [10] used these to define G_C-dimension, a refinement of projective dimension, for finitely generated modules. The G_C-dimension of a finitely generated R-module M is the length of the shortest resolution of M by so-called totally C-reflexive modules.

White introduced in [20] the G_C-projective modules and gave a functorial descriptions of the G_C-projective dimension of modules with respect to a semidualizing module over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [20]. Over a commutative Noetherian ring, the G_C-projective modules and the G_C-projective dimension were called C-Gorenstein projective modules and the C-Gorenstein projective dimension in [12], respectively. Note that the non-commutative versions of almost all the results in [20] also hold true.

In this paper, based on the results mentioned above,we further investigate the properties of the G_C-projective, injective and flat modules over general rings. and investigate the relation among them. At last, we study these properties under change of rings such that completions of rings, Morita equivalences and the localizations.

This paper is organized as follows.

In Section II, we give some definitions and and basic properties of G_C-projective, injective and flat modules such that semidualizing module , C-projective,injective and flat module .

In Section III, we study the relation and the properties among G_C-projective, injective and flat modules.

In Section IV, we consider the properties under change of rings. Specially, we consider the completions of rings, Morita equivalences and the localizations.

II. Preliminary Notes

In this section we give some definitions of G_C-projective, injective flat modules, and some known results about them. At first we introduce the semidualizing module and C-projective , injective and flat modules which are defined as follows:

Definition 1 [13]Let R, S be rings, an (S, R)-bimodule $C = {}_S C_R$ is semidualizing if the following conditions are satisfied.

(a1) ${}_S C$ admits a degreewise finite S-projective resolution.

(a2) C_R admits a degreewise finite R^{op}-projective resolution.

(b1) The homothety map ${}_S S_S \longrightarrow \mathrm{Hom}_{R^{op}}(C, C)$ is an isomorphism.

(b2) The homothety map ${}_R R_R \longrightarrow \mathrm{Hom}_S(C, C)$ is an isomorphism.

(c1) $\mathrm{Ext}^i_S(C, C) = 0$ for any $i \geq 1$.

(c2) $\mathrm{Ext}^i_R(C, C) = 0$ for any $i \geq 1$.

Unless otherwise stated, when $R = S$ is commutative, all semidualizing bimodules in this paper are symmetric in the sense that the two R-actions on C agree. In this case we will use the terminology "C is semidualizing over R". Note that when $R = S$ is commutative and noetherian, Definition 1 agrees with the established terminology; that is a finitely generated R-module C is semidualizing if the natural homothety map $R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism and

Jianmin Xing is with the College of Mathematics and Physics, Qingdao University of Science and Technology , Qingdao, 266061, P.R.China (e-mail: xingjianmin1978@163.com).
Rufeng Xing is with the College of Physical and Environmental Oceanography, Ocean University of China , Qingdao, 266100, P.R.China (e-mail: 100season@sina.com).

$\mathrm{Ext}^i_R(C,C) = 0$ for any $i \geq 1$. Two examples are the free module of rank 1, and over a Cohen-Macaulay local ring, the dualizing (canonical) module, when it exists. It is easy to see that a semidualizing module is finitely generated and even finitely presented.

Definition 2[13] Let R be a ring, ${}_RC_R$ be a semidualizing bimodule, a module in R-Mod is called C-projective if it has the form $C \otimes_R P$ for some projective module $P \in R$-Mod. A module in R-Mod is called C-injective if it has the form $\mathrm{Hom}_R(C, I)$ for some injective module $I \in R$-Mod . A module in R-Mod is called C-flat if it has the form $C \otimes_R F$ for some flat module $F \in R$-Mod . Set

$\mathcal{P}_C(R) = \{C \otimes_R P | {}_RP \text{ is projective}\}$,

$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) | {}_RI \text{ is injective}\}$ and

$\mathcal{F}_C(R) = \{C \otimes_R F | {}_RF \text{ is flat}\}$

Let $M \in R$-Mod. We denote $\mathrm{Add}_R M$ (resp. $\mathrm{Prod}_R M$) the subclass of R-Mod consisting of all modules isomorphic to direct summands of direct sums(resp. direct products)of copies of M. The following result was proved in [15, Proposition 2.4.].

Lemma 1[15]

(1) $\mathcal{P}_C(R) = \mathrm{Add}_R C$;

(2)$\mathcal{I}_C(R) = \mathrm{Prod}_R C^+$, where $C^+ = \mathrm{Hom}_R(C, E)$ with ${}_RE$ a injective cogenerator for Mod-R.

The following notions were introduced by Holm in [12] and White in [20] for commutative rings. The non-commutative versions of them were given in [15]. Now we give the commutative versions of them.

Definition 3[15]

(1) A complete $\mathcal{PP}_C$-resolution is a $\mathrm{Hom}_R(-, \mathrm{Add}_R C)$ exact exact sequence:

$$\mathcal{X} = \cdots \longrightarrow P_0 \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \cdots \quad (1)$$

in R-Mod with all P^i and P_i projective. A module $M \in R$-Mod is called G_C-projective if there exists a complete $\mathcal{PP}_C$ - resolution as in (1) with $M = \mathrm{coker}(P_1 \longrightarrow P_0)$. Set

$$\mathcal{GP}_C(R) = \text{ the class of } G_C-\text{projective modules in R-Mod.}$$

(2) A complete $\mathcal{I}_C\mathcal{I}$-resolution is a $\mathrm{Hom}_R(\mathrm{Prod}\ C^+, -)$ exact exact sequence:

$$\mathcal{Y} = \cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \quad (2)$$

in Mod-R with all I^i injective and $I_i \in \mathrm{Prod}_R C^+$. A module $M \in$ Mod-R is called G_C-injective if there exists a complete $\mathcal{I}_C\mathcal{I}$ - resolution as in (2) with $M = \mathrm{Im}(I_0 \longrightarrow I^0)$. Set

$$\mathcal{GI}_C(R) = \text{ the class of } G_C-\text{injective modules in Mod}-R.$$

(3) A complete $\mathcal{FF}_C$-resolution is a $\mathcal{I}_C(R)\otimes-$ exact exact sequence:

$$\mathcal{X} = \cdots \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots \quad (3)$$

in R-Mod with all F^i and F_i flat. A module $M \in R$-Mod is called G_C-flat if there exists a complete $\mathcal{FF}_C$-resolution as in (3) with $M = \mathrm{coker}(F_1 \longrightarrow P_0)$. Set

$$\mathcal{GF}_C(R) = \text{ the class of } G_C-\text{ flat modules in } R-\text{Mod}.$$

It is trivial that in case ${}_RC_R = {}_R R_R$,the G_C-projective (injective and flat) modules are just the usual Gorenstein projective (injective and flat) modules, respectively.

Using the definitions, we immediately get the following results.

Proposition 1

(1)If $(M_i)_{i\in I}$ is a family of G_C-projective modules, then $\oplus M_i$ is G_C-projective.

2)If $(F_i)_{i\in I}$ is a family of G_C-flat modules, then $\oplus F_i$ is G_C-flat.

III. THE G_C PROPERTY

In this section we always assume that R is a commutative ring without special instruction and C is a semidualizing R-bimodule, then we study the properties and relationship among G_C-projective, injective and flat modules.

Lemma 2 Let R be a commutative left coherent ring. Then

(1) M is an C-flat left R-module if and only if M^+ is an C-injective right R-module.

(2) M is an C-injective R-module if and only if M^+ is an C-flat right R-module.

Proof By virtue of the conclusion in Lemma 4.1 of [19].

Now we give the relation of G_C-flat modules and G_C-injective modules. The first conclusion is the Lemma 5.2 in [19].

Proposition 2[19] Let C be a semidualizing R-module. If M is an R-module, then M is in $\mathcal{GF}_C(R)$ if and only if its characteristic module M^+ is in $\mathcal{GI}_C(R)$.

Proposition 3 Let R be a commutative artinian ring , if M is a G_C- injective left R-module, then M^+ is a G_C-flat right R-module.

Proof There exists an complete $\mathcal{I}_C\mathcal{I}$-exact sequence

$$\cdots \to \mathrm{Hom}_R(C, I_1) \to \mathrm{Hom}_R(C, I_0) \to I^0 \to I^1 \to \cdots$$

with I_i, I^i injective for $i \geq 0$ and

$$M = \mathrm{coker}(\mathrm{Hom}_R(C, I_1) \longrightarrow \mathrm{Hom}_R(C, I_0))$$

. Let J be any injective left R- module. Then $J = \oplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [14, Theorem 6.6.4], and hence $\mathrm{Tor}\ {}^R_1(M^+, C \otimes J) = \oplus_\Lambda \mathrm{Tor}\ {}^R_1(M^+, C \otimes J_\alpha) = \oplus_\Lambda\ \mathrm{Ext}\ {}^1_R(C \otimes J_\alpha, M) = 0$ by [8, Theorem 3.2.13] for all $i \geq 1$. Therefore M^+ is a G_C-flat right R-module.

Proposition 4 Let R be a commutative ring and Q a projective R-module. If M is an G_C-projective left R- module, then $M \otimes_R Q$ is a G_C-projective left R-module.

Proof There is an exact sequence

$$\cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

with P_i, P^i projective and $M = \mathrm{coker}\ (P_1 \to P_0)$. Then the sequence

$$\cdots \to P_1 \otimes Q \to P_0 \otimes Q \to C \otimes_R P^0 \otimes Q \to C \otimes_R P^1 \otimes Q \to \cdots$$

is exact with $P_i \otimes Q, P^i \otimes Q$ projective by [18, Ch. 2, 1 Theorem 3]. Let Q' be any projective left R-module. Then $\mathrm{Ext}^1_R(M \otimes_R Q, C \otimes Q') = \mathrm{Hom}\ _R(Q, \mathrm{Ext}^1_R(M, C \otimes Q')) =$

0 by[17, p. 258, 9.20] for all $i \geq 1$. Hence $M \otimes_R Q$ is a G_C-projective R-module .

Proposition 5 Let R be a commutative ring and F a flat left R-module. If M is an G_C-flat left R- module, then $M \otimes_R F$ is a G_C-flat left R-module.

Proof There is an exact sequence

$$\cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

with F_i, F^i flat and M =coker $(F_1 \to F_0)$. Then the sequence

$$\cdots \to F_1 \otimes F \to F_0 \otimes F \to C \otimes_R F^0 \otimes F \to C \otimes_R F^1 \otimes F \to \cdots$$

is exact with $F_i \otimes F, F^i \otimes F$ flat by [18, Ch. 2, 1 Theorem 3]. Let I be any injective R-module and $\mathcal{F}$ be a flat resolution of $\mathrm{Hom}(C, I)$. Then

$$\begin{aligned} & \mathrm{Tor}_1^R(M \otimes_R F, \mathrm{Hom}(C, I)) \\ = \ & H_i((M \otimes_R F) \otimes \mathcal{F}) \simeq H_i(M \otimes_R (F \otimes \mathcal{F})) \\ \simeq \ & \mathrm{Tor}_1^R(M, F \otimes_R \mathrm{Hom}(C, I)) = 0 \end{aligned}$$

by[17, p. 258, 9.20] for all $i \geq 1$, since $F \otimes_R \mathrm{Hom}(C, I)) \simeq \mathrm{Hom}(C, F \otimes_R I)$ is a C-injective module by [8, Theorem 3.2.16] and [13, Lemma 1.12]. Hence $M \otimes_R F$ is a G_C-flat R-module.

Theorem 1 If R is commutative right coherent and C is faithfully semidualizing R-bimodule, then the class $\mathcal{GF}_C(R)$ of G_C-flat R- modules is projectively resolving and closed under direct summands.

Furthermore, if $M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots$ is a sequence of G_C-flat R- modules, then the direct limit $\varinjlim M_n$ is again G_C-flat.

Proof Using the dual of Theorem 2.8 in [20] and together with the Proposition 2 above, we see that $\mathcal{GF}_C(R)$ is projectively resolving. Now, comparing Proposition 2.5 with Proposition 1.4 in [11], we get that $\mathcal{GF}_C(R)$ is closed under direct summands.

Concerning the last statement, we know that if R is coherent and C is faithfully semidualizing, then the class $\mathcal{F}_C(R)$ is preenveloping on the category of R-modules by [13. Proposition 5.10.]. So we pick for each n a co-proper right $\mathcal{F}_C$-resolution F_n of M_n, as illustrated in the next diagram.

$$\begin{array}{ccccccccccc} F_0 & : & 0 & \to & M_0 & \to & C \otimes F_0^0 & \to & C \otimes F_0^1 & \to & \cdots \\ \vdots & & & & \downarrow & & \downarrow & & \downarrow & & \\ F_1 & : & 0 & \to & M_1 & \to & C \otimes F_1^0 & \to & C \otimes F_1^1 & \to & \cdots \\ \vdots & & & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & & & \vdots & & \vdots & & \vdots & & \end{array}$$

By Proposition 1.8 in [11], each map $M_n \longrightarrow M_{n+1}$ can be lifted to a chain map $F_n \longrightarrow F_{n+1}$ of complexes. Since we are dealing with sequences (and not arbitrary direct systems), each column above is again a direct system. Thus it makes sense to apply the exact functor $\varinjlim$ to the upon exact sequences, and doing so, we obtain an exact complex,

$$F = \varinjlim F_n = 0 \to \varinjlim M_n \to C \otimes \varinjlim F_n^0 \to \cdots$$

where each module $C \otimes F^k = C \otimes \varinjlim F_n^k, k = 0, 1, 2, \cdots$ is C-flat. When I is injective right R-module, then $\mathrm{Hom}_R(C, I) \otimes_R F_n$ is exact since

$$C \otimes F = C \otimes \mathrm{Hom}_Z(I, Q/Z) \simeq \mathrm{Hom}_Z(\mathrm{Hom}_R(C, I), Q/Z)$$

is a C-flat (left) R-module ,while the first isomorphism comes from that R is coherent and the the second isomorphisms holds by [13, Lemma 1.14], we get exactness of $\mathrm{Hom}_R(F_n, C \otimes F) = \mathrm{Hom}_R(F_n, \mathrm{Hom}_Z(\mathrm{Hom}_R(C, I), Q/Z)) = \mathrm{Hom}_Z(Hom_R(C, I) \otimes F_n, Q/Z)$ and hence of $\mathrm{Hom}_R(C, I) \otimes_R F_n$, since Q/Z is a faithfully injective Z-module. Since $\varinjlim$ commutes with the homology functor, we also get exactness of $\mathrm{Hom}_R(C, I) \otimes_R F = \varinjlim(\mathrm{Hom}_R(C, I) \otimes_R F_n)$ Thus we have constructed the "right half", F, of a complete $\mathcal{F}_C\mathcal{F}$-resolution for lim M_n.

Since M_n is G_C-flat , we also have

$$\mathrm{Tor}_i^R(\mathrm{Hom}_R(C, I), \varinjlim M) \simeq \varinjlim \mathrm{Tor}_i^R(\mathrm{Hom}_R(C, I), M) = 0$$

for $i > 0$, and all injective right modules I. Thus $\varinjlim M_n$ is G_C-flat.

Proposition 6 Let R be commutative artinian and C is semidualizing R-module. Then the class $\mathcal{GF}_C(R)$ of all G_C- flat right R-modules is closed under arbitrary direct products.

Proof Let $M = \Pi_{i \in I} M_i$, and $M_i \in \mathcal{GF}_C(R)$ for all $i \geq 1$. There exists an exact sequence

$$0 \to M_i \to C \otimes F_i^0 \to C \otimes F_i^1 \to C \otimes F_i^2 \to \cdots$$

for $i \geq 1$ and F_i^j flat for $j \geq 0$. Then

$$0 \to \Pi_{i \in I} M_i \to \Pi_{i \in I} C \otimes F_i^0 \to \Pi_{i \in I} C \otimes F_i^1 \to \cdots$$

is exact , where $\Pi_{i \in I} F_i$ is a flat right R-modules and $\Pi_{i \in I} C \otimes F_i^j \simeq C \otimes \Pi_{i \in I} F_i^j$, since C is finitely presented and R is left artinian . Let E be any injective left R-module. Then $E = \oplus_\Lambda E_\alpha$, where E_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [18, Theorem 6.6.4]. Thus

$$\begin{aligned} & \mathrm{Tor}_n^R(\Pi_{i \in I} M_i, \mathrm{Hom}_R(C, E)) \\ = \ & \oplus_\Lambda \mathrm{Tor}_n^R(\Pi_{i \in I} M_i, \mathrm{Hom}_R(C, E_\alpha)) \\ = \ & \oplus_\Lambda \Pi_{i \in I} \mathrm{Tor}_n^R(M_i, \mathrm{Hom}_R(C, E_\alpha)) = 0 \end{aligned}$$

by [8, Theorem 3.2.26] for all $n \geq 1$. Therefore M is an G_C-flat right R-module.

IV. CHANGE OF RINGS

Let (R, m) be a commutative local noetherian ring with residue field k and let $E(k)$ be the injective envelope of k. $\hat{R}$and $\hat{M}$ will denote the m-adic completion of a ring R and an R- module M, and M^v will denote the Matlis dual Hom $_R(M, E(k))$.

Lemma 3[13] Let $Q \longrightarrow R$ be a flat ring homomorphism between commutative rings. If E is semidualizing over Q, then $E \otimes_Q R$ is semidualizing over R.

Corollary 1 (1) Let R be a commutative ring and S a multiplicatively closed set of R, If C is a semidualizing R module,then $C[x]$ is a semidualizing $R[x]$-module and $S^{-1}C$ is a semidualizing $S^{-1}R$-module.

(2)Let R be a commutative noetherian ring , If C is a finitely generated semidualizing R-module, then $\hat{C}$ is a finitely generated semidualizing $\hat{R}$-module.

Proof By Lemma 3 and [8, Theorem 2.5.14].

Proposition 7 Let (R,m) be a commutative local noetherian ring and C be a semidualizing R-module, and M a finitely generated R-module. Then

(1)If $M \in \mathcal{GP}_C(R)$, then $\hat{M} \in \mathcal{GP}_{\hat{C}}(\hat{R})$.

(2) If $\hat{R}$ is a projective R-module and $\hat{M} \in \mathcal{GP}_{\hat{C}}(\hat{R})$, then $\hat{M} \in \mathcal{GP}_C(R)$.

Proof (1) There is an exact sequence $0 \to M \to C \otimes F^0 \to C \otimes F^1 \to \cdots$ in R-Mod with F^i is free by [20, Observation2.3.]. Then $0 \to \hat{M} \to \widehat{C \otimes F^0} \to \widehat{C \otimes F^1} \to \cdots$ is exact in $\hat{R}$-Mod by [8, Theorem 2.5.11]. Since $\widehat{C \otimes F^i} \simeq \hat{C} \otimes \hat{F^i}$ by [8, P.67,Exercise 7] for all $i \geq 0$, we have the exact sequence $0 \to \hat{M} \to \hat{C} \otimes \hat{F^0} \to \hat{C} \otimes \hat{F^1} \to \cdots$ in $\hat{R}$-Mod , where $\hat{C}$ is a finitely generated semidualizing $\hat{R}$-module by Corollary 1 and $\hat{F^i}$ is a free $\hat{R}$-Mod. Then $\text{Ext}^1(\hat{M}, \hat{C} \otimes \hat{R}) \simeq \text{Ext}^1(\hat{M}, \widehat{C \otimes R}) \simeq \text{Ext}^1(M \otimes \hat{R}, C \otimes R \otimes \hat{R}) \simeq \text{Ext}^1(M, C \otimes R) \otimes \hat{R} = 0$ by [8, Theorem 3.2.5] for all $i \geq 1$, we have $\hat{M} \in \mathcal{GP}_{\hat{C}}(\hat{R})$ by [20, Observation2.3.].

(2) There is an exact sequence $0 \to \hat{M} \to \hat{C} \otimes \bar{F^0} \to \hat{C} \otimes \bar{F^1} \to \cdots$ in $\hat{R}$-Mod with $\bar{F^i}$ is free . Then $\bar{F^i}$ is a free R-module since $\bar{F^i}$ is isomorphic to $\hat{R}^{(X)}$ for some set X and $\hat{R}^{(X)}$ is a projective R-module. It is easy to see that $\hat{C} \otimes \bar{F^i} \simeq C \otimes_R \hat{R}_{\hat{R}} \otimes \bar{F^i} \simeq C \otimes_R \bar{F^i}$. Since $0 = \text{Ext}^1_{\hat{R}}(\hat{M}, \hat{C} \otimes \hat{R}) \simeq \text{Ext}^1_{\hat{R}}(M \otimes \hat{R}, C \otimes \hat{R}) \simeq \text{Ext}^1_R(M, C) \otimes \hat{R}$ by [8,Theorem 3.2.5], we have $\text{Ext}^1_R(M, C \otimes R) = 0$ for all $i \geq 1$, since $\hat{R}$ is a faithfully flat R-module, and thus $\text{Ext}^1_R(\hat{M}, C \otimes F) \simeq \text{Ext}^1_R(\hat{R} \otimes_R M, C \otimes R) \simeq \text{Hom}_R(\hat{R}, \text{ Ext}^1_R(M, C \otimes R)) = 0$ by [17, p.258, 9.20] for all $i \geq 1$. Hence $\hat{M} \in \mathcal{GP}_C(R)$ by [20, Observation2.3.].

Proposition 8 Let (R,m) be a commutative local Noetherian ring and C is a semidualizing R-module, and M a finitely generated R-module. Then

(1)If $M \in \mathcal{GI}_C(R)$, then $\hat{M} \in \mathcal{GI}_{\hat{C}}(\hat{R})$.

(2) If $\text{Hom}_R(\hat{R}, M) \in \mathcal{GI}_{\hat{C}}(\hat{R})$, then $\text{Hom}_R(\hat{R}, M) \in \mathcal{GI}_C(R)$.

Proof (1) There is an exact sequence $\cdots \to \text{Hom}(C, I_1) \to \text{Hom}(C, I_0) \to M \to 0$ in R-Mod with I^i is injective. Then $\cdots \to \widehat{\text{Hom}(C, I_1)} \to \widehat{\text{Hom}(C, I_0)} \to \hat{M} \to 0$ in $\hat{R}$-Mod is exact by [8, Theorem 2.5.11]. Since $\widehat{\text{Hom}(C, I_i)} \simeq \text{Hom}(\hat{C}, \hat{I_i})$ by [8, Theorem 3.2.5] for all $i \geq 0$, we have the exact sequence $\cdots \to \text{Hom}(\hat{C}, \hat{I_1}) \to \text{Hom}(\hat{C}, \hat{I_0}) \to \hat{M} \to 0$ in $\hat{R}$-Mod , where $\hat{C}$ is a finitely generated semidualizing $\hat{R}$-module by Corollary 1 and $\hat{I_i}$ is an injective $\hat{R}$-Mod by the proof of Proposition 3.2. in [21]. Then

$$\text{Ext}^i_{\hat{R}}(\hat{M}, \text{Hom}(\hat{C}, \bar{I})) \simeq \text{Ext}^i_{\hat{R}}(M \otimes \hat{R}, \text{Hom}_R(C, \bar{I}))$$

$$\simeq \text{Ext}^i_R(M, \text{Hom}_{\hat{R}}(\hat{R}, \text{Hom}(C, \bar{I})) \simeq \text{Ext}^i_R(M, \text{Hom}(C, \bar{I})) = 0$$

by [8, Theorem 3.2.5] for all injective $\hat{R}$-module $\bar{I}$ and all $i \geq 1$, we have $\hat{M} \in \mathcal{GI}_{\hat{C}}(\hat{R})$ by [20, Observation2.3.].

(2) There is an exact sequence

$$\cdots \to \text{Hom}(\hat{C}, \bar{I_1}) \longrightarrow \text{Hom}(\hat{C}, \bar{I_0}) \to \text{Hom}_R(\hat{R}, M) \to 0$$

in $\hat{R}$-Mod with $\bar{I_i}$ is injective. Then $\bar{I_i}$ is an injective R-module by the proof of Proposition 3.2 in [21]. we also have $\text{Hom}_{\hat{R}}(\hat{C}, \bar{I_i}) \simeq \text{Hom}_R(C, \bar{I_i})$ by [17, p.258, 9.21]. Let I be any injective R-module. Then I is isomorphic to a summand of $E(k)^X$ for some set X, and hence $\hat{I} \simeq I \otimes_R \hat{R}$ is isomorphic to a summand of $E(k)^X \otimes_R \hat{R} \simeq E_{\hat{R}}(\hat{R}/\hat{m})^X \otimes_R \hat{R}$ by [8, Theorem 3.4.1]. It follows that $\hat{I}$ is an injective $\hat{R}$-module by [8, Theorem 3.2.16]. Hence

$$\begin{aligned} & \text{Ext}^i_R(\text{Hom}(C, I), \text{Hom}_R(\hat{R}, M)) \\ \simeq \quad & \text{Ext}^i_{\hat{R}}(\text{Hom}(C, I), \text{Hom}_{\hat{R}}(\hat{R}, \text{Hom}_R(\hat{R}, M))) \\ \simeq \quad & \text{Ext}^i_{\hat{R}}(\text{Hom}(C, I) \otimes \hat{R}, \text{Hom}_R(\hat{R}, M)) \\ \simeq \quad & \text{Ext}^i_{\hat{R}}(\text{Hom}(\hat{C}, \hat{I}), \text{Hom}_R(\hat{R}, M)) = 0 \end{aligned}$$

by [17, p.258, 9.21] for all $i \geq 1$. So $\text{Hom}_R(\hat{R}, M) \in \mathcal{GI}_C(R)$.

Proposition 9 Let R and S be equivalent rings via equivalences $F : R\text{-Mod} \longrightarrow S\text{-Mod}$ and $G : S\text{-Mod} \longrightarrow R\text{-Mod}$.Then

(1) $M \in \mathcal{GP}_C(R)$ if and only if $F(M) \in \mathcal{GP}_C(S)$ for all $M \in R$-Mod;

(2) $M \in \mathcal{GI}_C(R)$ if and only if $F(M) \in \mathcal{GI}_C(S)$ for all $M \in R$-Mod;

(3) $M \in \mathcal{GF}_C(R)$ if and only if $F(M) \in \mathcal{GF}_C(S)$ for all $M \in R$-Mod.

Proof (1)($\Rightarrow$) There is a complete $\mathcal{PP}_C$-resolution of the form $\mathbf{P} = \cdots \to P_1 \longrightarrow P_0 \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \cdots$ in R-Mod with all P^i and P_i projective such that M =coker$(P_1 \longrightarrow P_0)$. Then $F(\mathbf{P}) = \cdots \to F(P_1) \longrightarrow F(P_0) \longrightarrow F(C) \otimes_S F(P^0) \longrightarrow F(C) \otimes_S F(P^1) \longrightarrow \cdots$ in S-Mod with all $F(P^i)$ and $F(P_i)$ projective such that $F(M)$ =coker$(F(P_1) \longrightarrow F(P_0))$. It is easy to see that $F(C)$ is a semidualizing S-module. Let Q be any projective S-module. Then $\text{Hom}_S(F(\mathbf{P}), C \otimes_S Q) \simeq \text{Hom}_R(\mathbf{P}, G(C) \otimes G(Q))$ is exact such that $G(C)$ is a semidualizing R-module. Hence $F(M) \in \mathcal{GP}_C(S)$.

($\Leftarrow$) By $GF(M) \simeq M$.

(2)and(3) By analogy with the proof of (1).

Corollary 2[21] Let R and S be equivalent rings via equivalences $F : R\text{-Mod} \longrightarrow S\text{-Mod}$ and $G : S\text{-Mod} \longrightarrow R\text{-Mod}$.Then

(1) For all $M \in R$-Mod, ${}_RM$ is G-projective if and only if ${}_SF(M)$ is G-projective;

(2) For all $M \in R$-Mod, ${}_RM$ is G-injective if and only if ${}_SF(M)$ is G-injective;

(3) For all $M \in R$-Mod, ${}_RM$ is G-flat if and only if ${}_SF(M)$ is G-flat.

Proof Easy.

Let R be a commutative ring and S a multiplicatively closed set of R. Then $S^{-1}R = (R \times S)/\sim = [a/s | a \in R, s \in S]$ is a ring and $S^{-1}M = (M \times S)/\sim = [x/s | x \in M, s \in S]$ is a $S^{-1}R$-module. If P is prime ideal of R and $S = R - P$. Then we will denote $S^{-1}M, S^{-1}R$ by M_P, R_P,respectively.

At first, we give two Lemmas which is used in the following section.

Lemma 4[17] Let A be an commutative noetherian with subset S,and A, B be R-modules with A finitely generated.

There are isomorphisms, for all $n \geq 0$,

$$\mathrm{Ext}^n_{S^{-1}R}(S^{-1}A, S^{-1}B) \cong S^{-1}\mathrm{Ext}^n_R(A,B)$$

Lemma 5[21] Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then $\bar{A}$ is a projective R-module if and only if $\bar{A}$ is a projective $S^{-1}R$ -module for any $\bar{A} \in S^{-1}R$-Mod.

For convenience, nextly we note the $G_{S^{-1}C}$-projective(injective,flat) $S^{-1}R$-module by the G_C-projective(injective,flat) $S^{-1}R$-module .

Proposition 10 Let R be a commutative ring and S a multiplicatively closed set of R, then

(1)If $S^{-1}R$ is a projective R-module and A is a finitely generated G_C-projective R-module , then $S^{-1}A$ is an G_C-projective $S^{-1}R$-module ;

(2)If $S^{-1}R$ is a faithfully flat R-module and $\overline{B}$ is a finitely generated G_C-projective R-module if and only if $\overline{B}$ is an finitely generated G_C-projective $S^{-1}R$-module for any $\overline{B} \in S^{-1}R$-mod.

Proof (1) There exists an complete $\mathcal{PP}_C$-exact sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \otimes P^0 \longrightarrow C \otimes P^1 \longrightarrow \cdots$ with P_i, P^i projective for $i \geq 0$ and M =coker$(P_1 \longrightarrow P_0)$. Then there exists an complete $\mathcal{PP}_C$ exact sequence $\cdots \longrightarrow S^{-1}P_1 \longrightarrow S^{-1}P_0 \longrightarrow S^{-1}C \otimes S^{-1}P^0 \longrightarrow S^{-1}C \otimes S^{-1}P^1 \longrightarrow \cdots$ in $S^{-1}R$-Mod, with $S^{-1}A$ =coker$(S^{-1}P_1 \longrightarrow S^{-1}P_0)$ and $S^{-1}P_i, S^{-1}P^i$ are projective $S^{-1}R$-module for $i \geq 0$. Let $\overline{Q}$ be any projective $S^{-1}R$-module, then $\overline{Q}$ is projective R-module by Lemma 5. Since A is an G_C-projective , we have $\mathrm{Ext}^i_R(A, C \otimes \overline{Q}) = 0$ for all $i \geq 1$. So $\mathrm{Ext}^i_{S^{-1}R}(S^{-1}A, S^{-1}C \otimes \overline{Q}) \simeq \mathrm{Ext}^i_{S^{-1}R}(S^{-1}A, S^{-1}C \otimes S^{-1}\overline{Q}) \simeq S^{-1}\ \mathrm{Ext}^i_R(A, C \otimes \overline{Q}) = 0$ by Lemma 4 for all $i \geq 1$. Hence $S^{-1}A$ is an G_C-projective $S^{-1}R$-module .

(2) $(\Longrightarrow)$ By (1), Since $B \simeq S^{-1}B$ by [16.Prop.5.17].

$(\Longleftarrow)$ There exists an complete $\mathcal{PP}_C$-exact sequence $\cdots \longrightarrow \bar{P}_1 \longrightarrow \bar{P}_0 \longrightarrow S^{-1}C \otimes \bar{P}^0 \longrightarrow S^{-1}C \otimes \bar{P}^1 \longrightarrow \cdots$ in $S^{-1}R$-Mod with $\bar{P}_i, \bar{P}^i$ projective for $i \geq 0$ and $\bar{B}$ =coker$(\bar{P}_1 \longrightarrow \bar{P}_0)$ then $\bar{P}_i, \bar{P}^i$ are projective R-module by Lemma 5. Let Q be any projective R-module, then $S^{-1}Q$ is a projective $S^{-1}R$-module. So $S^{-1}\mathrm{Ext}^i_R(\bar{B}, C \otimes Q) \simeq \mathrm{Ext}^i_{S^{-1}R}(\bar{B}, S^{-1}C \otimes S^{-1}Q)) = 0$, then $\mathrm{Ext}^i_R(\bar{B}, C \otimes Q) = 0$ since $S^{-1}R$ is faithfully flat R-module. Therefore $\bar{B}$ is an G_C-projective R-module.

Lemma 6 Let R be a commutative ring and S a multiplicatively closed set of R and $S^{-1}R$ be a finitely generated projective R-module , If I is a injective R-module, then $\mathrm{Hom}_R(S^{-1}R,\mathrm{Hom}(C, I))$ is a C-injective $S^{-1}R$-module.

Proof Since $S^{-1}R$ be a finitely generated projective R-module, then

$$\begin{aligned} & \mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I)) \\ \simeq\ & S^{-1}\mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}_R(C, I)) \\ \simeq\ & \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}\mathrm{Hom}_R(C, I)) \\ \simeq\ & \mathrm{Hom}_{S^{-1}R}(S^{-1}R, \mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I)) \\ \simeq\ & \mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I)). \end{aligned}$$

Since $S^{-1}I$ is injective $S^{-1}R$-modules , hence $\mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I))$ is a C-injective $S^{-1}R$-module.

Proposition 11 Let R be a commutative ring and S a multiplicatively closed set of R . If $S^{-1}R$ is a finitely generated projective R-module and C is a finitely generated semidualizing R-module , then

(1) If A is an G_C-injective R-module, then $\mathrm{Hom}_R(S^{-1}R, A)$ is an G_C-injective $S^{-1}R$-module;

(2) For any $B \in R$-Mod, $\mathrm{Hom}_R(S^{-1}R, B)$ is an G_C-injective R-module if and only if $\mathrm{Hom}_R(S^{-1}R, B)$ is an G_C-injective $S^{-1}R$-module.

Proof(1) There exists an complete $\mathcal{I}_C\mathcal{I}$- exact sequence $\cdots \to \mathrm{Hom}(C, I_1) \to \mathrm{Hom}(C, I_0) \to I^0 \to I^1 \to \cdots$ with I_i, I^i injective for $i \geq 0$ and

$$A = \mathrm{coker}(\mathrm{Hom}(C, I_1) \to \mathrm{Hom}(C, I_0)).$$

Then there exists an complete $\mathcal{I}_C\mathcal{I}$- exact sequence

$$\cdots \to \mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I_1)) \to \mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I_0))$$
$$\to \mathrm{Hom}_R(S^{-1}R, I^0) \to \mathrm{Hom}_R(S^{-1}R, I^0) \to \cdots$$

in $S^{-1}R$-Mod, with

$$\mathrm{Hom}_R(S^{-1}R, A) = \mathrm{coker}(\mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I_1))$$
$$\to \mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I_0)))$$

and $\mathrm{Hom}_R(S^{-1}R, I_i)$ is an injective $S^{-1}R$-module for $i \geq 0$ by [8.Theorem 3.2.9] and $\mathrm{Hom}_R(S^{-1}R, \mathrm{Hom}(C, I_i))$ is an C-injective $S^{-1}R$-module by Lemma 6. Let $\bar{I}$ be any injective $S^{-1}R$-module , then $\bar{I}$ is an injective R-module by [4. Lemma 1.2]. So

$$\begin{aligned} & \mathrm{Ext}^i_{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}), \mathrm{Hom}_R(S^{-1}R, A)) \\ \simeq\ & \mathrm{Ext}^i_R(\mathrm{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}), A)) \\ \simeq\ & \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, \mathrm{Hom}_R(S^{-1}R, \bar{I})), A) = 0 \end{aligned}$$

by [14.P.258.9.21] for all $i \geq 1$, and Hence $\mathrm{Hom}_R(S^{-1}R, A)$ is an G_C-injective $S^{-1}R$-module.

(2)$(\Longrightarrow)$ is obvious.

$(\Longleftarrow$) There is an exact sequence

$$0 \to \mathrm{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_1) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_0)$$
$$\to \bar{I}^0 \to \bar{I}^1 \to \cdots$$

in $S^{-1}R$-Mod with $\bar{I}_i$ and $\bar{I}^i$ injective for all$i \geq 0$ and $\mathrm{Hom}_R(S^{-1}R, B)$ =ker$(\bar{I}^0 \to \bar{I}^1)$. Then $\bar{I}_i$ and $\bar{I}^i$ are injective R-module. But we have

$$\mathrm{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_i) \simeq \mathrm{Hom}_R(C, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, \bar{I}_i))$$
$$\simeq \mathrm{Hom}_R(C, \bar{I}_i)$$

Hence $\mathrm{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_i)$ is a C-injective R-module. Let I be any injective R-module, then $S^{-1}I$ is an injective $S^{-1}R$-module. So

$$\begin{aligned} & \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), \mathrm{Hom}_R(S^{-1}R, B)) \\ \simeq\ & \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), \mathrm{Hom}_{S^{-1}R}(S^{-1}R, \mathrm{Hom}_R(S^{-1}R, B)) \\ \simeq\ & \mathrm{Ext}^i_{S^{-1}R}(S^{-1}\mathrm{Hom}_R(C, I), \mathrm{Hom}_R(S^{-1}R, B)) \\ \simeq\ & \mathrm{Ext}^i_{S^{-1}R}(\mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I), \mathrm{Hom}_R(S^{-1}R, B)) = 0 \end{aligned}$$

by [17.P.107. Theorem 3.84] and [14.P.258.9.21] for all $i \geq 1$. Hence Hom $_R(S^{-1}R, B)$ is an G_C-injective R-module .

Proposition 12 Let R be a commutative ring and S a multiplicatively closed set of R .If C be a semidualizing R-module and $S^{-1}R$ is a finitely generated projective R-module, then

(1) If A is a G_C-flat R-module, then $S^{-1}A$ is a G_C-flat R-module;

(2) If A is an G_C-flat R-module, then $S^{-1}A$ is an G_C-flat $S^{-1}R$-module;

(3) For any $\bar{B} \in S^{-1}R-$Mod, $\bar{B}$ is an G_C-flat R-module if and only if $\bar{B}$ is an G_C-flat $S^{-1}R$-module.

Proof (1)There is an exact sequence $\cdots \to F_1 \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$ in R-Mod where $F_i$and F^i are flat for $i \geq 0$ and A =coker$(F_1 \longrightarrow F_0)$. Then $\cdots \to S^{-1}F_1 \longrightarrow S^{-1}F_0 \longrightarrow S^{-1}C \otimes_R S^{-1}F^0 \longrightarrow S^{-1}C \otimes_R S^{-1}F^1 \longrightarrow \cdots$ is exact and $S^{-1}F_i$ and $S^{-1}F^i$ are flat $S^{-1}R$-module for all $i \geq 0$. Hence $S^{-1}F_i$ and $S^{-1}F^i$ are flat R-module for all $i \geq 0$. Since $S^{-1}R$ is a finitely generated projective R-module, $S^{-1}R \oplus P \simeq R^{(n)}$, where P is projective. So $(S^{-1}C \otimes_R S^{-1}F^i) \oplus (P \otimes C \otimes_R S^{-1}F^i) \simeq (S^{-1}R \otimes C \otimes_R S^{-1}F^i) \oplus (P \otimes C \otimes_R S^{-1}F^i) \simeq R^{(n)} \otimes C \otimes_R S^{-1}F^i \simeq C \otimes_R (S^{-1}F^i)^{(n)}$, hence $S^{-1}C \otimes_R S^{-1}F^i$ is C-flat R-module for all $i \geq 0$ by [13, Proposition5.5.]. Let I be any injective R-module. Then $\mathrm{Tor}_i^R(\mathrm{Hom}_R(C,I), S^{-1}A) \simeq \mathrm{Tor}_i^R(S^{-1}\mathrm{Hom}_R(C,I), A) = 0$ by [16.Prop.5.17] , since $S^{-1}\mathrm{Hom}_R(C,I) \simeq \mathrm{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I) \simeq \mathrm{Hom}_R(C, S^{-1}I)$ by [13 , Lemma 1.2] and so $S^{-1}\mathrm{Hom}_R(C,I)$ is C-injective R-module. Hence $S^{-1}A$ is an G_C-flat R-module.

(2)There is an exact sequence $\cdots \to F_1 \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$ in R-Mod where $F_i$and F^i are flat for $i \geq 0$ and A =coker$(F_1 \longrightarrow F_0)$. Then $\cdots \to S^{-1}F_1 \longrightarrow S^{-1}F_0 \longrightarrow S^{-1}C \otimes_R S^{-1}F^0 \longrightarrow S^{-1}C \otimes_R S^{-1}F^1 \longrightarrow \cdots$ is exact and $S^{-1}F_i$ and $S^{-1}F^i$ are flat $S^{-1}R$-module for all $i \geq 0$. Let $\bar{I}$ be any injective $S^{-1}R$-module. Then $\bar{I}$ be any injective R-module by [3.Lemma 1.2]. So

$$\begin{aligned} & \mathrm{Tor}_i^{S^{-1}R}(\mathrm{Hom}(S^{-1}C, \bar{I}), S^{-1}A) \\ \simeq\ & \mathrm{Tor}_i^{S^{-1}R}(S^{-1}\mathrm{Hom}(C, \bar{I}), S^{-1}A) \\ \simeq\ & S^{-1}\mathrm{Tor}_i^R(\mathrm{Hom}(C, \bar{I}), A) = 0 \end{aligned}$$

for all $i \geq 1$. Hence $S^{-1}A$ is an G_C-flat $S^{-1}R$-module.

(3)($\Longrightarrow$) by (2)

$\Longleftarrow$ There is an exact sequence $\cdots \to \bar{F_1} \longrightarrow \bar{F_0} \longrightarrow S^{-1}C \otimes_R \bar{F^0} \longrightarrow S^{-1}C \otimes_R \bar{F^1} \longrightarrow \cdots$ in $S^{-1}R$-Mod where $\bar{F_i}$ and $\bar{F^i}$ are flat for $i \geq 0$ and $\bar{B}$ =coker$(\bar{F_1} \longrightarrow \bar{F_0})$. Then $\bar{F_i}$ and $\bar{F^i}$ are flat R-module and $S^{-1}C \otimes_R \bar{F^1}$ is a C-flat R-module by the proof of (1). Let I be any injective R-module. Then

$$\begin{aligned} & S^{-1}\mathrm{Tor}_i^R(\mathrm{Hom}(C,I), \bar{B}) \\ \simeq\ & \mathrm{Tor}_i^{S^{-1}R}(S^{-1}\mathrm{Hom}(C,I), \bar{B}) \\ \simeq\ & \mathrm{Tor}_i^{S^{-1}R}(\mathrm{Hom}(S^{-1}C, S^{-1}I), \bar{B}) = 0 \end{aligned} .$$

So $\bar{B}$ is an G_C-flat R-module.

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