Vibration Characteristics of Functionally Graded Material Skew Plate in Thermal Environment

Gulshan Taj M. N. A., Anupam Chakrabarti, Vipul Prakash

***Abstract*—**In the present investigation, free vibration of functionally graded material (FGM) skew plates under thermal environment is studied. Kinematics equations are based on the Reddy's higher order shear deformation theory and a nine noded isoparametric Lagrangian element is adopted to mesh the plate geometry. The issue of C_1 continuity requirement related to the assumed displacement field has been circumvented effectively to develop C_0 finite element formulation. Effective mechanical properties of the constituents of the plate are considered to be as position and temperature dependent and assumed to vary in the thickness direction according to a simple power law distribution. The displacement components of a rectangular plate are mapped into skew plate geometry by means of suitable transformation rule. One dimensional Fourier heat conduction equation is used to ascertain the temperature profile of the plate along thickness direction. Influence of different parameters such as volume fraction index, boundary condition, aspect ratio, thickness ratio and temperature field on frequency parameter of the FGM skew plate is demonstrated by performing various examples and the related findings are discussed briefly. New results are generated for vibration of the FGM skew plate under thermal environment, for the first time, which may be implemented in the future research involving similar kind of problems.

***Keywords*—**Functionally graded material, finite element method, higher order shear deformation theory, skew plate, thermal vibration.

I. INTRODUCTION

WITH the nature of being smooth and gradual variation of material properties along the preferred/chosen direction, the concept of functional grading in structural components registered its extensive applications in diverse fields of engineering. Among many, aerospace, nuclear, automobiles and biotechnology industries are few to cite. Typical FGM structure offers possible decrease of in-plane and through-the thickness transverse stresses in addition to the improved thermal behavior owing to low thermal conductivity of ceramic component. Apparently, for the safe and optimal design of FGMs, it is necessary to understand the vibration characteristics of such structures under thermal environment. Hence the research development of such topic has drawn considerable interest among many scientists in the past. Here, the authors are intended to discuss the works related to the present topic, thereby to find out the lapse exists in the

Gulshan Taj, M.N.A., PhD student; Anupam Chakrabarti, Associate Professor; and Vipul Prakash, Associate Professor, are with the Department of Civil Engineering, Indian Institute of Technology (IIT), Roorkee 247 667, India (e-mail: gulshantaj19@yahoo.co.in, anupam1965@yahoo.co.uk, vipulfeq@iitr.ernet.in).

previous literature.

In view to record the vibration response of FGM plates under thermal and mechanical loading conditions, many theories were evolved and implemented in the framework of numerical and analytical tools. Of these, displacement based models dominate the literature, which may be combined with the finite element and other popular meshless methods. Thermo-elastic response of FGM cylinders and plates was investigated by Reddy and Chin [1] using first order shear deformation theory (FSDT). Here, the temperature dependent properties are picked up to perform the dynamic thermo-elastic response of one dimensional axi-symmetric functionally graded cylinder. It was found that thermo-mechanical coupling plays significant role on the radial stresses of the cylinder. Kumar et al. [2] analyzed the free vibration of functionally graded material plates based on the higher order shear deformation theory (HSDT). Effects of side-to-thickness ratio, modulus ratio and aspect ratio on the non-dimensional natural frequencies are discussed in detail. They concluded that the natural frequency of different FGM constituents lies between those of pure metal and ceramic plates. Exact relationship between the natural frequencies and buckling stresses of FGM plate is established by Matsunaga [3]. Two dimensional higher order theory is used to derive the fundamental equations of the plate. Noda [4] solved a thermal stress problem that arises in the FGM plate under severe temperature conditions. In addition, crack propagation path due to thermal shock was also discussed with some numerical examples.

Talha and Singh [5] adopted the HSDT model and finite element method to study the vibration and static behavior of FGM plates. Lagrangian finite element with 13 degrees of freedom per node was utilized in the formulation and considers the quadratic variation of transverse displacement in the kinematic model. It was found that gradient in the material properties plays a vital role in predicting the response of the FGM plate in bending and vibration analysis. Further, Talha and Singh [6] extended their work to study the thermo-mechanical behavior of FGM plates subjected to various boundary conditions and loadings. Hadji et al. [7] presented the four variable refined plate theory (RPT) which is similar to classical plate theory (CPT) for free vibration analysis of FGM sandwich rectangular plates. Improvement in RPT is required when applied to a laminate structure in order to satisfy the inter-laminar shear stress continuity at the layer interfaces. Global collocation method, the FSDT and TSDT are used to study the natural frequencies of functionally

graded plates by Ferreira et al. [8]. The accuracy of the frequency parameter was controlled by the location of the collocation points and the parameter c present in the multi quadric basis functions. Uymaz and Aydogdu [9] investigated the vibration analysis of FGM rectangular plate with different types of boundary conditions. The proposed theory may be used to predict the frequency parameter of moderately thick to very thick plates. Abrate [10] examined the static, buckling and free vibration of functionally graded plates and it was found that natural frequencies of the FGM plates are always proportional to those of isotropic plates. It was concluded that the analysis of FGM plates can be predicted by knowing the behavior of corresponding homogeneous plates. A three dimensional solution proposed by Vel and Batra [11] was used to solve the free and forced vibrations of simply supported functionally graded rectangular plates. Mori-Tanaka and Self- consistent schemes are used to calculate the effective material properties of the plate. The results are compared with different plate theories such as classical plate theory (CPT), first order shear deformation theory (FSDT), third order shear deformation theory (TSDT) and observed that FSDT gives accurate results than HSDT as far as functionally graded plates are concerned.

Huang and Shen [12] investigated the dynamic response and vibration of FGM plates under thermal environment, based on HSDT and von Karman non linearity. Equation of motion was solved by perturbation technique and they concluded that volume fraction index and temperature field plays important role in the nonlinear response of functionally graded plate. Further, Yang and Shen [13] analyzed the transient response of functionally graded plates under different thermal environments. Modal superposition technique, Galerkin approach and one dimensional differential quadrature were applied to study the response of the plate under lateral dynamic loads. It was concluded that natural frequencies of functionally graded plates with intermediate properties do not necessarily match with intermediate frequencies of the plate. Talha and Singh [14] studied the vibration characteristics of FGM plates under thermo-mechanical load. HSDT is adopted with modification in the transverse displacement and variation of temperature in the thickness direction was assumed. The nonlinear vibration analysis of silicon nitride/stainless steel FGM plate under thermal environment was examined by Soundarrajan et al. [15]. Finite element method with direct iteration technique was used to derive the linear and nonlinear frequencies of the plate. The effective properties of the constituents are calculated by Mori-Tanaka homogenization method and variation of non linear frequency ratio with amplitude was highlighted.

Qian et al. [16] investigated the transient response of isotropic and orthotropic plates under external loads based on higher order normal and shear deformable plate theory (HNSDT). Park and Kim [17] studied the vibration and buckling behavior of the plate based on FSDT and von Karman non linearity. Finite element with incremental method

was used to derive the strain-displacement relationship of the plate. It was concluded that the behavior of FGM plate is different from those of the isotropic plate and volume fraction of the material plays important role in the post buckling and vibration characteristics of FGM plate. Khorramabadi et al. [18] derived the frequency of simply supported FGM plates based on FSDT, TSDT and Navier's method. They concluded that TSDT gives high accuracy than first order shear deformation theory for thick plates. Makhecha et al. [19] studied the dynamic response of thick skew sandwich plates using HSDT. A C_0 continuous serendipity quadrilateral shear flexible plate element with thirteen degrees of freedom per node was employed. Direct integration method was used to solve the governing equation of motion and the influence of skew angle on dynamic response of the plate under mechanical and thermal loads was discussed in brief.

Vibration analysis of FGM plate under constant, linear and nonlinear temperature rise was studied by Shahrjerdi et al. [20]. They observed that the second order theory approaches results very close to other shear deformation theories and the effects of material composition, plate geometry and temperature field on frequency parameter are presented. Lei and Liansheng [21] analyzed the axi-symmetric nonlinear vibration of FGM circular plate under thermal environment with clamped boundary condition. It was assumed that the mechanical and thermal properties varies in the thickness direction of the plate and follows a simple power law distribution. Ritz-Kantorovich method was used to convert the von Karman equations into a set of nonlinear ordinary differential equation and shooting method was employed to solve the differential equation. Effects of amplitude, thermal load, material index on vibration behavior of plate are discussed. Janghorban and Zare [22] examined the vibration analysis of functionally graded plate with cutouts and skew boundary and the role of different parameters such as cutout size, type of loading and different boundary conditions on vibration of plate were reported.

A detailed study on available literatures reveals that there exists a dearth of data for free vibration analysis of functionally graded material skew plate under thermal load by using an efficient numerical tool. Due to this reason, the scope of the present paper is to study the free vibration characteristics of functionally graded skew plate under thermal environment based on HSDT in conjunction with C_0 finite formulation. A nine noded isoparametric Lagrangian element is used to model the plate and Voigt rule of mixture is used to estimate the effective properties of the constituents of the materials. One dimensional Fourier heat conduction equation is used to describe the temperature profile along the thickness direction of the plate. Temperature dependent properties of the constituents are considered. Wide ranges of numerical problems are solved by considering various material, geometric properties and skew angles.

II. FUNCTIONALLY GRADED MATERIALS

A. Estimation of Effective Properties

The FGM skew plate of dimension a x b x h with skew angle Ψ (Fig. 1) is considered in the present study. The top portion of the plate ($z = +h/2$) is ceramic rich and bottom portion of the plate ($z = -h/2$) is metal rich. Numerous number of micromechanical models are developed by researchers to study the effective properties of functionally graded plates in preferred direction (usually along the thickness direction).
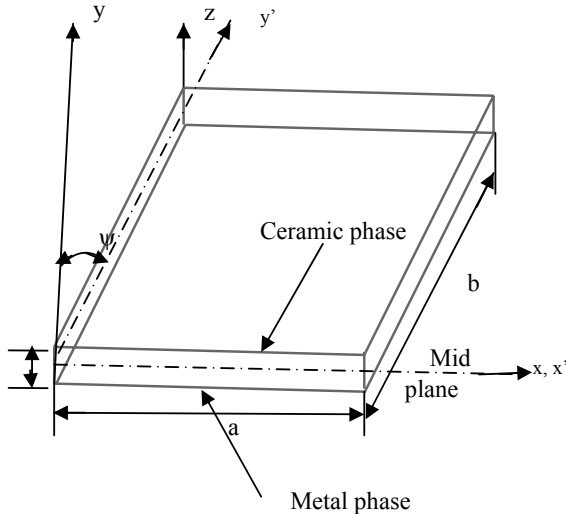

Fig. 1 Configuration of FGM skew plate

In the present study, Voigt rule of mixture is used to ascertain the effective properties of material along the thickness direction. According to Voigt rule of mixture, the effective properties at any point within the FGM plate can be expressed as

$$P_{eff} = P_c(T,z)V_c(z) + P_m(T,z)V_m(z) \tag{1}$$

where P_{eff} is the effective properties of the FGM; P_c, P_m are the temperature dependent properties of the ceramic and metal, respectively and V_c and V_m are the volume fraction of the ceramic and metal constituents of the FGM respectively. In (1), volume fractions of ceramic and metal are considered as spatial function whereas the properties of the constituents are considered as functions of temperature and position. In addition to (1), the volume fraction of the constituents are assumed to vary according to simple power-law as

$$V_c(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^n$$
$$V_m(z) = 1 - V_c(z), (0 \le n \le \infty) \tag{2}$$

where n is the non-negative number known as volume fraction index which defines the smooth and continuous variation of material properties in the thickness direction. The user may vary the value of n to get the optimum material distribution along the thickness direction. The temperature dependent properties of the material according to Touloukian [23] can be expressed as

$$P = P_0(\frac{P_{-1}}{T} + 1 + P_1T + P_2T^2 + P_3T^3) \tag{3}$$

where P_0, P_{-1}, P_1, P_2, P_3 are the coefficients of temperature ($T=300K$) and are constants in the cubic fit of the material property. By using (1) and (2), the modulus of Elasticity (E), thermal conductivity (k), co-efficient of thermal expansion (α) and density (ρ) can be represented as

$$E(z,T) = \left\{E_c(T) - E_m(T)\right\}V_c + E_m$$
$$\alpha(z,T) = \left\{\alpha_c(T) - \alpha_m(T)\right\}V_c + \alpha_m$$
$$k(z) = (k_c - k_m)V_c + k_m \tag{4}$$
$$\rho(z) = (\rho_c - \rho_m)V_c + \rho_m$$

It is to be noted that, Young's modulus and thermal expansion co-efficient are considered to be dependent on temperature and position; whereas the thermal conductivity and density are assumed to be temperature independent and position dependent.

B. Thermal Analysis

The temperature through the thickness is governed by the one-dimensional Fourier equation of heat conduction as given below

$$\frac{d}{dz}\left[k(z)\frac{dT}{dz}\right] = 0, T = T_c \text{ at } z = h/2 \text{ and } T = T_m \text{ at } z = -h/2$$

where T_c and T_m denote the temperature of ceramic and metal respectively. The solution of heat conduction equation is obtained by means of polynomial series as

$$T(z) = T_m + (T_c - T_m)\eta(z,h) \tag{5}$$

where

$$C = 1 - \frac{k_{cm}}{(k+1)k_m} + \frac{k^2_{cm}}{(2k+1)k^2_m} - \frac{k^3_{cm}}{(3k+1)k^3_m} + \frac{k^4_{cm}}{(4k+1)k^4_m} -$$
$$\frac{k^5_{cm}}{(5k+1)k^5_m} \text{ and } k_{cm} = k_c - k_m$$

III. MATHEMATICAL FORMULATION

A. Kinematics

Various forms of displacement fields are proposed in the literature to study the behavior of FGM plates under different loading conditions. In the FSDT proposed by Mindlin [24], linear variation of shear strain in the thickness direction is considered. Hence shear correction factor is necessary to

predict the actual variation of shear strain in the transverse direction. To overcome the problem, HSDT is developed which accounts for parabolic variation of transverse shear strain and does not require any shear correction factor. According to HSDT [25], the displacement field at any generic point can be expressed as

$$u(x,y,z) = u_0 + z\theta_x - \frac{4z^3}{3h^2}\left(\theta_x + \frac{\partial w}{\partial x}\right)$$

$$v(x,y,z) = v_0 + z\theta_y - \frac{4z^3}{3h^2}\left(\theta_y + \frac{\partial w}{\partial y}\right) \qquad (6)$$

$$w(x,y) = w_0$$

where u, v, and w are the displacement fields in the x, y, and z directions, respectively. u_0, v_0, and w_0 are the displacements at the middle plane of the plate along the x, y, and z directions; θ_x, θ_y are the bending rotations defined at the middle plane about the y and x axes, respectively. The C_1 continuity requirement of the above theory has been formulated by considering the terms involving the derivatives of the transverse displacement as separate field variables i.e., $\gamma_x = \left(\theta_x + \frac{\partial w}{\partial x}\right)$ and $\gamma_y = \left(\theta_y + \frac{\partial w}{\partial y}\right)$. Hence in the present study an efficient C_0 nine noded element with seven nodal unknowns (u_0, v_0, w_0, θ_x, θ_y, γ_x, and γ_y) at each node is proposed in the formulation.

B. Constitutive Law

The constituent relation of FGM plate can be represented as

$$\begin{Bmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{yz}\\\sigma_{xz}\\\sigma_{xy}\end{Bmatrix} = \left([Q]\begin{Bmatrix}\varepsilon_{xx}\\\varepsilon_{yy}\\\gamma_{yz}\\\gamma_{xz}\\\gamma_{xy}\end{Bmatrix} - \begin{Bmatrix}1\\1\\0\\0\\0\end{Bmatrix}\alpha(z,T)\Delta T\right) \qquad (7)$$

$$\text{in which } [Q] = \begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\Q_{21} & Q_{22} & 0 & 0 & 0\\0 & 0 & Q_{33} & 0 & 0\\0 & 0 & 0 & Q_{44} & 0\\0 & 0 & 0 & 0 & Q_{55}\end{bmatrix}$$

where Q_{ij} contains the terms elastic moduli (E) and Poisson's ratio (γ), in which E is the function of depth and temperature as given below.

$$Q_{11} = Q_{22} = \frac{E(z,T)}{1-\gamma^2}, \; Q_{12} = Q_{21} = \frac{\gamma E(z,T)}{1-\gamma^2}, \; Q_{33} = Q_{44} = Q_{55} = \frac{E(z,T)}{2(1+\gamma)}$$

C. Transformation Matrix

In skew plate, one of the boundaries of plate may not be parallel to global axes (x, y) of the plate. Hence global displacements (u, v, and w) should be represented in terms of the displacement at the skew boundary (u^l, v^l, and w^l) by means of suitable transformation. Transformation matrix [T] for nodes lying on the skew edges can be expressed as

$$[T] = \begin{pmatrix}\cos\psi & -\sin\psi & 0 & 0 & 0 & 0 & 0\\\sin\psi & \cos\psi & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & \cos\psi & -\sin\psi & 0 & 0\\0 & 0 & 0 & \sin\psi & \cos\psi & 0 & 0\\0 & 0 & 0 & 0 & 0 & \cos\psi & -\sin\psi\\0 & 0 & 0 & 0 & 0 & \sin\psi & \cos\psi\end{pmatrix}$$

D. Governing Equation

For free vibration problem the acceleration at any point within the element may be expressed in terms of the middle plane parameters as

$$\{\ddot{f}\} = \frac{\partial^2}{\partial t^2}\{\bar{f}\} = -\omega^2\begin{Bmatrix}u_0\\v_0\\w_0\end{Bmatrix} = -\omega^2[F]\{f\} \qquad (8)$$

where $\{f\} = \begin{bmatrix}u_0 & v_0 & w_0 & \theta_x & \theta_y & \gamma_x & \gamma_y\end{bmatrix}^T$ and $[F]$ is the matrix of order (3x7) and contains the terms involving z and h as shown below.

$$[F] = \begin{bmatrix}1 & 0 & 0 & z & 0 & \frac{-4z^3}{3h^2} & 0\\0 & 1 & 0 & 0 & z & 0 & \frac{-4z^3}{3h^2}\\0 & 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}$$

Further $\{f\}$ matrix can be expressed in-terms of global displacement vector {X} as,

$$\{f\} = [C]\{X\} \qquad (9)$$

where [C] is the matrix of order (7x63) and contains interpolation function terms.

The strain energy and kinetic energy of the FGM plate can be written as

$$U = \frac{1}{2}\int\{\varepsilon\}^T\{\sigma\}dV \qquad (10)$$

$$T = \frac{1}{2}\int_v\rho\{\ddot{f}\}^T\{\ddot{f}\}dv$$

$$= \frac{1}{2}\int_v\rho[C]^T[F^T][C]\{F\}dv \qquad (11)$$

where mass matrix $[m] = \iint_A[C]^T[L][C]\,dA$ and matrix [L] can be written as

$$[L] = \int_z\rho[F]^T[F]\,dz \qquad (12)$$

By the principle of virtual displacement governing equation for free vibration can be obtained by using equations (10) and (11) and may be written as

$$\left(\left[K^{'}\right]-\omega^2\left[M\right]\right)\{X\}=\{0\} \tag{13}$$

in which $\left[K^{'}\right]=[K]-\Delta T\left[K_g\right]$

where $[M]$, $[K^{'}]$, $[K]$, $[Kg]$, and ΔT are mass matrix, reduced stiffness matrix due to thermal load, linear stiffness matrix, geometric stiffness matrix and critical buckling temperature parameter, respectively.

IV. RESULTS AND DISCUSSION

The present study, here, has been focused mainly on the vibration response of functionally graded skew plates under different thermal environments with more emphasis on consequence of skew angle on frequency parameter. Mathematical model is developed in FORTRAN environment in conjunction with a numerical tool finite element method. A nine noded isoparametric Lagrangian element with seven degrees of freedom per node is used to model the plate element. Here, it is worth to mention that, the present element does not invite any C_1 continuity requirement due to the choice of suitable displacement field to develop strain-displacement relationship. In addition, it is assumed that throughout the deformation process the element behaves in perfectly elastic manner. To show the proposed formulation does not have any limitation on boundary type, different combinations of boundary conditions are employed to perform the validation as well as numerical part. Before proceeding to establish the new results with respect to FGM skew plates, the formulation developed herein is validated against the available literature results. Since, no results are available on vibration analysis of plates having skew boundary under thermal field, the results for rectangular FGM plates published by Huang and Shen [12] are taken to validate the present model. In the present study, Young's modulus and thermal expansion are considered as temperature dependent; thermal conductivity and density are considered as temperature independent. Different parameters such as skew angle, boundary condition, temperature field, aspect ratio, thickness ratio are considered to predict the vibration characteristics of FGM skew plate under different thermal loading conditions. Effect of each parameter on vibration analysis of FGM skew plate is briefly discussed and the important conclusions are highlighted at the end. The non-dimensional frequency parameter used in the present study is given below.

$$\Omega = w\left(\frac{a^2}{h}\right)\left(\frac{\rho_0\left(1-\gamma^2\right)}{E_0}\right)^{0.5}$$

where ρ_0 and E_0 are the reference values of bottom constituent (metal) of the FGM plate at T_0=300K. For the sake of ease understanding, the result part has been deployed in to two parts. In the first phase, the validation as well as convergence study is done by taking temperature dependent properties of the constituents. In the later part, new results are generated for FGM skew plates by considering different parameters.

TABLE I
TEMPERATURE DEPENDENT PROPERTIES OF CONSTITUENTS, FROM [1]

Constituents	Properties	P_0	P_{-1}	P_1	P_2	P_3	$P(T=300K)$
ZrO$_2$	E (Pa)	244.27e+09	0	-1.371e-03	1.214e-06	-3.681e-10	168.063e+09
	α (1/K)	12.766e-06	0	-1.491e-03	1.006e-05	-6.778e-11	18.591e-06
Ti-6Al-4V	E (Pa)	122.56e+09	0	-4.586e-04	0	0	105.698e+09
	α (1/K)	7.5788e-06	0	6.638e-04	-3.147e-06	0	6.941e-06
Si$_3$N$_4$	E (Pa)	348.43e+09	0	-3.070e-04	2.160e-07	-8.946e-11	322.2715e+09
	α (1/K)	5.8723e-06	0	9.095e-04	0	0	7.4746e-06
SUS$_3$O$_4$	E (Pa)	201.04e+09	0	3.079e-04	-6.534e-07	0	207.7877e+09
	α (1/K)	12.330e-06	0	8.086e-04	0	0	15.321e-06

A. Validation Study

A square Si$_3$N$_4$/ SUS$_3$O$_4$ plate with dimension $a = b = 0.2$ and thickness $h = 0.025$ under thermal environment is considered for comparison study. The thermal conductivity and mass density of the constituents are: ρ_m= 8166 kg/m^3, k_m=12.04 W/m K, ρ_c= 2370 kg/m^3, and k_c=9.19 W/m K, where the subscripts 'm' and 'c' denotes the metal and ceramic part of the plate. The influence of temperature on Poisson's ratio is very less and hence it is assumed as constant (γ=0.28). FGMs are mainly used in high-temperature environments and it is therefore necessary to use the properties both temperature and position dependent. This is possible by using a simple rule of mixture for the stiffness co-efficients coupled with the temperature-dependent properties of the constituents [1]. Different types of FGM used in the present study and the corresponding temperature coefficients are tabulated in Table I.

The temperature at the top of the plate Tc= 400K and bottom of the plate Tm=300K is maintained. The non dimensional linear frequency Ω for simply supported Si$_3$N$_4$/ SUS$_3$O$_4$ plate under thermal environment is reported in Table II. The results obtained by Huang and Shen [12] are used to compare the results which are based on HSDT taking non-linearity into account. A perturbation technique is used to

solve the equation of motion by the reference authors. A close range between the two results is obvious from Table II. Also, numerical problems are solved for different mesh sizes and it is found that 5x5 mesh division is sufficient for the convergence of results with desired accuracy. Hence a mesh division of 5x5 is used to generate the results in all the cases considered. Two cases of thermal environment, where temperature of bottom constituent (metal) is taken as 300 K and temperature of top constituent (ceramic) is taken as 400K and 600K for temperature in-dependent and temperature dependent condition are performed. The value of volume fraction index (n) is varied from ceramic to metal part as per (2) to establish its consequence on frequency. The calculated

percentage of error between the two methods is also accomplished in Table II. From Table II, the maximum percentage of error is reported under Tc= Tm= 300K case for ceramic segment, while the minimum error was reported for the case of Tc=400K and Tm=300K where temperature independent properties are taken with the value of n=0(ceramic). Meanwhile, nil error has been observed for the material with temperature dependent properties for which the bottom and top temperature of the plate are taken as 300K and 400K, respectively. It is expected that, the solution methodologies adopted by the authors may be the possible cause for the deviation amongst the results.

TABLE II
CONVERGENCE AND VALIDATION STUDY

Temperature field	References	Volume fraction index (n)				
		Ceramic	0.5	1.0	2.0	Metal
T_c=300K	Present [a]	13.034	8.878	7.726	6.892	5.334
T_m=300K	Huang and Shen [12]	12.495	8.675	7.555	6.777	5.405
	% of error	4.31	2.34	2.26	1.70	1.31
T_c=400K	Present	12.811	8.759	7.638	7.827	5.290
T_m=300K (T.I)	Huang and Shen [12]	12.397	8.615	7.474	6.698	5.311
	% of error	3.34	1.67	2.19	4.05	0.04
T_c=600K	Present	12.353	8.516	7.459	6.695	5.203
T_m=300K (T.I)	Huang and Shen [12]	12.213	8.425	7.305	6.523	5.104
	% of error	1.15	1.08	2.11	2.64	1.94
T_c=400K	Present	12.405	8.532	7.465	6.693	5.307
T_m=300K (T.D)	Huang and Shen [12]	12.397	8.615	7.474	6.693	5.311
	% of error	0.06	0.96	0.12	0.00	0.08
T_c=600K	Present	12.142	8.252	7.186	6.423	5.068
T_m=300K(T.D)	Huang and Shen [12]	11.984	8.269	7.171	6.398	4.971
	% of error	1.32	0.21	0.21	0.39	1.95

T.I-Temperature- independent; T.D-Temperature-dependent
[a] 5x5 mesh size

TABLE III
NATURAL FREQUENCY PARAMETER Ω FOR SQUARE Si_3N_4/ SUS_30_4 SKEW (SSSS) PLATE IN THERMAL ENVIRONMENT

Ψ	n	Mode					
		1	2	3	4	5	6
0°	Ceramic	2.0061	4.8528	4.8528	7.4805	9.2543	9.2543
	0.5	1.3756	3.3344	3.3644	5.1483	6.3693	6.3723
	1.0	1.2022	2.9125	2.9125	4.4991	5.5643	5.5686
	2.0	1.0776	2.6100	2.6100	4.0304	4.9825	4.9865
	Metal	0.8530	2.0840	2.0840	3.2201	3.9890	3.9870
15°	Ceramic	2.1240	4.7543	5.4808	7.5202	9.7136	10.096
	0.5	1.4581	3.2678	3.7680	5.1773	6.6868	6.9537
	1.0	1.2749	2.8547	3.2916	4.5249	5.8417	6.0765
	2.0	1.1431	2.5585	2.9496	4.0537	5.2303	5.4405
	Metal	0.9053	2.0428	2.3565	3.2386	4.1868	4.3519
30°	Ceramic	2.5542	5.1870	6.9332	8.0161	9.9596	11.208
	0.5	1.7585	3.5693	4.7719	5.5237	6.9207	7.7743
	1.0	1.5392	3.1192	4.1695	4.8291	6.0239	6.7724
	2.0	1.3806	2.7958	3.7355	4.3260	5.3316	6.0061
	Metal	1.0948	2.2338	2.9878	4.4569	4.3087	4.8462
45°	Ceramic	3.6349	6.4917	9.6328	9.7160	10.1312	13.260
	0.5	2.5108	4.4738	6.6475	6.7682	6.9805	9.1631
	1.0	2.1998	3.9107	5.8130	5.8982	6.0986	8.0147
	2.0	1.9736	3.5047	5.2054	5.2267	5.4597	7.1702
	Metal	1.5677	2.8051	4.1637	4.2141	4.3754	5.7330

A. New Results-Functionally Graded Skew Plate

Example 1. In order to study the effect of skew angle of the plate on frequency parameter, Si_3N_4/SUS_3O_4 square skew plate with thickness ratio $(a/h) = 10$ is considered. Different boundary conditions viz., simply supported (SSSS), clamped (CCCC) and simply supported-clamped (SCSC) are incorporated and the results are revealed in Tables III-V, respectively. In addition, various values of skew angles (15°, 30°, and 45°) also are considered. The temperature field of $Tc = 400K$ and $Tm = 300K$ is applied on the top and bottom of the plate to generate new results. The obtained non-dimensional frequency parameter for the first six modes is given for different values of skew angle (0°- 45°). It can be inferred that, increase in skew angle of the plate increases the frequency for all the six modes and this trend is irrespective of the value of volume fraction index (n) considered. Increase in skew angle reduces the distance between non-skew edges thus reducing the effective area and thereby increases the frequency parameter value. Next, in all the cases, a close range of frequency parameter is observed for skew angles 0° and 15°, beyond which noticeable difference is discerned. This difference is independent of the value of volume fraction index and type of boundary condition taken for the analysis. It is also noted that, increase in volume fraction index from ceramic to metal portion of the plate reduces the frequency parameter. The reason owing to the above statement is that the increase in volume fraction index corresponds to the less ceramic portion (i.e., bottom of the plate) and hence the less stiffness of the plate. Therefore it may be concluded that the volume fraction index is one of the vital parameter and plays significant role in understanding the vibration characteristics of FGM skew plate. Further, among the different boundary conditions assumed, highest frequency is observed for CCCC plate and lowest frequency is reported for SSSS plate. It can be ascertained that imposing constrains on the boundary of the plate increases the frequency of the plate.

TABLE IV
NATURAL FREQUENCY PARAMETER Ω FOR SQUARE Si_3N_4/SUS_3O_4 SKEW (CCCC) PLATE IN THERMAL ENVIRONMENT

Ψ	n	Mode					
		1	2	3	4	5	6
0°	Ceramic	3.4690	6.6929	6.6929	9.4163	11.3369	11.445
	0.5	2.3888	4.6153	4.6153	6.4762	7.8247	7.9011
	1.0	2.0860	4.0300	4.0300	5.6718	6.8302	6.8973
	2.0	1.8668	3.6025	2.6025	5.0671	6.0975	6.1575
	Metal	1.4906	2.8829	2.8829	4.0596	4.8900	4.9372
15°	Ceramic	3.6667	6.6507	7.4396	9.5553	11.8853	12.353
	0.5	2.5255	4.5868	5.1316	6.5934	8.2046	8.5294
	1.0	2.2054	4.0050	4.4806	5.7564	7.1613	7.4451
	2.0	1.9734	3.5798	4.0043	5.1419	6.3917	6.6448
	Metal	1.5761	2.8648	3.2057	4.1199	5.1272	5.3296
30°	Ceramic	4.3621	7.3539	9.1829	10.3505	13.6765	13.772
	0.5	3.0068	5.0746	6.3381	7.1463	9.4779	9.5115
	1.0	2.6257	4.4305	5.5331	6.2382	8.2556	8.3003
	2.0	2.3484	3.9581	4.9412	5.5689	7.3231	7.4028
	Metal	1.8769	3.1691	3.9595	4.4641	5.9134	5.9434
45°	Ceramic	5.9966	9.2809	12.591	12.8447	15.4032	16.404
	0.5	4.1392	6.4117	8.7041	8.8735	10.7022	11.350
	1.0	3.6140	5.5963	7.5955	7.7429	9.3196	9.9030
	2.0	3.2289	4.9933	6.7713	6.9045	8.2639	8.8197
	Metal	2.5839	4.0027	5.4338	5.5428	6.6769	7.0822

TABLE V
NATURAL FREQUENCY PARAMETER Ω FOR SQUARE Si_3N_4/ SUS_3O_4 SKEW (SCSC) PLATE IN THERMAL ENVIRONMENT

Ψ	n	Mode					
		1	2	3	4	5	6
0°	Ceramic	2.6811	5.7285	5.7647	8.4221	10.2993	10.325
	0.5	1.8427	3.9444	3.9694	5.8041	7.0982	7.1179
	1.0	1.6095	3.4452	3.4673	5.0700	6.1975	6.2157
	2.0	1.4415	3.0838	3.1036	4.5356	5.5403	5.5569
	Metal	1.1475	2.4643	2.4801	3.6285	4.4402	4.4515
15°	Ceramic	2.8234	5.6934	6.3988	8.5206	10.8052	11.168
	0.5	1.9418	3.9212	4.4074	5.8728	7.4485	7.6998
	1.0	1.6965	3.4254	3.8497	5.1299	6.5035	6.7224
	2.0	1.5195	3.0661	3.4452	4.5888	5.8132	6.0075
	Metal	1.2098	2.4501	2.7547	3.6719	4.6596	4.8167
30°	Ceramic	3.3437	6.2614	7.9522	9.1659	11.5464	12.557
	0.5	2.3035	4.3159	5.4813	6.3212	8.0209	8.6608
	1.0	2.0135	3.7706	4.7877	5.5214	6.9841	7.5608
	2.0	1.8035	3.3744	4.2829	4.9373	6.1887	6.7543
	Metal	1.4370	2.6976	3.4282	3.9531	4.9940	5.4192
45°	Ceramic	4.5859	7.8831	11.092	11.283	11.7433	14.852
	0.5	3.1667	5.4409	7.6525	7.7839	8.1784	10.264
	1.0	2.7697	4.7536	6.6793	6.7973	7.1313	8.9705
	2.0	2.4805	4.2511	5.9650	6.0753	6.3227	8.0066
	Metal	1.9785	4.4031	4.7907	4.8722	5.0881	6.4167

Example 2. Effect of temperature field on frequency parameter for Si_3N_4/SUS_3O_4 square plate with thickness (*h*) 0.1 is furnished in Table VI. Different boundary conditions (CCCC, SSSS, and SCSC) are adopted and the value of volume fraction index is kept constant as 1.0. Here, thermal conductivity and density of the plate are considered as position dependent and temperature independent, whereas Young's modulus (E) and thermal expansion are considered as position and temperature dependent. It is assumed that the temperature is constant along the in-plane direction of the plate and varies according to one dimensional heat conduction equation in the thickness direction. The temperature range of 0° - 1200° *K* is selected to generate the frequency parameter for different skew angles. Two observations are made from the example. First, the increase in temperature field reduces the frequency of the plate for all the cases of boundary condition. This output is expected, because at high temperature the Young's modulus of the plate becomes weak, thus reducing the corresponding frequency of the plate. Next, there is an significant increase of frequency parameter is observed as the skew angle of plate rises from 0° to 45°. As expected, the plate with clamped boundary ensures higher frequency compared to the plate having simply supported and simply supported-clamped boundary.

In some cases, it is observed that buckling occurs before the plate starts vibrating in its natural frequency. In order to study such situation, uniform temperature rise is applied over the plate to extract its critical buckling temperature. Then the temperature below the critical buckling temperature one is applied to get the thermal frequency of the plate. The frequency results for $Zr0_2$/ Ti-6Al-4V skew (SSSS) plate for skew angles from 0° to 45° is depicted in Table VII. The first

six modes of frequencies were reported under different cases of thermal differences and linear variation of volume fraction index is considered. The observations with respect to skew angle, boundary condition, temperature observed from Table VI holds true for this example also.

TABLE VI
NATURAL FREQUENCY PARAMETER Ω FOR SQUARE Si_3N_4/ SUS_3O_4 SKEW PLATE UNDER DIFFERENT CASES OF TEMPERATURE DIFFERENCE

Ψ	B.C	Temperature difference (ΔT)			
		0	400	800	1200
0°	CCCC	2.1079	2.0343	1.9574	1.8771
	SSSS	1.2357	1.1319	1.0176	0.8888
	SCSC	1.6367	1.5476	1.4527	1.3506
15°	CCCC	2.2274	2.1538	2.0772	1.9972
	SSSS	1.5622	1.4924	1.4164	1.3333
	SCSC	2.0345	1.9690	1.8935	1.8164
30°	CCCC	2.6477	2.5742	2.4982	2.4194
	SSSS	1.5622	1.4924	1.4164	1.3333
	SCSC	2.0344	1.9660	1.8935	1.8164
45°	CCCC	3.6363	3.5624	3.4867	3.4092
	SSSS	2.2105	2.1773	2.1343	2.0817
	SCSC	2.7812	2.7419	2.6945	2.6396

Example3. The variation of frequency parameter for Zr02/Ti-6Al-4V plate with several aspect ratios (a/b= 0.5, 1.0, 1.5, and 2.0) having simply supported boundary is depicted in Figs. 2 -5, and for clamped boundary is shown in Figs. 6-9.

TABLE VII
NATURAL FREQUENCY PARAMETER Ω FOR SQUARE ZrO_2/ Ti-6AL-4V SKEW (SSSS) PLATE IN THERMAL ENVIRONMENT

Ψ	ΔT (°C)	Mode					
		1	2	3	4	5	6
0°	100	6.6238	16.0508	16.0508	24.7913	30.6590	30.6803
	300	6.2144	15.6369	15.6369	24.3646	30.2231	30.2451
	600	5.5438	14.9944	14.9944	23.7107	29.5569	29.5802
	900	4.7800	14.3232	14.3232	23.0370	28.8755	28.9009
15°	100	7.0262	15.7338	18.1418	24.9345	32.1866	33.4773
	300	6.6531	15.3595	17.7539	24.5455	31.7831	33.0639
	600	6.0493	14.7787	17.1555	23.9456	31.1652	32.4348
	900	5.3764	14.1722	16.5353	23.3377	30.5315	31.7953
30°	100	8.4876	17.1955	22.9816	26.6122	33.3158	31.0082
	300	8.2126	16.9253	22.6758	26.3281	33.3205	36.9234
	600	7.7684	16.4981	22.2046	25.8907	33.3253	36.6705
	900	7.2796	16.0424	21.7176	25.4370	33.3264	36.2176
45°	100	12.133	21.5628	32.0284	32.6081	33.6084	44.1183
	300	11.997	21.4734	31.9400	32.0975	33.4177	44.0362
	600	11.745	21.2859	31.7710	33.0998	33.3140	43.8861
	900	11.455	21.0377	31.5596	32.7674	33.6872	43.7040

Thermal conductivity and density are considered as temperature independent and only position dependent. The values are ρ_m= 4429 kg/m³, k_m=7.82 W/m K, ρ_c= 3000 kg/m³, and k_c=1.80 W/m K, where the subscripts '*m*' and '*c*' denotes the metal and ceramic part of the plate. It can be seen that increase in aspect ratio tends to increase the frequency of the plate and maximum frequency is recorded for plate aspect ratio (*a/b*) =2.0 for simply supported and clamped boundary conditions. Also, the difference between frequency parameters of plate with skew angle 0° and 15° is insignificant, while the plate with 45° skew angle shows noticeable difference of frequency values. Furthermore, as observed from previous cases, the clamped FGM skew plate ensures maximum frequency parameter compared to simply supported boundary.
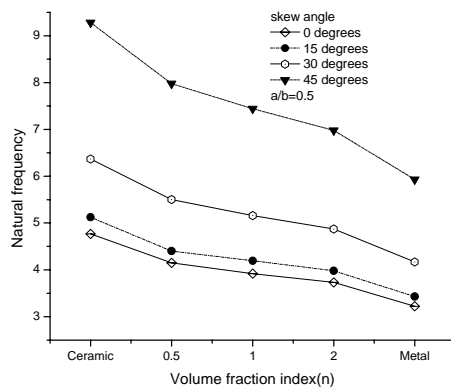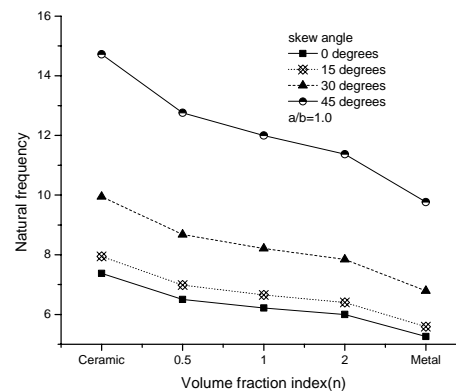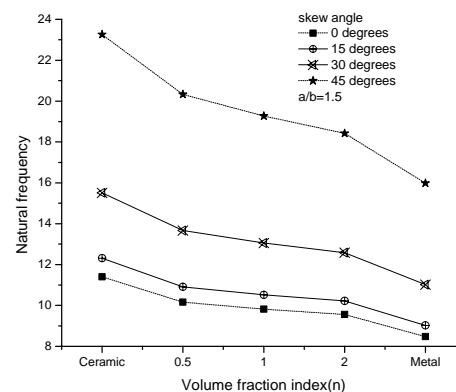

Fig. 3 Natural frequency parameter Ω for Si_3N_4/ SUS_3O_4 (SSSS) plate in thermal environment (*a/b*=1.0)

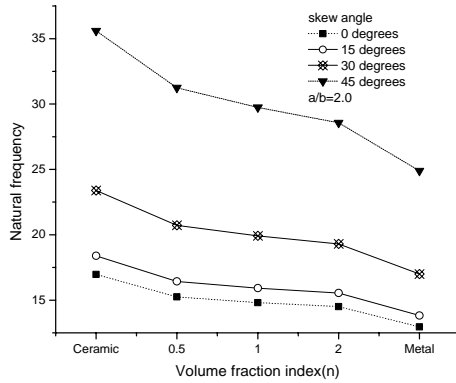Fig. 2 Natural frequency parameter Ω for Si_3N_4/ SUS_3O_4 (SSSS) plate in thermal environment (*a/b*=0.5)

Fig. 4 Natural frequency parameter Ω for Si_3N_4/ SUS_3O_4 (SSSS) plate in thermal environment (*a/b*=1.5)

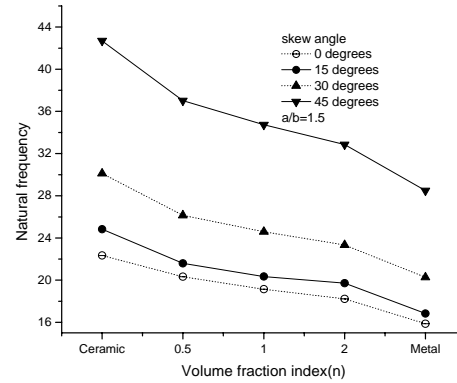Fig. 5 Natural frequency parameter Ω for Si_3N_4/ SUS_30_4 (SSSS) plate in thermal environment (a/b=2.0)

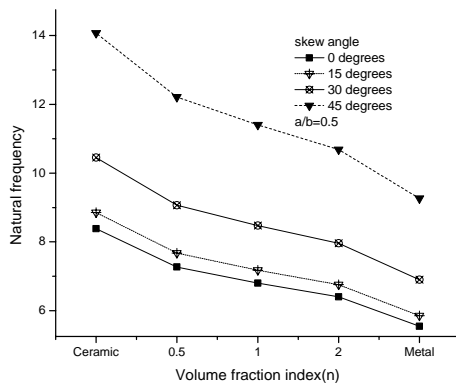Fig. 8 Natural frequency parameter Ω for Si_3N_4/ SUS_30_4 (CCCC) plate in thermal environment (a/b=1.5)

Fig. 6 Natural frequency parameter Ω for Si_3N_4/ SUS_30_4 (CCCC) plate in thermal environment (a/b=0.5)

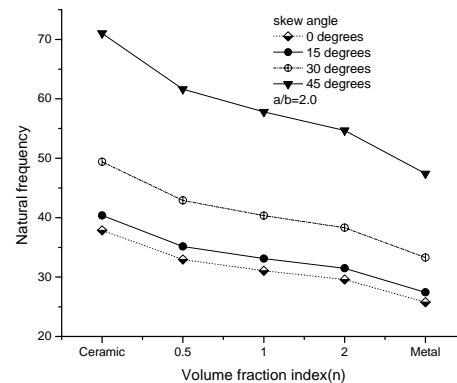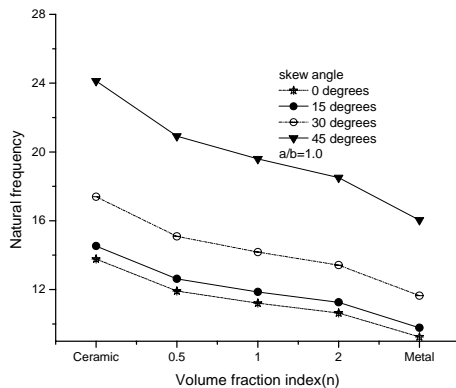Fig. 9 Natural frequency parameter Ω for Si_3N_4/ SUS_30_4 (CCCC) plate in thermal environment (a/b=2.0)

Fig. 7 Natural frequency parameter Ω for Si_3N_4/ SUS_30_4 (CCCC) plate in thermal environment (a/b=1.0)

Next, the effect of thickness ratio (a/h) on frequency is demonstrated for $Zr0_2$/Ti-6Al-4V plate having clamped boundary. Figs. 10 and 11 furnish the value of non-dimensional frequency parameter for thickness ratio (a/h) = 5 and 10 respectively. Linear variation of volume fraction index (n=1.0) is considered to generate results. The top of the plate is kept at 600K, whereas the bottom of the plate is kept at 300K. Results are shown for square FGM plate by setting the value of skew angle to zero. In addition various ranges of skew angles (15°, 30° and 45°) are also considered. For specific thickness ratio (a/h), plate with skew angle beyond 45° shows remarkable change of frequency parameter. Also, it is found that increase in thickness ratio of the plate increases the frequency of the plate regardless of the skew angle and volume fraction index.
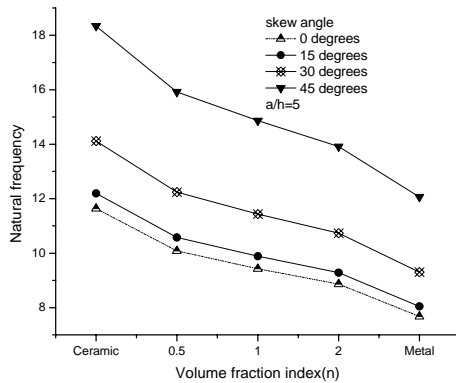
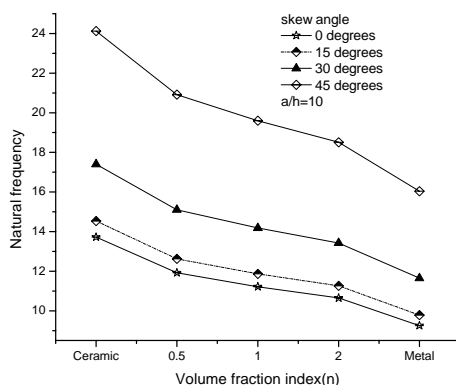Fig. 10 Natural frequency parameter Ω for Si_3N_4/ SUS_3O_4 (CCCC) plate in thermal environment (a/h=5.0)

Fig. 11 Natural frequency parameter Ω for Si_3N_4/ SUS_3O_4 (CCCC) plate in thermal environment (a/h=10.0)

V. CONCLUDING REMARKS

The free vibration analysis of FGM skew plates under thermal environment is reported in this paper. Effective properties of the constituents are calculated by Voigt rule of mixture and simple power law is used to ascertain the variation of material properties in the thickness direction. An efficient nine noded C_0 finite element model recently developed by the authors [26]-[30] based on the HSDT is utilized in the present study which does not require any shear correction factor. The efficacy of the model is shown by comparing the results with standard available literature results. Parametric study is carried out to observe the influence of factors like, aspect ratio, thickness ratio, skew angle and volume fraction index on free vibration response of the FGM skew plate. Temperature dependent material properties of the constituents are considered. It is interpreted that the volume fraction index and skew angle plays significant role in predicting the vibration of FGM skew plate subjected to thermal load. Further increase in skew angle beyond 45° has noticeable remark on frequency of the plate. Vibration analysis of FGM sandwich skew plates under thermal environment is a topic of interest in the field of FGM. This subject could be the future work by the authors.

REFERENCES

[1] J.N. Reddy, and C.D. Chin,"Thermo mechanical analysis of functionally graded cylinders and plates," *J. of Ther. Stress.,* vol.21, pp. 593-626, 1998.

[2] J.Suresh Kumar, B. Sidda Reddy, and C. Eswara Reddy, "Higher order theory for free vibration analysis of functionally graded material plates," *ARPN J. of Eng. and App. Sci.,* vol.6, pp. 105-111, 2011.

[3] Hiroyuki Matsunaga, "Free vibration and stability of functionally graded plates according to a 2-D higher-order deformation theory," *Comp. Struct.,* vol. 82, pp. 499-512, 2008.

[4] N. Noda, "Thermal stresses in materials with temperature-dependent properties," *App. Mech. Revision* vol.44, pp. 383-97, 1991.

[5] Mohammed Talha, and B.N.Singh, "Static response and free vibration analysis of FGM plates using higher order shear deformation theory," *App. Math. Model.,* vol.34, pp. 3991-4011, 2010.

[6] M. Talha, and B.N. Singh, "Thermo-mechanical deformation behavior of functionally graded rectangular plates subjected to various boundary conditions and loadings," *Int. J. Aerospace and Mech. Eng.,* vol. 6, no.1, pp. 14-25, 2012.

[7] L. Hadji, H.A.Atmane, and A.Tounsi, "Free vibration of functionally graded sandwich plates using four-variable refined plate theory," *App. Math. and mech.,* vol.32, pp. 925-942, 2011.

[8] A.J.M. Ferreria, R.C. Batra, and C.M.C.Roque, "Natural frequencies of functionally graded plates by a meshless method" *Comp. Struct.,* vol. 75, pp.593-600, 2006.

[9] Bahar uymaz and Metin aydogdu, "Three-Dimensional Vibration Analyses of Functionally Graded Plates under Various Boundary Conditions," *J. Rein. Plas. Comp.,* vol. 26, pp. 1847-1863, 2007.

[10] Serge Abrate, "Free vibration, buckling, and static deflection of functionally graded plates," *Comp. Sci. and Tech.,* vol.66, pp.2383–2394, 2006.

[11] Senthil S. Vel, and R.C.Batra, "Three dimensional exact solution for the vibration of functionally graded rectangular plates," *J. of Sound and Vibr.,* vol.272, pp. 703-730, 2004.

[12] Xiao-Lin Huang, and Hui-Shen Shen, "Non-linear vibration and dynamic response of functionally graded plates in thermal environments," *Int. J. of Solids and Struct.,* vol.41, pp. 2403-2427, 2004.

[13] J. Yang, and H.S.Shen, "Vibration characteristics and transient response of shear deformable functionally graded plates in thermal environments," *J. of Sound and Vib.,* vol.255, pp. 579-602, 2002.

[14] M. Talha, and B.N. Singh, "Thermo-mechanical induced vibration characteristics of shear deformable functionally graded ceramic-metal plates using finite element method," *Proc. of the Institution of Mech. Engineers Part C J. of Mech. Eng. Sci.,* vol. 225, pp. 50-65, 2011.

[15] N. Soundararajan, T.Prakash, and M.Ganapathi, "Nonlinear free flexural vibrations of functionally graded rectangular and skew plates under thermal environments," *Finite Ele. in Ana. and Design* vol.42, pp.152-168, 2005.

[16] L.F.Qian, R.C. Batra, and L.M.Chen, " Free and forced vibrations of thick rectangular plates using higher order shear and normal deformable plate theory and meshless Petrov-Galerkin method," *CMES* vol.4, pp. 519-534, 2003.

[17] Jae-Sang Park and Ji-Hwan Kim, "Thermal postbuckling and vibration analyses of functionally graded plates," *J. of Sound and Vib.,* vol.289, pp.77–93, 2006.

[18] Jae-Sang Park and Ji-Hwan Kim, "Thermal postbuckling and vibration analyses of functionally graded plates," *J. of Sound and Vib.,* vol.289, pp.77–93, 2006.

[19] D.P.Makhecha, B.P. Patel, and M.Ganapathi, "Transient dynamics of thick skew sandwich laminates under thermal/mechanical loads," *Journal of Plastics and Composites* 20: 1524-1545, 2001.

[20] Ali Shahrjerdi, F.Mustapha, and S.M.Sapuan, "Thermal free vibration analysis of temperature-dependent functionally graded plates using second order shear deformation," *Key Eng., Mat.,* vol.471-472, pp.133-139, 2011.

[21] W.U. Lei, and M.A.Liansheng, "Thermal vibration of functionally graded circular plates," *Key Eng. Mat.,* vol.353-358, pp. 1777-1780, 2007.

[22] Maziar Janghorban, and Amin Zare, "Thermal effect on free vibration analysis of functionally graded arbitrary straight-sided plates with different cutouts," *Latin Americal J. Solids and Struct.,* vol.8, pp. 245 – 257, 2011.

[23] Y.S. Touloukian, "Thermophysical properties of high temperature solid materials," MacMillan, New York, 1967.

[24] R.D., Mindlin, "Influence of rotator inertia and shear on flexural vibration of isotropic elastic plates," *J. of App. Mech.,* vol.73, pp. 31-38, 1951.

[25] J.N.Reddy, "A simple higher-order theory for laminated composite plate," *J. of App. Mech.,* vol.51, pp.745-752, 1984.

[26] M.N.A. Gulshan Taj, Anupam chakrabarti, "Static and dynamic analysis of functionally graded skew plates," *ASCE J. Eng. Mech.* Vol. 139, no. (7), pp. 848-857, 2013.

[27] M.N.A. Gulshan Taj, Anupam Chakrabarti, and Abdul Hamid Sheikh, "Analysis of functionally graded plates using higher order shear deformation theory (Periodical style- accepted for publication)," App. Math. Modeling, to be published.

[28] M.N.A. Gulshan Taj, and Anupam Chakrabarti, "Dynamic analysis of functionally graded skew shell panel," *Latin American J. of Solids and Struct.,* vol.10, no.6, pp., 1243-1266, 2013.

[29] M.N.A. Gulshan Taj, and Anupam Chakrabarti, "An Efficient Finite Element Approach for Bending Analysis of Functionally Graded Ceramic-Metal Skew Shell Panels (Periodical style- submitted for publication)," submitted for publication.

[30] M.N.A. Gulshan Taj, and Anupam Chakrabarti, "Bending analysis of functionally graded skew sandwich plates with through-the thickness displacement variations (Periodical style- submitted for publication)," submitted for publication.