International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:1, 2014

Weak Convergence of Mann Iteration for a Hybrid Pair of Mappings in a Banach Space

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Abstract—We prove the weak convergence of Mann iteration for a hybrid pair of maps to a common fixed point of a selfmap f and a multivalued f nonexpansive mapping T in Banach space E.

Keywords—Common fixed point, Mann iteration, Multivalued mapping, weak convergence.

I. Introduction

Let E be a Banach space and K, a nonempty subset of E. We denote by 2^E, the family of all subsets of E; $CB(E)$, the family of nonempty closed and bounded subsets of E and $C(E)$, the family of nonempty compact subsets of E. Let $f : K \to K$ be a selfmap. Let H be a Hausdorff metric on $CB(E)$. That is, for $A, B \in CB(E)$,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\},$$

where

$$d(x,B) = \inf\{\|x-y\| : y \in B\}.$$

A multivalued mapping $T : K \to 2^K$ is called f *nonexpansive* if

$$H(Tx,Ty) \le \|fx - fy\|,$$

for all $x, y \in K$.

If $f = I_K$, the identity mapping on K , then we call T is a *multivalued nonexpansive* mapping.

A point x is *a fixed point* of T if $x \in Tx$. A point x is called *a common fixed point* of f and T if $fx = x \in Tx$. $F(T) = \{x \in K : x \in Tx\}$ stands for the fixed point set of a mapping T and $F = F(T)\cap F(f) = \{x \in K : fx = x \in Tx\}$ stands for the common fixed point set of maps f and T.

Recently, Song and Wang [5] introduced the following Mann iterates of a Multivalued mapping T:

Let K be a nonempty convex subset of E, $\alpha_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Let $T : K \to CB(K)$ be a multivalued mapping. Let $x_0 \in K$, and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \tag{1}$$

where $y_n \in Tx_n$ such that

$$\|y_{n+1} - y_n\| \le H(Tx_{n+1}, Tx_n) + \gamma_n, n = 0, 1, 2, \cdots .$$

Song and Wang [5] established the following theorem on the convergence of Mann iteration.

Theorem 1. [5] Let E be a Banach space satisfying Opial's condition and K be a nonempty, weakly compact and convex subset of E. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mappings for which $F(T) \ne \emptyset$ and for which $T(y) = \{y\}$ for each $y \in F(T)$. For $x_0 \in K$, let $\{x_n\}$ be the Mann iteration defined by (1). Assume that

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ weakly converges to a fixed point of T.

The aim of this paper is to prove the weak convergence of Mann iteration for a hybrid pair of maps to a common fixed point of a selfmap f and a multivalued f nonexpansive mapping T in Banach space E. Our results extend the results of Song and Wang [5] to a hybrid pair of maps.

II. Preliminaries

Throughout this paper E denotes real Banach space. We denote the weak convergence of $\{x_n\}$ to x in E by $x_n \rightharpoonup x$, and that of strong convergence by $x_n \to x$.

A Banach space E is said to satisfy *Opial's condition* [3] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for each $y \in E$ with $x \ne y$.

Every Hilbert space and l^p $(1 < p < \infty)$ space satisfy Opial's condition [3].

A multivalued mapping $T : K \to CB(K)$ is said to satisfy Condition I [4] if there is a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x,Tx) \ge \varphi(d(x,F(T)) \text{ for all } x \in K.$$

A selfmap $f : E \to E$ is said to be *weakly continuous* if $fx_n \rightharpoonup fx$ whenever $x_n \rightharpoonup x$. A map $f : K \to E$ is said be *demiclosed* at 0 if for each sequence $\{x_n\}$ in K converging weakly to x and $\{fx_n\}$ converging strongly to 0, we have $fx = 0$.

Lemma 1. [2] Let (E,d) be a complete metric space, and $A, B \in CB(E)$ and $a \in A$. Then for each positive number ε, there exists $b \in B$ such that

$$d(a,b) \le H(A,B) + \varepsilon.$$

Lemma 2. [6] Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in [0,1]$ with

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

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Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$ for all integers $n \geq 1$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

We will construct the following iteration.

Let K be a nonempty subset of a metric space X. Let $f : K \to K$, $T : K \to CB(K)$ with $f(K)$ is convex and $Tx \subseteq f(K)$ for all $x \in K$. Let $\alpha_n \in [0,1]$, and $\gamma_n \in (0,\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$ and $y_0 \in Tx_0$. Let $z_0 = fx_0$ and

$$\begin{aligned} z_1 = fx_1 &= (1-\alpha_0)fx_0 + \alpha_0 y_0 \\ &= (1-\alpha_0)z_0 + \alpha_0 y_0. \end{aligned}$$

From Lemma 1, there exists $y_1 \in Tx_1$ such that

$$\|y_1 - y_0\| \leq H(Tx_1, Tx_0) + \gamma_0.$$

Let

$$z_2 = fx_2 = (1-\alpha_1)z_1 + \alpha_1 y_1.$$

Inductively, we have

$$z_{n+1} = fx_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \tag{2}$$

where $y_n \in Tx_n$ such that

$$\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n, \ n = 0, 1, 2, \cdots.$$

III. Main Results

Proposition 1. [1] Let K be a nonempty subset of a Banach space E. Let $f : K \to K$ be a selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multivalued f nonexpansive mapping and $Tx \subseteq f(K)$ for all $x \in K$. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps T and f, defined by (2) and assume also that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then $\lim_{n\to\infty} \|z_n - y_n\| = 0$ and $\lim_{n\to\infty} d(z_n, Tx_n) = 0$.

Proof. From the definition of the Mann iteration $\{z_n\}$ given by (2), it follows that $z_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n$, where $y_n \in Tx_n$ such that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq H(Tx_{n+1}, Tx_n) + \gamma_n \\ &\leq \|z_{n+1} - z_{n+1}\| + \gamma_n, \ n = 0, 1, 2, \cdots. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|z_{n+1} - z_n\|) \leq \limsup_{n\to\infty} \gamma_n = 0.$$

Hence, all conditions of Lemma 2 are satisfied. Hence, by Lemma 2, we obtain $\lim_{n\to\infty} \|z_n - y_n\| = 0$.

Since $y_n \in Tx_n$ for all $n = 0, 1, 2, \cdots$, we have $d(z_n, Tx_n) \leq \|z_n - y_n\|$.

Hence, $\lim_{n\to\infty} d(z_n, Tx_n) = 0$.

Definition 1. A multivalued mapping $T : K \to CB(K)$ is said to satisfy *Condition I* associated with a selfmap $f : K \to K$ if there is a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0,\infty)$ such that

$$d(fx, Tx) \geq \varphi(d(fx, F), \text{ for all } x \in K.$$

where $F = \{x \in K : fx = x \in Tx\}$.

If $f = I_K$, in Definition 1, then T is said to satisfy 'Condition I'(Senter and Dotson [4]).

Theorem 2. Let K be a nonempty closed subset of a Banach space E. Let $f : K \to K$ be a continuous selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multivalued f nonexpansive mapping for which $Tx \subseteq f(K)$ for all $x \in K$; $F = F(T) \cap F(f) \neq \emptyset$, and satisfies condition I associated with f. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps T and f, defined by (2) and assume also that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

If $T(y) = \{y\}$ for each $y \in F(T)$, then the Mann iteration $\{z_n\}$ strongly converges to a common fixed point of f and T.

Proof. It follows from Proposition 1 that $\lim_{n\to\infty} d(z_n, Tx_n) = 0$.

Now let $p \in F = F(T) \cap F(f)$. Then,

$$\begin{aligned} \|z_{n+1} - p\| &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n H(Tx_n, Tp) \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \\ &= \|z_n - p\|, \ n = 0, 1, 2, \cdots. \end{aligned}$$

which gives

$$d(z_{n+1}, F) \leq d(z_n, F), \ n = 0, 1, 2, \cdots.$$

Then the sequence $\{d(z_n, F)\}$ is a non increasing sequence of nonnegative reals and hence $\lim_{n\to\infty} d(z_n, F)$ exists.

Since T satisfies Condition I associated with f, we have

$$\begin{aligned} \lim_{n\to\infty} \varphi(d(z_n, F)) &= \lim_{n\to\infty} \varphi(d(fx_n, F)) \\ &\leq \lim_{n\to\infty} d(fx_n, fx_n) \\ &= \lim_{n\to\infty} d(z_n, fx_n) = 0. \end{aligned}$$

Hence, $\lim_{n\to\infty} \varphi(d(z_n, F)) = 0$.

Since φ is non decreasing function, we get $\lim_{n\to\infty} d(z_n, F) = 0$. Hence, for $\varepsilon > 0$, there exist natural number n_0 such that if $n \geq n_0$, $d(z_n, F) < \frac{\epsilon}{4}$. In particular, $\inf\{\|z_{n_0} - y\| : y \in F\} < \frac{\epsilon}{4}$. So there must exist a $z \in F$ such that $\|z_{n_0} - z\| < \frac{\epsilon}{2}$. Now for $m, n \geq n_0$, we have,

$$\begin{aligned} \|z_{n+m} - z_n\| &\leq \|z_{n+m} - z\| + \|z_n - z\| \\ &\leq 2\|z_n - z\| \\ &< 2(\frac{\varepsilon}{2}) = \varepsilon. \end{aligned}$$

Hence, $\{z_n\}$ is a Cauchy sequence in a closed subset F of a Banach space E, therefore it converges to a point, say $p \in K$. Hence, $d(p, F) = 0$. Since F is closed, $p \in F$.

Hence, the conclusion of the theorem follows.

Corollary 1. [5] Let K be a nonempty, closed and convex subset of a Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued nonexpansive mapping for which $F(T) \neq \emptyset$ and for which $T(y) = \{y\}$ for each $y \in F(T)$ and satisfies condition I. For $x_0 \in K$, let $\{x_n\}$ be the Mann iteration defined by (1). Assume that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

Proof. Follows from Theorem 2 by taking $f = I_K$.

Theorem 3. Let E be a Banach space satisfying Opial's condition and K be a nonempty weakly compact subset of E. Let $f : K \to K$ be a weakly continuous selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multivalued f nonexpansive mapping for which $Tx \subseteq f(K)$ for all $x \in K$; $F(T) \cap F(f) \neq \emptyset$, and $d(x, Tx) \leq d(fx, Tx)$ for all $x, y \in K$. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps T and f, defined by (2) and assume also that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

If $T(y) = \{y\}$ for each $y \in F(T)$ and $I - f$ is demiclosed at 0, then the Mann iteration $\{z_n\}$ weakly converges to a common fixed point of f and T.

Proof. It follows from Proposition 1 that $\lim_{n\to\infty} d(z_n, Tx_n) = 0$.
Further, since $d(x_n, Tx_n) \leq d(z_n, Tx_n)$ we get $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
Now let $p \in F(T) \cap F(f)$. Then,

$$\begin{aligned}\|z_{n+1} - p\| &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n H(Tx_n, Tp) \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \\ &= \|z_n - p\|, \ n = 0, 1, 2, \cdots .\end{aligned}$$

Then the sequence $\{\|z_n - p\|\}$ is a decresing sequence of nonnegative reals and hence $\lim_{n\to\infty} \|z_n - p\|$ exists for each $p \in F(T) \cap F(f)$.

From the weak compactness of K, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ as $j \to \infty$, for some $q \in K$. By the weak continuity of f, we have $z_{n_j} \rightharpoonup fq = y$ (say) as $j \to \infty$. Suppose $q \notin Tq$. By the compactness of Tq, for any given x_{n_j}, there exists $w_j \in Tq$ such that $\|x_{n_j} - w_j\| = d(x_{n_j}, Tq)$. Since Tq is compact, w_k has a convergent subsequence. For simplicity sake, we write that subsequence as w_j itself. So, $w_j \to w \in Tq$. Then $q \neq w$. Now the Opial's property of E implies that,

$$\begin{aligned}\limsup_{j\to\infty} \|z_{n_j} - w\| &\leq \limsup_{j\to\infty}[\|z_{n_j} - w_j\| + \|w_j - w\|] \\ &\leq \limsup_{j\to\infty} \|z_{n_j} - w_j\| \\ &\leq \limsup_{j\to\infty}[d(z_{n_j}, Tx_{n_j}) + H(Tx_{n_j}, Tq)] \\ &= \limsup_{j\to\infty} \|z_{n_j} - y\| \\ &< \limsup_{j\to\infty} \|z_{n_j} - w\|,\end{aligned}$$

a contradiction. Hence, $q = w$, and so $q \in Tq$.
Since,

$$\|fx_{n_j} - x_{n_j}\| \leq d(z_{n_j}, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}),$$

we have $(I - f)(x_{n_j}) \to 0$ strongly as $k \to \infty$. By the demiclosedness of $I - f$, we have $(I - f)(q) = 0$ and hence $q = fq = y$.

Hence, $fy = y \in Ty$.

Hence, y is a common fixed point of f and T.

Next we show that $z_n \rightharpoonup y$ as $n \to \infty$. Suppose not. There exists another subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup z \neq y$. Then, by similar argument as above, we have $fz = z \in Tz$.
From Opial's property, we have

$$\begin{aligned}\lim_{n\to\infty} \|z_n - z\| &= \limsup_{k\to\infty} \|z_{n_k} - z\| \\ &< \limsup_{k\to\infty} \|z_{n_k} - y\| \\ &= \limsup_{j\to\infty} \|z_{n_j} - y\| \\ &< \limsup_{j\to\infty} \|z_{n_j} - z\| \\ &= \lim_{n\to\infty} \|z_n - z\|,\end{aligned}$$

a contradiction. Hence, $z_n \rightharpoonup y$ as $n \to \infty$.

Thus, the conclusion of the theorem follows.

Corollary 2. If $f = I_K$, then we get Theorem 1. Hence, Theorem 3 extends Theorem 1 to a pair of maps.

References

[1] G. V. R. Babu, G.N. Alemayehu, Strong Convergence of Mann Iteration for a Hybrid Pair of Mappings in a Banach Space, Mathematica Aeterna, 3 (2013)(7), 501–507.
[2] S. B. Nadler Jr., Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–487.
[3] Z. Opial, Weak Convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.
[4] H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974) 375–380.
[5] Y. Song, H. Wang, Convergence of iterative algorithms for multivalued mappings in Banach spaces, Nonlinear Analysis 70 (2009) 1547–1556.
[6] T. Suzuki, Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces, Fixed Point Theory Appl. 2005 (1) (2005) 103–123, doi:10.1155 \ FPTA.2005.103.