International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:1, 2014

Entropic Measures of a Probability Sample Space and Exponential Type (α, β) Entropy

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Abstract—Entropy is a key measure in studies related to information theory and its many applications. Campbell for the first time recognized that the exponential of the Shannon's entropy is just the size of the sample space, when distribution is uniform. Here is the idea to study exponentials of Shannon's and those other entropy generalizations that involve logarithmic function for a probability distribution in general. In this paper, we introduce a measure of sample space, called 'entropic measure of a sample space', with respect to the underlying distribution. It is shown in both discrete and continuous cases that this new measure depends on the parameters of the distribution on the sample space - same sample space having different 'entropic measures' depending on the distributions defined on it. It was noted that Campbell's idea applied for Rènyi's parametric entropy of a given order also. Knowing that parameters play a role in providing suitable choices and extended applications, paper studies parametric entropic measures of sample spaces also. Exponential entropies related to Shannon's and those generalizations that have logarithmic functions, i.e. are additive have been studies for wider understanding and applications. We propose and study exponential entropies corresponding to non additive entropies of type (α, β), which include Havard and Charvât entropy as a special case.

Keywords—Sample space, Probability distributions, Shannon's entropy, Rènyi's entropy, Non-additive entropies .

I. Introduction

LET $\Delta_n = \{P = (p_1, \ldots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}, n \geq 2$ be a set of n-complete probability distributions.
For any probability distribution $P = (p_1, \ldots, p_n) \in \Delta_n$, Shannon's entropy [9], is defined as

$$H(P) = -\sum_{i=1}^{n} p(x_i) \log p(x_i) \tag{1}$$

Various generalized entropies have been introduced in the literature, taking the Shannon entropy as basic and have found applications in various disciplines such as economics, statistics, information processing and computing etc.

Generalizations of Shannon's entropy started with Rènyi's entropy [8] of order-α, given by

$$H_\alpha(P) = \frac{1}{(1-\alpha)} \log \Big[\sum_{i=1}^{n} (p(x_i))^\alpha\Big], \quad \alpha \neq 1, \alpha > 0 \tag{2}$$

Campbell [1] studied exponentials of the Shannon's and Rènyi's entropies, given by

$$E(P) = e^{H(P)} \tag{3}$$

and

$$E_\alpha(P) = e^{H_\alpha(P)} \tag{4}$$

where $H(P)$ and $H_\alpha(P)$ represent respectively the Shannon's and Rènyi's entropies. It may also be mentioned that Koski and Persson [5] studied

$$E_{(\alpha,\beta)}(P) = e^{H_{(\alpha,\beta)}(P)} \tag{5}$$

exponential of Kapur's entropy [4] given by

$$H_{\alpha,\beta}(P) = \frac{1}{(\beta-\alpha)} \log \frac{\sum_{i=1}^{n} (p(x_i))^\alpha}{\sum_{i=1}^{n} (p(x_i))^\beta}, \quad \alpha \neq \beta, \alpha, \beta > 0 \tag{6}$$

It is interesting to notice that, in the case of discrete uniform distribution $P \in \Delta_n$, (3), (4) and (5) all reduce to n, just the 'size of sample space of the distribution'.

In fact, when we consider corresponding entropies in the continuous case,uniform distribution in a finite interval (a, b), the exponential of these entropies are equal to length $[b - a]$ of the sample space.

Measures, as we know, are important for concepts and their applications. Here we raise a question: Is there a measure of the sample space in terms of the probability distribution defined over it? In this paper we introduce such a measure and study it.

Further, it is well known that, parameters in measures play a significant role in widening their applications and meaningfulness. In this paper we introduce a measure for general sample space of a probability distribution, involving parameters also.

It may be recalled that Shannon's and Rènyi's entropies are additive and involve logarithmic function. The idea has significantly been advanced in non-additive measures by Havrda and Charvât [2] and Sharma and Taneja [10]. Exponential entropies corresponding to these expend this study. Since Sharma and Taneja's entropy holds Havrda and Charvât entropy as a particular case, we take for studying exponential 'type (α, β)' entropy, corresponding to Sharma and Taneja [10] entropy of type (α, β) which is a two parametric non-additive generalization of Shannon's entropy given by:

$$H^{(\alpha,\beta)}(P) = \frac{\sum_{i=1}^{n} (p(x_i))^\alpha - \sum_{i=1}^{n} (p(x_i))^\beta}{2^{(1-\alpha)} - 2^{(1-\beta)}}, \tag{7}$$

where

$$\alpha \neq \beta,\ \alpha, \beta > 0.$$

This paper is organized as follows: In section II we define a measure called "entropic measure of sample

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space of a probability distribution", in general, based on Shannon's entropy. In section III we propose a generalized "order-α entropic measure" for sample space of a probability distribution. In section IV. we introduce exponential "type (α, β)" entropy and discuss its limiting and particular cases. In section V. we study some properties of exponential "type (α, β)" entropy and brief our conclusions are presented in Section VI.

II. Entropic measure of the sample space of a probability distribution

We proceed with the following formal definition:

Definition 1(Entropic Measure of a Sample Space): Given a probability distribution P on a sample space S , the entropic measure of S with respect to the distribution P is defined as:

$$E(S,P) = e^{H(P)} \tag{8}$$

where $H(P)$ is Shannon's entropy of the distribution.

Note 1: As pointed out earlier, $E(S,P)$ gives size/order of the sample space when the distribution is uniform on a finite sample space. The condition of 'finiteness' of the sample space, as illustrated by examples below, may not be necessary. In fact an infinite sample space, depending upon the distribution defined on it may have finite entropic measure.

Note 2: The idea of size of the sample space is expandable to multivariate cases also. If we consider bi-variate situation specified by two discrete random variables X and Y:

$$X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n), \tag{9}$$

their joint occurrences are given by the set of points

$$XY = (x_i y_j | i = 1, \ldots, n; j = 1, \ldots, m), \tag{10}$$

when both the distributions on X and Y are uniform, and their resulting joint distribution is also uniform, U, and

$$E(XY, U) = e^{H(XY)} = nm \tag{11}$$

which is also the size of product sample space.
This is also the case, if X and Y were continuous random variables uniformly distributed in finite intervals, (a, b) and (c, d), as can be quickly verified.
Note 3: The entropic measure $E(S,P)$ that we define, as will be seen below, is finally a function of the parameters of the distribution, which makes it interesting further.

Examples of discrete distributions: The entropic measures of sample space S, under the following distributions are different depending only on the parameters of the distributions:
(i) *Geometric distribution* [3]: For $S = \{i|i = 0, 1, \ldots, \infty\}$,

$$p_i = qp^i, \quad p + q = 1 \tag{12}$$

then

$$H(P) = -\frac{1}{q}[p \log p + q \log q]. \tag{13}$$

From Definition in (1), we get

$$E(S, G(P)) = \frac{1}{1-p}\Big(\frac{1}{p}\Big)^{\frac{p}{1-p}} \tag{14}$$

where $G(P)$ stands for geometric distribution, is a function of parameter p only.

(ii) *Inverse λ-Power distribution* [3]: For $S = \{i|i = 1, \ldots, \infty\}$ and $\lambda > 1$,

$$p_i = \frac{i^{-\lambda}}{\zeta(\lambda)}, \quad \zeta(\lambda) = \sum_{i=1}^{\infty} i^{-\lambda} \tag{15}$$

then

$$H(P) = \log \zeta(\lambda) - \lambda \frac{\zeta^{'}(\lambda)}{\zeta(\lambda)}. \tag{16}$$

Using Definition in (1), we get

$$E(S, \zeta(\lambda)) = \zeta(\lambda) e^{-\lambda \frac{\zeta^{'}(\lambda)}{\zeta(\lambda)}} \tag{17}$$

where $\zeta(\lambda)$ represents inverse λ-power distribution, is a function of parameter λ only.

Examples of continuous distributions:
(i) *Two sided power distribution* [6]: For $-\infty < a \le m \le b < \infty$ and $n > 0$

$$f(x) = \begin{cases} \frac{n}{b-a}\Big(\frac{x-a}{m-a}\Big)^{n-1}, & \text{if } a < x \le m, \\ \frac{n}{b-a}\Big(\frac{b-x}{b-m}\Big)^{n-1}, & \text{if } m \le x \le b, \end{cases} \tag{18}$$

then

$$H(P) = \log(b-a) - \log n + \frac{n-1}{n}. \tag{19}$$

From Definition in (1), we get

$$E(S, Ts(P)) = \frac{(b-a)\exp\big(\frac{n-1}{n}\big)}{n} \tag{20}$$

where $Ts(P)$ represents two sided power distribution, is a function of parameters a, b only.

(ii) *Two piece normal distribution* [6]: For $-\infty < \mu < \infty$, $\sigma_1 > 0$ and $\sigma_2 > 0$,

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_1+\sigma_2}\exp\{-\frac{(x-\mu)^2}{2\sigma_1{}^2}\}, & \text{if } x \le \mu, \\ \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_1+\sigma_2}\exp\{-\frac{(x-\mu)^2}{2\sigma_2{}^2}\} & \text{if } x > \mu, \end{cases} \tag{21}$$

then

$$H(P) = \frac{1}{2} - \log\Big(\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_1+\sigma_2}\Big). \tag{22}$$

Using Definition in (1), we get

$$E(S, Tp(P) = \sqrt{\frac{\pi}{2}}\exp\Big(\frac{1}{2}\Big)(\sigma_1 + \sigma_2) \tag{23}$$

where $Tp(P)$ represents two piece normal distribution, is a function of parameters σ_1, σ_2 only.

(iii) *Exponential distribution* [3]: For $0 \le x < \infty$ and $\lambda > 0$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0 \text{ ,} \\ 0, & \text{if } x < 0, \end{cases} \tag{24}$$

then

$$H(P) = \log(\lambda) - 1. \quad (25)$$

From Definition in (1), we get

$$E(S, Ed(P)) = \frac{\lambda}{e} \quad (26)$$

where $Ed(P)$ represents exponential distribution, is a function of parameter λ only.

(iv) *Asymmetric Laplace distribution* [6]: For $-\infty < \theta < \infty$ and $\phi_1, \phi_2 > 0$,

$$f(x) = \begin{cases} \frac{1}{2\phi_1}\exp(-\frac{|x-\theta|}{\phi_1}), & \text{if } x \geq \theta, \\ \frac{1}{2\phi_2}\exp(-\frac{|x-\theta|}{\phi_2}), & \text{if } x < \theta, \end{cases} \quad (27)$$

then

$$H(P) = 1 + \log(2) + \frac{(\log\phi_1 + \log\phi_2)}{2}. \quad (28)$$

Using Definition in (1), we get

$$E(S, Al(P)) = 2e\sqrt{\phi_1\phi_2} \quad (29)$$

where $Al(P)$ represents two piece Asymmetric Laplace distribution, is a function of parameters ϕ_1, ϕ_2 only.

(v) *Generalized Pareto distribution* [6]: For $x > 0$ (if $c \leq 0$ and $k > 0$) or for $0 < x < \frac{k}{c}$ (if $c > 0$ and $k > 0$),

$$f(x) = \frac{1}{k}(1 - \frac{cx}{k})^{\frac{1}{c}-1} \quad (30)$$

then

$$H(P) = 1 - c + \log k. \quad (31)$$

From Definition in (1), we get

$$E(S, Gpd(P) = k\exp(1 - c) \quad (32)$$

where $Gp(P)$ represents generalized Pareto distribution, is a function of parameters k, c only.

(vi) *Gaussian distribution* [7]: For $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma^2 > 0$,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{(x-\mu)^2}{2\sigma^2}} \quad (33)$$

then

$$H(P) = \frac{1}{2}\log(2\pi e\sigma^2). \quad (34)$$

Using Definition in (1), we get

$$E(S, Gd(P)) = \sigma\sqrt{(2\pi e)} \quad (35)$$

where $Gd(P)$ represents Gaussian distribution, is a function of parameter σ only.
So far we studied a measure which contained no extraneous parameter.
In the next section, we propose a generalized order-α entropic measure of a sample space.

III. Generalized order-α Entropic Measure of a Sample Space

As we mentioned earlier, parametric generalization, in particular Rènyi's order-α entropy has been studied with quite some interest. In this section, we introduce "order-α entropic measure" of a sample space in respect of an underlying probability distribution.

Definition 2 (Order-α Entropic Measure of a Sample Space): Given a probability distribution P on a sample space S , order-α entropic measure of S, is defined as:

$$E_\alpha(S, P) = e^{H_\alpha(P)} \quad (36)$$

where $H_\alpha(P)$ is Rènyi's entropy of the distribution P.

Examples of order-α entropic measure of discrete distributions:
In these examples, we take the various distributions considered earlier.
(i) *Geometric distribution* [6]:

$$H_\alpha(X) = \frac{1}{1-\alpha}\log\Big[\frac{(1-p)^\alpha}{1-p^\alpha}\Big]. \quad (37)$$

Using Definition in (2), we get

$$E_\alpha(S, G(P)) = \Big[\frac{(1-p)^\alpha}{1-p^\alpha}\Big]^{\frac{1}{1-\alpha}} \quad (38)$$

where $G(P)$ stands for geometric distribution, is a function of parameters α, p only.

(ii) *Inverse λ-Power distribution* [6]:

$$H_\alpha(P) = \log\Big[\frac{\zeta(\lambda\alpha)}{\zeta(\lambda)^\alpha}\Big]^{\frac{1}{1-\alpha}}. \quad (39)$$

From Definition in (2), we get

$$E_\alpha(S, \zeta(\lambda)) = \Big[\frac{\zeta(\lambda\alpha)}{\zeta(\lambda)^\alpha}\Big]^{\frac{1}{1-\alpha}} \quad (40)$$

where $\zeta(\lambda)$ represents inverse λ-power distribution, is a function of parameters α, λ only.

Examples of order-α entropic measure of continuous distributions:
(i) *Two sided power distribution* [6]:

$$H_\alpha(P) = \log(b-a) + \frac{\alpha\log n - \log(\alpha n - \alpha + 1)}{1-\alpha}. \quad (41)$$

Using Definition in (2), we get

$$E_\alpha(S, Ts(P)) = \frac{(b-a)n^{\frac{\alpha}{1-\alpha}}}{(\alpha n - \alpha + 1)^{\frac{1}{1-\alpha}}} \quad (42)$$

where $Ts(P)$ represents two sided power distribution, is a function of parameters a, b, α only.

(ii) *Two piece normal distribution* [6]:

$$H_\alpha(P) = -\log\Big(\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_1+\sigma_2}\Big) - \frac{log(\alpha)}{2(1-\alpha)}. \quad (43)$$

From Definition in (2), we get

$$E_\alpha(S, Tp(P) = \sqrt{\frac{\pi}{2}}\alpha^{\frac{1}{2(\alpha-1)}}(\sigma_1 + \sigma_2) \tag{44}$$

where $Tp(P)$ represents two piece normal distribution, is a function of parameters $\alpha, \sigma_1, \sigma_2$ only.

(iii) *Exponential distribution* [7]:

$$H_\alpha(P) = \log\lambda - \frac{\log\alpha}{1-\alpha}. \tag{45}$$

Using Definition in (2), we get

$$E_\alpha(S, Ed(P)) = \lambda\alpha^{(\alpha-1)} \tag{46}$$

where $Ed(P)$ represents exponential distribution, is a function of parameters α, λ only.

(iv) *Asymmetric Laplace distribution* [6]:

$$H_\alpha(P) = \frac{1}{1-\alpha}\log\Big[\frac{{\phi_1}^{1-\alpha} + {\phi_2}^{1-\alpha}}{\alpha 2^\alpha}\Big]. \tag{47}$$

From Definition in (2), we get

$$E_\alpha(S, Al(P)) = \Big[\frac{{\phi_1}^{1-\alpha} + {\phi_2}^{1-\alpha}}{\alpha 2^\alpha}\Big]^{\frac{1}{1-\alpha}} \tag{48}$$

where $Al(P)$ represents asymmetric Laplace distribution, is a function of parameters α, ϕ_1, ϕ_2 only.

(v) *Generalized Pareto distribution* [6]:

$$H_\alpha(P) = \log k - \frac{\log(\alpha - \alpha c + c)}{1-\alpha} \tag{49}$$

Using Definition in (2), we get

$$E_\alpha(S, Pd(P)) = k(\alpha - \alpha c + c)^{\frac{1}{\alpha-1}} \tag{50}$$

where $Pd(P)$ represents generalized Pareto distribution, is a function of parameters k, α, c only.

(vi) *Gaussian distribution* [7]:

$$H_\alpha(P) = \frac{1}{2}\log(2\pi\sigma^2) - \frac{\log\alpha}{2(1-\alpha)}. \tag{51}$$

From Definition in (2), we get

$$E_\alpha(S, Gd(P)) = \sigma\sqrt{(2\pi)}\alpha^{2(\alpha-1)} \tag{52}$$

where $Gd(P)$ represents Gaussian distribution, is a function of parameters σ, α only.
In the next section, we first propose a exponential two-parametric generalization of Shannon's entropy that we refer to as exponential "type(α, β)" entropy and discuss its limiting and particular cases also.

IV. Exponential "type(α, β)" entropy

Corresponding to Sharma and Taneja "type (α, β)" entropy, the exponential "type(α, β)" entropy is defined as follows:

Definition 3: Expontial type(α, β) entropy of a discrete distribution P is given by:

$$E^{(\alpha,\beta)}(P) = \frac{\Big[e^{\Big(\sum_{i=1}^n (p_i)^\alpha - \sum_{i=1}^n (p_i)^\beta\Big)} - 1\Big]}{e^{(2^{1-\alpha} - 2^{1-\beta})} - 1} \tag{53}$$

where

$$\alpha \neq \beta,\ \alpha, \beta > 0.$$

This has interesting particular cases that we briefly mention below.
Limiting and Particular cases:

(a) When $\alpha \to \beta$, measure (48) reduces to

$$H^\beta(P) = -2^{\beta-1}\sum_{i=1}^n (p(x_i))^\beta \log p(x_i) \tag{54}$$

This measure was given by Sharma and Taneja [10].
(b) When $\alpha \to \beta$ and further taking $\beta \to 1$, measure (48) reduces to Shannon's entropy.
(c)When $\alpha = 1$, measure (48) reduces to

$$E^{1,\beta}(P) = \frac{\Big[e^{1 - \sum_{i=1}^n (p_i)^\beta} - 1\Big]}{e^{1-2^{1-\beta}} - 1}, \quad \beta \neq 1, \beta > 0 \tag{55}$$

This can be considered as exponential "type-β" entropy corresponding to Havrda and Charvât entropy [2] of type β given by

$$H^\beta(P) = \frac{\sum_{i=1}^n (p_i)^\beta - 1}{2^{1-\beta} - 1}, \tag{56}$$

In the next section, we study some properties of $E^{(\alpha,\beta)}(P)$, the exponential "type(α, β) " entropy.

V. Properties of the exponential "type(α, β)" entropy

The quantity introduced in the preceding section is an 'entropy'. Such a name will be justified, if it shares some major properties with Shannon's and other entropies in the literature. We study some such properties in the next three theorems.

Theorem 1: The measure of information $E^{(\alpha,\beta)}(P)$, $P \in \Delta_n$, where $\Delta_n = \{P = (p_1, \ldots, p_n) : p_i \geq 0, \sum_{i=1}^n p_i = 1\}$ has the following properties:

1) Symmetry:
$E^{(\alpha,\beta)}(P) = E^{(\alpha,\beta)}(p_1, \ldots, p_n)$ is a symmetric function of $(p_1, \ldots, p_n)$.
2) Normalized:
$E^{(\alpha,\beta)}(\frac{1}{2}, \frac{1}{2}) = 1$.
3) Expansible:
$E^{(\alpha,\beta)}(p_1, \ldots, p_n, 0) = E^{(\alpha,\beta)}(p_1, \ldots, p_n)$.
4) Decisive:
$E^{(\alpha,\beta)}(1, 0) = E^{(\alpha,\beta)}(0, 1) = 0$.

5) Continuity:
$E^{(\alpha,\beta)}(p_1,\ldots,p_n)$ is continuous in the region $p_i \geq 0$ for all $\alpha,\beta > 0$.

Proof: (1) to (4):these properties are obvious and can be verified easily.
5). We know that $\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta$ is continuous in the region $p_i \geq 0$ for all $\alpha,\beta > 0$.
Hence, $E^{(\alpha,\beta)}(P)$, is also continuous in the region $p_i \geq 0$ for all $\alpha,\beta > 0$.

Theorem 2: The measure $E^{(\alpha,\beta)}(P)$ is non-negative for all $\alpha,\beta > 0$.
Proof: We consider the following cases:
Case(i): When $\alpha > 1$ and $0 < \beta < 1$,

$$e^{\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta} - 1 < 0$$

and

$$e^{(2^{1-\alpha}-2^{1-\beta})} - 1 < 0$$

we get

$$E^{(\alpha,\beta)}(P) > 0. \tag{57}$$

Case(ii): When $\beta > 1$ and $0 < \alpha < 1$,

$$e^{\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta} - 1 > 0$$

and

$$e^{(2^{1-\alpha}-2^{1-\beta})} - 1 > 0$$

we get

$$E^{(\alpha,\beta)}(P) > 0. \tag{58}$$

Case(iii): When $\alpha > 1$ and $\beta > 1$,
(a) Let $\alpha > \beta > 1$

$$e^{\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta} - 1 < 0$$

and

$$e^{(2^{1-\alpha}-2^{1-\beta})} - 1 < 0$$

we get

$$E^{(\alpha,\beta)}(P) > 0. \tag{59}$$

(b) Let $\beta > \alpha > 1$

$$e^{\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta} - 1 > 0$$

and

$$e^{(2^{1-\alpha}-2^{1-\beta})} - 1 > 0$$

we get

$$E^{(\alpha,\beta)}(P) > 0. \tag{60}$$

Case(iv): When $\alpha < 1$ and $\beta < 1$,
(a) Let $\alpha < \beta < 1$

$$e^{\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta} - 1 > 0$$

and

$$e^{(2^{1-\alpha}-2^{1-\beta})} - 1 > 0$$

we get

$$E^{(\alpha,\beta)}(P) > 0. \tag{61}$$

(b)Let $\beta < \alpha < 1$

$$e^{\sum_{i=1}^{n}(p_i)^\alpha - \sum_{i=1}^{n}(p_i)^\beta} - 1 < 0$$

and

$$e^{(2^{1-\alpha}-2^{1-\beta})} - 1 < 0$$

we get

$$E^{(\alpha,\beta)}(P) > 0 \tag{62}$$

From (57), (58), (59), (60), (61), (62), we conclude that

$$E^{(\alpha,\beta)}(P) > 0 \text{ for all } \alpha,\beta > 0.$$

To prove the next theorem, we shall use the following definition of a concave function.

Definition 4 (Concave Function): A function $f(.)$ over the points in a convex set $\mathbf{R}$ is *concave* if for all $r_1, r_2 \in R$ and $\mu \in (0,1)$

$$\mu f(r_1) + (1-\mu) f(r_2) \leq f(\mu r_1 + (1-\mu) r_2) \tag{63}$$

The function $f(.)$ is convex if the above inequality holds with $\geq$ in place of $\leq$.

Theorem 3: The measure $E^{(\alpha,\beta)}(P)$ is a concave function of the probability distribution $P = (p_1,\ldots,p_n), p_i \geq 0, \sum_{i=1}^{n} p_i = 1$, when one of the parameters $\alpha,\beta (> 0)$ is greater than unity and the other is less than or equal to unity, i.e., either $\alpha > 1$ and $0 < \beta \leq 1$ or $\beta > 1$ and $0 < \alpha \leq 1$.
Proof: Associated with the random variable $X = (x_1, x_2, \ldots, x_n)$, let us consider r distributions

$$P_k(X) = \{p_k(x_1), \ldots, p_k(x_n)\}, \tag{64}$$

where

$$\sum_{i=1}^{n} p_k(x_i) = 1,\ k = 1,2,\ldots,r.$$

Next let there be r numbers $(a_1, a_2, \ldots, a_r)$ such that $a_k \geq 0$, $\sum_{i=1}^{n} a_k = 1$ and define

$$P_0(X) = \{p_0(x_1), \ldots, p_0(x_n)\},$$

where

$$p_0(x_i) = \sum_{k=1}^{r} a_k p_k(x_i),\ i = 1,2,\ldots,n. \tag{65}$$

Obviously $\sum_{i=1}^{n} p_0(x_i) = 1$ and thus $P_0(X)$ is a bonafide distribution of X.
If $\alpha > 1$ and $0 < \beta \leq 1$, then we have

$$\sum_{k=1}^{r} a_k E^{(\alpha,\beta)}(P_k) - E^{(\alpha,\beta)}(P_0)$$

$$= \sum_{k=1}^{r} a_k E^{(\alpha,\beta)}(P_k) - \frac{\left[e^{\left([\sum_{i=1}^{r} a_i p_i]^\alpha - [\sum_{i=1}^{n} a_i p_i]^\beta\right)} - 1\right]}{e^{(2^{1-\alpha}-2^{1-\beta})} - 1}$$

$$\leq \sum_{k=1}^{r} a_k E^{(\alpha,\beta)}(P_k) - \frac{\left[e^{\left(\sum_{i=1}^{r} a_i (p_i)^\alpha - \sum_{i=1}^{n} a_i (p_i)^\beta\right)} - 1\right]}{e^{(2^{1-\alpha}-2^{1-\beta})} - 1}$$

i.e.,

$$\sum_{i=1}^{r} a_i E^{(\alpha,\beta)}(P_i) \leq E^{(\alpha,\beta)}(P) \tag{66}$$

By symmetry in α and β, the above result is also true for $\beta > 1$ and $0 < \alpha \leq 1$.

VI. Conclusion

In this paper, for the first time, concept of measure of a sample space with associated probability distribution has been introduced. This idea has quite some potential for further study and exploration both in statistics as well as in information theoretic applications.Using parametric generalization provides further desirable flexibilities.

Acknowledgment

The authors wish to express their sincere thanks to the referees for the valuable suggestions which helped in improving the presentation of the paper.

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