International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:11, 2013

Intuitionistic Fuzzy Implicative Ideals with Thresholds (λ,μ) of BCI-Algebras

Qianqian Li, Shaoquan Sun

***Abstract*—**The aim of this paper is to introduce the notion of intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of BCI-algebras and to investigate its properties and characterizations.

***Keywords*—**BCI-algebra, intuitionistic fuzzy set, intuitionistic fuzzy ideal with thresholds (λ,μ), intuitionistic fuzzy implicative ideal with thresholds (λ,μ).

I. Introduction

A BCI-algebra is an important class of logical algebra and was introduced by Iséki [1], [2]. K. Atannassov [3] introduced the concept of intuitionistic fuzzy sets. At present this concept has been applied to many mathematical branches. In 2003, K. Hur [4] applied the concept to the theory of rings, and introduced the concepts of intuitionistic fuzzy subgroups and subrings. M. Jiang and X.L. Xin [5] later introduced the concepts of (λ,μ) intuitionistic fuzzy subrings (Ideals), some meaningful results are obtained. In [6], we have given the concepts of intuitionistic fuzzy subalgebras with thresholds (λ,μ) and intuitionistic fuzzy ideals with thresholds (λ,μ) of BCI-algebras, in this paper, we apply the concept of intuitionistic fuzzy sets to the ideals theory of BCI-algebras, and introduce the notions of intuitionistic fuzzy implicative ideals with thresholds (λ,μ) of BCI- algebras. We give several properties and characterizations of intuitionistic fuzzy implicative ideals with thresholds (λ,μ) of BCI- algebras.

II. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCI- algebra if it satisfies the following axioms:

(BCI-1) $((x*y)*(x*z))*(z*y)=0$,

(BCI-2) $(x*(x*y))*y=0$,

(BCI-3) $x*x=0$,

(BCI-4) $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x,y,z\in X$. In a BCI-algebra X, we can define a partial ordering $\le$ by putting $x\le y$ if and only if $x*y=0$.

In any BCI-algebra X, the following hold:

Qianqian Li is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 150-63912851; e-mail: 839086541@qq.com).

Shaoquan Sun is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 185-61681686; e-mail: qdsunsaoquan@163.com).

(1) $(x*y)*z=(x*z)*y$,

(2) $x*0=x$,

(3) $0*(x*y)=(0*x)*(0*y)$,

(4) $(x*z)*(y*z)\le x*y$,

(5) $x*(x*(x*y))=x*y$,

(6) $x\le y$ implies $x*z\le y*z$ and $z*y\le z*x$,

for all $x,y,z\in X$.

In this paper, X always means a BCI-algebra unless otherwise specified.

A nonempty subset K of X is called an ideal of X if $(I_1): 0\in K, (I_2): x*y\in K$ and $y\in K$ imply $x\in K$. A nonempty subset K of X is called a implicative ideal of X if it satisfies (I_1) and (I_3) $(((x*y)*y)*(0*y))*z\in K$ and $z\in K$ imply $x*((y*(y*x))*(0*(0*(x*y))))\in K$.

Definition 1 [3] Let S be any set. An intuitionistic fuzzy subset A of S is an object of the following form

$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ where $\mu_A: S\to[0,1]$ and $\nu_A: S\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in S$ respectively and for every $x\in S$, $0\le\mu_A(x)+\nu_A(x)\le 1$.

Denote $\langle I\rangle=\{\langle a,b\rangle: a,b\in[0,1]\}$.

Definition 2 Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ be an intuitionistic fuzzy set in a set S. For $\langle\alpha,\beta\rangle\in\langle I\rangle$, the set $A_{\langle\alpha,\beta\rangle}=\{x\in S:\mu_A(x)\ge\alpha, \nu_A(x)\le\beta\}$ is called a cut set of A.

Definition 3 [6] Let $\lambda,\mu\in(0,1]$ and $\lambda<\mu$.

An intuitionistic fuzzy set A in X is said to be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X if the following are satisfied:

(IF_1) $\mu_A(0)\vee\lambda\ge\mu_A(x)\wedge\mu$,

(IF_2) $\nu_A(0)\wedge\mu\le\nu_A(x)\vee\lambda$,

(IF_3) $\mu_A(x)\vee\lambda\ge\mu_A(x*y)\wedge\mu_A(y)\wedge\mu$,

(IF_4) $\nu_A(x)\wedge\mu\le\nu_A(x*y)\vee\nu_A(y)\vee\lambda$,

for all $x,y\in X$.

Proposition 1 [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. If $x\le y$ holds in X, then $\mu_A(x)\vee\lambda\ge\mu_A(y)\wedge\mu$, $\nu_A(x)\wedge\mu\le\nu_A(y)\vee\lambda$.

Proposition 2 [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. If the inequality $x*y\le z$ holds in X, then for all $x,y,z\in X$,

$\mu_A(x)\vee\lambda\ge\mu_A(y)\wedge\mu_A(z)\wedge\mu,$

$\nu_A(x)\wedge\mu\le\nu_A(y)\vee\nu_A(z)\vee\lambda.$

III. Intuitionistic Fuzzy Implicative Ideals with Thresholds (λ,μ) of BCI- Algebras

Definition 4 Let $\lambda,\mu\in(0,1]$ and $\lambda<\mu$.

An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X if it satisfies $(IF_1),(IF_2)$ and

(IF_5) $\mu_A\Big(x*\big((y*(y*x))*(0*(0*(x*y)))\big)\Big)\vee\lambda$

$\ge\mu_A\Big((((x*y)*y)*(0*y))*z\Big)\wedge\mu_A(z)\wedge\mu,$

(IF_6) $\nu_A\Big(x*\big((y*(y*x))*(0*(0*(x*y)))\big)\Big)\wedge\mu$

$\le\nu_A\Big((((x*y)*y)*(0*y))*z\Big)\vee\nu_A(z)\vee\lambda.$

Proposition 3 Any intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X, but the converse does not hold.

Proof. Assume that A is an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X and put $y=0$ in (IF_5) and (IF_6), we get

$$\mu_A(x)\vee\lambda=\mu_A\Big(x*\big((0*(0*x))*(0*(0*(x*0)))\big)\Big)\vee\lambda$$
$$\ge\mu_A\Big((((x*0)*0)*(0*0))*z\Big)\wedge\mu_A(z)\wedge\mu$$
$$\ge\mu_A(x*z)\wedge\mu_A(z)\wedge\mu,$$
$$\nu_A(x)\wedge\mu=\nu_A\Big(x*\big((0*(0*x))*(0*(0*(x*0)))\big)\Big)\wedge\mu$$
$$\le\nu_A\Big((((x*0)*0)*(0*0))*z\Big)\vee\nu_A(z)\vee\lambda$$
$$=\nu_A(x*z)\vee\nu_A(z)\vee\lambda.$$

This means that A satisfies (IF_3) and (IF_4). Combining (IF_1) and (IF_2), A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X.

To show the last half part, we see the following example.

Example 1 Let $X=\{0,1,2\}$ with Cayley table gived by

TABLE I
Result of computation

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ where $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ by $\mu_A(0)=2/3,\ \mu_A(1)=\mu_A(2)=1/3,$ $\nu_A(0)=1/4,\nu_A(1)=\nu_A(2)=1/2.$ Let $\lambda=1/8$ and $\mu=3/4.$ By routine calculations give that A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. But it is not an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) as

$$\mu_A\Big(1*\big((2*(2*1))*(0*(0*(1*2)))\big)\Big)\vee\lambda=\mu_A(1)$$
$$<\mu_A(0)=\mu_A\Big((((1*2)*2)*(0*2))*0\Big)\wedge\mu_A(0)\wedge\mu.$$

The characterization of intuitionistic fuzzy implicative ideals with thresholds (λ,μ) of X are given by the following proposition.

Proposition 4 An intuitionistic fuzzy set A of X is an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X if and only if, for all $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or an implicative ideal of X.

Proof. Let A be an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X and $A_{\langle\alpha,\beta\rangle}\neq\varnothing$ for some $\alpha,\beta\in(\lambda,\mu]$. It is clear that $0\in A_{\langle\alpha,\beta\rangle}$. Let $(((x*y)*y)*(0*y))*z\in A_{\langle\alpha,\beta\rangle}$ and $z\in A_{\langle\alpha,\beta\rangle}$, then

$\mu_A\Big((((x*y)*y)*(0*y))*z\Big)\ge\alpha,\ \mu_A(z)\ge\alpha,$

$\nu_A\Big((((x*y)*y)*(0*y))*z\Big)\le\beta,\nu_A(z)\le\beta.$

It follows from (IF_5) and (IF_6),

$$\mu_A\Big(x*\big((y*(y*x))*(0*(0*(x*y)))\big)\Big)\vee\lambda$$
$$\ge\mu_A\Big((((x*y)*y)*(0*y))*z\Big)\wedge\mu_A(z)\wedge\mu\ge\alpha,$$
$$\nu_A\Big(x*\big((y*(y*x))*(0*(0*(x*y)))\big)\Big)\wedge\mu$$
$$\le\nu_A\Big((((x*y)*y)*(0*y))*z\Big)\vee\nu_A(z)\vee\lambda\le\beta.$$

Namely, $\mu_A\Big(x*\big((y*(y*x))*(0*(0*(x*y)))\big)\Big)\ge\alpha,$

$\nu_A\Big(x*\big((y*(y*x))*(0*(0*(x*y)))\big)\Big)\le\beta$ and

$x*\big((y*(y*x))*(0*(0*(x*y)))\big)\in A_{\langle\alpha,\beta\rangle}.$

This shows that $A_{\langle\alpha,\beta\rangle}$ is an implicative ideal of X.

Conversely, suppose that for each $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or an implicative ideal of X. For any $x\in X$, let $\alpha=\mu_A(x)\wedge\mu,\beta=\nu_A(x)\vee\lambda.$ Then $\mu_A(x)\ge\alpha,\nu_A(x)\le\beta,$ hence $x\in A_{\langle\alpha,\beta\rangle}$ and $A_{\langle\alpha,\beta\rangle}$ is an implicative ideal of X, therefore $0\in A_{\langle\alpha,\beta\rangle}$, i.e., $\mu_A(0)\ge\alpha,$ and $\nu_A(0)\le\beta.$ We get $\mu_A(0)\vee\lambda\ge\mu_A(0)\ge\alpha=\mu_A(x)\wedge\mu,$

$\nu_A(0)\wedge\mu\le\nu_A(0)\le\beta=\nu_A(x)\vee\lambda,$

i.e., $\mu_A(0)\vee\lambda\ge\mu_A(x)\wedge\mu$ and $\nu_A(0)\wedge\mu\le\nu_A(x)\vee\lambda,$

for all $x\in X$.

Now we only need to show that A satisfies (IF_5) and (IF_6). Let

$\alpha=\mu_A\left((((x*y)*y)*(0*y))*z\right)\wedge\mu_A(z)\wedge\mu,$

$\beta=\nu_A\left((((x*y)*y)*(0*y))*z\right)\vee\nu_A(z)\vee\lambda.$

Then

$\mu_A\left((((x*y)*y)*(0*y))*z\right)\ge\alpha,\ \mu_A(z)\ge\alpha,$

$\nu_A\left((((x*y)*y)*(0*y))*z\right)\le\beta,\ \nu_A(z)\le\beta.$

Hence $(((x*y)*y)*(0*y))*z\in A_{\langle\alpha,\beta\rangle}$ and $z\in A_{\langle\alpha,\beta\rangle}$. Since $A_{\langle\alpha,\beta\rangle}$ is an implicative ideal of X, thus

$x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\in A_{\langle\alpha,\beta\rangle}$, i.e.,

$\mu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\ge\alpha,$

$\nu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\le\beta.$

We get

$\mu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\vee\lambda$

$\ge\mu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)$

$\ge\alpha=\mu_A\left((((x*y)*y)*(0*y))*z\right)\wedge\mu_A(z)\wedge\mu,$

$\nu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\wedge\mu$

$\le\nu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)$

$\le\beta=\nu_A\left((((x*y)*y)*(0*y))*z\right)\vee\nu_A(z)\vee\lambda.$

This means that A satisfies (IF_5) and (IF_6). Hence, A is an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X.

Proposition 5 Let J be an implicative ideal of X. Then there exists an intuitionistic fuzzy implicative ideal A with thresholds (λ,μ) of X such that $A_{\langle\alpha,\beta\rangle}=J$ for some $\alpha,\beta\in(\lambda,\mu]$.

Proof. Define $A=\left\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\right\}$ by

$$\mu_A(x)=\begin{cases}\alpha & if\ x\in J,\\ \lambda & if\ x\notin J,\end{cases}$$

$$\nu_A(x)=\begin{cases}\beta & if\ x\in J,\\ \mu & if\ x\notin J,\end{cases}$$

where α,β are two fixed numbers in $(\lambda,\mu]$.

Since J is an implicative ideal of X, if $(((x*y)*y)*(0*y))*z\in J$ and $z\in J$, then

$x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\in J.$

Hence

$\mu_A\left((((x*y)*y)*(0*y))*z\right)=\mu_A(z)$

$=\mu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)=\alpha,$

$\nu_A\left((((x*y)*y)*(0*y))*z\right)=\nu_A(z)$

$=\nu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)=\beta,$

thus

$\mu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\vee\lambda$

$\ge\mu_A\left((((x*y)*y)*(0*y))*z\right)\wedge\mu_A(z)\wedge\mu,$

$\nu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\wedge\mu$

$\le\nu_A\left((((x*y)*y)*(0*y))*z\right)\vee\nu_A(z)\vee\lambda.$

If at least one of $(((x*y)*y)*(0*y))*z\in$ and z is not in J, then at least one of $\mu_A\left((((x*y)*y)*(0*y))*z\right)$ and $\mu_A(z)$ is λ, and at least one of $\nu_A\left((((x*y)*y)*(0*y))*z\right)$ and $\nu_A(z)$ is μ.

Therefore,

$\mu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\vee\lambda$

$\ge\mu_A\left((((x*y)*y)*(0*y))*z\right)\wedge\mu_A(z)\wedge\mu,$

$\nu_A\left(x*\left((y*(y*x))*\left(0*\left(0*(x*y)\right)\right)\right)\right)\wedge\mu$

$\le\nu_A\left((((x*y)*y)*(0*y))*z\right)\vee\nu_A(z)\vee\lambda.$

This means that A satisfies (IF_5) and (IF_6). Since $0\in J$,

$\mu_A(0)\vee\lambda=\alpha\ge\mu_A(x)\wedge\mu,\ \nu_A(0)\wedge\mu=\beta\le\nu_A(x)\vee\lambda,$

for all $x\in X$ and so A satisfies (IF_1) and (IF_2). Thus, A is an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X. It is clear that $A_{\langle\alpha,\beta\rangle}=J$.

Definition 5 [6] Let $\lambda,\mu\in(0,1]$ and $\lambda<\mu$.

An intuitionistic fuzzy ideal A with thresholds (λ,μ) in X is said to be an intuitionistic fuzzy closed ideal with thresholds (λ,μ) of X if the following are satisfied:

$\mu_A(0*x)\vee\lambda\ge\mu_A(x)\wedge\mu,$

$\nu_A(0*x)\wedge\mu\le\nu_A(x)\vee\lambda,$

for all $x\in X$.

Proposition 6 Let A be an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X. If A is an intuitionistic fuzzy closed ideal with thresholds (λ,μ) of X, then for all $x,y\in X$,

$$\mu_A\left(x*\left(y*\left(y*x\right)\right)\right)\vee\lambda\geq\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu,$$
$$\nu_A\left(x*\left(y*\left(y*x\right)\right)\right)\wedge\mu\leq\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda.$$

Proof. Assume that A is both an intuitionistic fuzzy implicative ideal and an intuitionistic fuzzy closed ideal with thresholds (λ,μ) of X. Substituting 0 for z in (IF_5) and (IF_6), we get

$$\mu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\vee\lambda$$
$$\geq\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu,$$
$$\nu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\wedge\mu$$
$$\leq\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda.$$

By Definition 5, we have

$$\mu_A\left(0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\right)\vee\lambda$$
$$\geq\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu,$$
$$\nu_A\left(0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\right)\wedge\mu$$
$$\leq\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda.$$

Since

$$\left(x*\left(y*\left(y*x\right)\right)\right)*\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)$$
$$\leq\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)*\left(y*\left(y*x\right)\right)$$
$$=0*\left(0*\left(0*\left(x*y\right)\right)\right)$$
$$=0*\left(x*y\right)$$

and

$$0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)$$
$$=\left(\left(0*\left(x*y\right)\right)*\left(0*y\right)\right)*\left(0*\left(0*y\right)\right)$$
$$=\left(\left(0*\left(0*\left(0*y\right)\right)\right)*\left(x*y\right)\right)*\left(0*y\right)$$
$$=\left(\left(0*y\right)*\left(x*y\right)\right)*\left(0*y\right)$$
$$=0*\left(x*y\right),$$

we get

$$\left(x*\left(y*\left(y*x\right)\right)\right)*\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)$$
$$\leq0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right).$$

by Proposition 2, we obtain

$$\mu_A\left(x*\left(y*\left(y*x\right)\right)\right)\vee\lambda$$
$$=\left(\mu_A\left(x*\left(y*\left(y*x\right)\right)\right)\vee\lambda\right)\vee\lambda$$
$$\geq\left(\mu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\right.$$
$$\left.\wedge\mu_A\left(0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\right)\wedge\mu\right)\vee\lambda$$
$$=\left(\mu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\vee\lambda\right)$$
$$\wedge\left(\mu_A\left(0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\right)\vee\lambda\right)\wedge\left(\mu\vee\lambda\right)$$
$$\geq\left(\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu\right)$$
$$\wedge\left(\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu\right)\wedge\mu$$
$$=\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu,$$
$$\nu_A\left(x*\left(y*\left(y*x\right)\right)\right)\wedge\mu$$
$$=\left(\nu_A\left(x*\left(y*\left(y*x\right)\right)\right)\wedge\mu\right)\wedge\mu$$
$$\leq\left(\nu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\right.$$
$$\left.\vee\nu_A\left(0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\right)\vee\lambda\right)\wedge\mu$$
$$=\left(\nu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\wedge\mu\right)$$
$$\vee\left(\nu_A\left(0*\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\right)\wedge\mu\right)\vee\left(\lambda\wedge\mu\right)$$
$$\leq\left(\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda\right)$$
$$\vee\left(\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda\right)\vee\lambda$$
$$=\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda.$$

Namely,

$$\mu_A\left(x*\left(y*\left(y*x\right)\right)\right)\vee\lambda\geq\mu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\wedge\mu,$$
$$\nu_A\left(x*\left(y*\left(y*x\right)\right)\right)\wedge\mu\leq\nu_A\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)\vee\lambda.$$

Definition 6 Let S be any set. If

$$A=\left\{\left\langle x,\mu_A(x),\nu_A(x)\right\rangle:x\in S\right\},B=\left\{\left\langle x,\mu_B(x),\nu_B(x)\right\rangle:x\in S\right\}$$

be any two intuitionistic fuzzy subsets of S **,** then

$$A\cap B=\left\{\left\langle x,(\mu_A\cap\mu_B)(x),(\nu_A\cup\nu_B)(x)\right\rangle:x\in S\right\}$$
$$=\left\{\left\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\right\rangle:x\in S\right\}$$

Proposition 7 Let A and B be two intuitionistic fuzzy implicative ideals with thresholds (λ,μ) of X .Then $A\cap B$ is also an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of X.

Proof. For all $x,y,z\in X$, by Definition 4, we have

$$\mu_{A\cap B}(0)\vee\lambda=\left(\mu_A(0)\wedge\mu_B(0)\right)\vee\lambda$$
$$=\left(\mu_A(0)\vee\lambda\right)\wedge\left(\mu_B(0)\vee\lambda\right)$$
$$\geq\left(\mu_A(x)\wedge\mu\right)\wedge\left(\mu_B(x)\wedge\mu\right)$$
$$=\left(\mu_A(x)\wedge\mu_B(x)\right)\wedge\mu$$
$$=\mu_{A\cap B}(x)\wedge\mu,$$
$$\nu_{A\cap B}(0)\wedge\mu=\left(\nu_A(0)\vee\nu_B(0)\right)\wedge\mu$$
$$=\left(\nu_A(0)\wedge\mu\right)\vee\left(\nu_B(0)\wedge\mu\right)$$
$$\leq\left(\nu_A(x)\vee\lambda\right)\vee\left(\nu_B(x)\vee\lambda\right)$$
$$=\left(\nu_A(x)\vee\nu_B(x)\right)\vee\lambda$$

$$= \nu_{A\cap B}(x) \vee \lambda,$$

$$\mu_{A\cap B}\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\vee\lambda$$
$$=\Big(\mu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)$$
$$\wedge\mu_B\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\Big)\vee\lambda$$
$$=\Big(\mu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\vee\lambda\Big)$$
$$\wedge\Big(\mu_B\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\vee\lambda\Big)$$
$$\geq\Big(\mu_A\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\wedge\mu_A(z)\wedge\mu\Big)$$
$$\wedge\Big(\mu_B\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\wedge\mu_B(z)\wedge\mu\Big)$$
$$=\Big(\mu_A\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)$$
$$\wedge\mu_B\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\Big)\wedge\left(\mu_A(z)\wedge\mu_B(z)\right)\wedge\mu$$
$$=\mu_{A\cap B}\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\wedge\mu_{A\cap B}(z)\wedge\mu.$$

$$\nu_{A\cap B}\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\wedge\mu$$
$$=\Big(\nu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)$$
$$\vee\nu_B\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\Big)\wedge\mu$$
$$=\Big(\nu_A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\wedge\mu\Big)$$
$$\vee\Big(\nu_B\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y\right)\right)\right)\right)\right)\wedge\mu\Big)$$
$$\leq\Big(\nu_A\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\vee\nu_A(z)\vee\lambda\Big)$$
$$\vee\Big(\nu_B\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\vee\nu_B(z)\vee\lambda\Big)$$
$$=\Big(\nu_A\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)$$
$$\vee\nu_B\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\Big)\vee\left(\nu_A(z)\vee\nu_B(z)\right)\vee\lambda$$
$$=\nu_{A\cap B}\left(\left(\left(\left(x*y\right)*y\right)*\left(0*y\right)\right)*z\right)\vee\nu_{A\cap B}(z)\vee\lambda.$$

Hence $A\cap B$ is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X.

Definition 7 Let A and B be two intuitionistic fuzzy sets of a set X. The Cartesian product of A and B is defined by $A\times B=\left\{\left\langle\mu_{A\times B}(x,y),\nu_{A\times B}(x,y)\right\rangle : x,y\in X\right\}$ where

$$\mu_{A\times B}(x,y)=\mu_A(x)\wedge\mu_B(y),\ \nu_{A\times B}(x,y)=\nu_A(x)\vee\nu_B(y).$$

Proposition 8 Let A and B be two intuitionistic fuzzy positive implicative ideals with thresholds (λ,μ) of X. Then $A\times B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of $X\times X$.

Proof. For all $(x,y)\in X\times X$, by Definition 4, we get

$$\mu_{A\times B}(0,0)\vee\lambda=\left(\mu_A(0)\wedge\mu_B(0)\right)\vee\lambda$$
$$=\left(\mu_A(0)\vee\lambda\right)\wedge\left(\mu_B(0)\vee\lambda\right)$$
$$\geq\left(\mu_A(x)\wedge\mu\right)\wedge\left(\mu_B(y)\wedge\mu\right)$$
$$=\mu_{A\times B}(x,y)\wedge\mu,$$

for all $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$, we have

$$\mu_{A\times B}\Big(x_1*\left(\left(y_1*\left(y_1*x_1\right)\right)*\left(0*\left(0*\left(x_1*y_1\right)\right)\right)\right),$$
$$x_2*\left(\left(y_2*\left(y_2*x_2\right)\right)*\left(0*\left(0*\left(x_2*y_2\right)\right)\right)\right)\Big)\vee\lambda$$
$$=\Big(\mu_A\left(x_1*\left(\left(y_1*\left(y_1*x_1\right)\right)*\left(0*\left(0*\left(x_1*y_1\right)\right)\right)\right)\right)$$
$$\wedge\mu_B\left(x_2*\left(\left(y_2*\left(y_2*x_2\right)\right)*\left(0*\left(0*\left(x_2*y_2\right)\right)\right)\right)\right)\Big)\vee\lambda$$
$$=\Big(\mu_A\left(x_1*\left(\left(y_1*\left(y_1*x_1\right)\right)*\left(0*\left(0*\left(x_1*y_1\right)\right)\right)\right)\right)\vee\lambda\Big)$$
$$\wedge\Big(\mu_B\left(x_2*\left(\left(y_2*\left(y_2*x_2\right)\right)*\left(0*\left(0*\left(x_2*y_2\right)\right)\right)\right)\right)\vee\lambda\Big)$$
$$\geq\Big(\mu_A\left(\left(\left(\left(x_1*y_1\right)*y_1\right)*\left(0*y_1\right)\right)*z_1\right)\wedge\mu_A(z_1)\wedge\mu\Big)$$
$$\wedge\Big(\mu_B\left(\left(\left(\left(x_2*y_2\right)*y_2\right)*\left(0*y_2\right)\right)*z_2\right)\wedge\mu_B(z_2)\wedge\mu\Big)$$
$$=\Big(\mu_A\left(\left(\left(\left(x_1*y_1\right)*y_1\right)*\left(0*y_1\right)\right)*z_1\right)$$
$$\wedge\mu_B\left(\left(\left(\left(x_2*y_2\right)*y_2\right)*\left(0*y_2\right)\right)*z_2\right)\Big)\wedge\left(\mu_A(z_1)\wedge\mu_B(z_2)\right)\wedge\mu$$
$$=\mu_{A\times B}\Big(\left(\left(\left(\left(x_1*y_1\right)*y_1\right)*\left(0*y_1\right)\right)*z_1\right),$$
$$\left(\left(\left(\left(x_2*y_2\right)*y_2\right)*\left(0*y_2\right)\right)*z_2\right)\Big)\wedge\mu_{A\times B}(z_1,z_2)\wedge\mu.$$

Similarly it can be proved that

$$\nu_{A\times B}(0,0)\wedge\mu\leq\nu_{A\times B}(x,y)\vee\lambda,$$
$$\nu_{A\times B}\Big(x_1*\left(\left(y_1*\left(y_1*x_1\right)\right)*\left(0*\left(0*\left(x_1*y_1\right)\right)\right)\right),$$
$$x_2*\left(\left(y_2*\left(y_2*x_2\right)\right)*\left(0*\left(0*\left(x_2*y_2\right)\right)\right)\right)\Big)\wedge\mu$$
$$\leq\nu_{A\times B}\Big(\left(\left(\left(\left(x_1*y_1\right)*y_1\right)*\left(0*y_1\right)\right)*z_1\right),$$
$$\left(\left(\left(\left(x_2*y_2\right)*y_2\right)*\left(0*y_2\right)\right)*z_2\right)\Big)\vee\nu_{A\times B}(z_1,z_2)\vee\lambda.$$

Hence $A\times B$ is an intuitionistic fuzzy implicative ideal with thresholds (λ,μ) of $X\times X$.

References

[1] K.Iséki, "On BCI-algebras", Math. Sem. Notes, vol. 8, pp. 125-130, 1980.
[2] K.Iséki and S. Tanaka, "An introduction to the theory of BCK-algebras", Math.Japon, vol. 23, pp. 1-26, 1978.
[3] T.K. Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol. 20, pp. 87-96, 1986.
[4] Hur K, Kang H W and Song H K, "Intuitionistic fuzzy subgroups and subrings", Honam Math. J., vol. 25, pp. 19-41, 2003.
[5] M. Jiang, X.L. Xin, "(λ, μ) Intuitionistic Fuzzy Subrings (Ideals)", Fuzzy Systems and Mathematics, vol. 27, pp. 1-8, 2013.
[6] S.Q. Sun, Q.Q. Li, "Intuitionistic Fuzzy Subalgebras (Ideals) with Thresholds (λ, μ) of BCI-algebras", submitted for publication.