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The Existence and Uniqueness of Positive Solution for Nonlinear Fractional Differential Equation Boundary Value Problem

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Abstract—In this paper, the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problem is concerned by a fixed point theorem of a sum operator. Our results can not only guarantee the existence and uniqueness of positive solution, but also be applied to construct an iterative scheme for approximating it. Finally, the example is given to illustrate the main result.

Keywords—Fractional differential equation, Boundary value problem, Positive solution, Existence and uniqueness, Fixed point theorem of a sum operator.

I. Introduction

FRACTIONAL differential equations are used in various fields, such as mechanics, physics, chemistry, engineering, economics and biological sciences, etc.; see $[1-9]$ and the references therein. In recent years, there are many papers discuss the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem by the use of Leray-Shauder theory, fixed-point theorems, etc., see $[10-14]$. However, there are few papers consider the existence of a unique positive solution for nonlinear fractional differential equation boundary value problem, see $[15, 16]$.

In particular, by means of mixed monotone method, Xu, Jiang and Yuan [16] obtained the uniqueness of solution to the singular boundary value problem for fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)), \ \ 0 < t < 1, \ 3 < \alpha \leq 4, \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \quad (1)$$

They need

$$f(t,u) = q(t)[\varphi(u) + \psi(u)], \varphi : [0, +\infty) \to [0, +\infty)$$

is continuous and nondecreasing, $\psi : (0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing and $q \in C((0, 1), (0, +\infty))$ satisfies $\int_0^1 s^{2-\eta(2-\alpha)}(1-s)^{\alpha-2-2\eta}q(s)\mathrm{d}s < +\infty, \eta \in (0, 1)$.

Naturally, when $f(t, u)$ can not be denoted by $f(t, u) = q(t)[\varphi(u) + \psi(u)]$, how to consider the unique positive solution of it? In this paper, by means of a fixed point theorem for a sum operator, we obtain the existence and uniqueness of positive solution to the nonsingular boundary value problem for fractional differential equation (1) with the assumption $f(t, u) = g(t, u) + h(t, u)$. Moreover, we can construct an iterative scheme to approximate the unique solution, which is important for evaluation and application.

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II. Preliminaries and previous results

In this section, we present some definitions, lemmas and basic results that will be used in the proof of our main result.

Definition 1 [3] The integral

$$I_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t, \quad x > 0$$

is called the Riemann-Liouville fractional integral of order α, where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2 [3] For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}x})^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t,$$

is called the Riemann-Liouville fractional derivative of order α, where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number α.

Lemma 1 [16] Given $y \in C[0, 1]$ and $3 < \alpha \leq 4$, the unique solution of the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) = y(t), \ \ 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \quad (2)$$

is

$$u(t) = \int_0^1 G(t, s)y(s)\mathrm{d}s, \ \ t \in [0, 1]$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, \\ \qquad 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-2}(1-s)^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, \\ \qquad 0 \leq t \leq s \leq 1. \end{cases} \quad (3)$$

Here $G(t, s)$ is called the Green function of the fractional differential equation (2).

Lemma 2 [16] The Green function $G(t,s)$ defined by (3) has the following property:

$$\frac{\alpha-2}{\Gamma(\alpha)}(1-s)^{\alpha-2}s^2t^{\alpha-2}(1-t)^2 \le G(t,s) \le \frac{1}{\Gamma(\alpha)}M_0t^{\alpha-2}(1-t)^2 \quad (4)$$

for $t,s\in(0,1)$. where $M_0=\max\{\alpha-1,(\alpha-2)^2\}$.

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed point theorem which will be used later.

Suppose $(E,||\cdot||)$ is a real Banach space which is partially ordered by a cone $P\subset E$, i.e. $x\le y$ if and only if $y-x\in P$. If $x\le y$ and $x\ne y$, then we denote $x<y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P\subset E$ is a cone if it satisfies (i)$x\in P,\lambda\ge 0\Rightarrow \lambda x\in P$; (ii)$x\in P,-x\in P\Rightarrow x=\theta$.

Putting $P^0=\{x\in P|x$ is an interior point of $P\}$, a cone P is said to be solid if P^0 is non-empty. Moreover, P is called normal if there exists a constant $N>0$ such that, for all $x,y\in E,\theta\le x\le y$ implies $||x||\le N||y||$; in this case N is called the normality constant of P. We say that an operator $A:E\to E$ is increasing if $x\le y$ implies $Ax\le Ay$.

For all $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\le y\le \mu x$. Clearly $\sim$ is an equivalence relation. Given $w>\theta$ (i.e. $w\ge\theta$ and $w\ne\theta$), we denote the set $P_w=\{x\in E|x\sim w\}$ by P_w. It is easy to see that $P_w\subset P$ for $w\in P$.

Theorem 1 [17] Let P be a normal cone in a real Banach space E, $A:P\to P$ be an increasing γ-concave operator and $B:P\to P$ be an increasing sub-homogeneous operator. Assume that
(i) there is $w>\theta$ such that $Aw\in P_w$ and $Bw\in P_w$;
(ii) there exists a constant $\delta_0>0$ such that $Ax\ge\delta_0 Bx,\forall x\in P$.
Then operator equation $Ax+Bx=x$ has a unique solution x^* in P_w. Moreover, constructing successively the sequence $y_n=Ay_{n-1}+By_{n-1},n=1,2,\cdots$ for any initial value $y_0\in P_w$, we have $y_n\to x^*$ as $n\to\infty$.

III. Main results

In this section, we apply Theorem 1 to investigate the fractional differential equation (1) with the assumption$f(t,u)=g(t,u)+h(t,u)$, and we obtain the new result on the existence and uniqueness of positive solution.

In this paper, we will work in the Banach space $C[0,1]=\{x:[0,1]\to R$ is continuous$\}$ with the standard norm $\|x\|=\sup\{|x(t)|:t\in[0,1]\}$. Notice that this space can be endowed with a partial order given by $x,y\in C[0,1],\ \ x\le y\Leftrightarrow x(t)\le y(t)$ for $t\in[0,1]$.

Let $P=\{x\in C[0,1]|x(t)\ge 0,t\in[0,1]\}$ be the standard cone. Evidently, P is a normal cone in $C[0,1]$ and the normality constant is 1.

Theorem 2 Assume that
(H1) $g,h:[0,1]\times[0,\infty)\to[0,\infty)$ are continuous and increasing with respect to the second argument, $h(t,0)\not\equiv 0$;
(H2) there exists a constant $\gamma\in(0,1)$ such that $g(t,\lambda x)\ge\lambda^\gamma g(t,x),\forall t\in[0,1],\lambda\in(0,1),x\in[0,\infty)$, and $h(t,\mu x)\ge\mu h(t,x)$ for $\mu\in(0,1),t\in[0,1],x\in[0,\infty)$;
(H3) there exists a constant $\delta_0>0$ such that $g(t,x)\ge\delta_0 h(t,x),t\in[0,1],x\ge 0$. Then the fractional differential equation (1) has a unique positive solution u^* in P_w, where $w(t)=t^{\alpha-2}(1-t)^2,t\in[0,1]$. Moreover, for any initial value $u_0\in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t)=\int_0^1 G(t,s)f(s,u_n(s))\mathrm{d}s,\quad n=0,1,2,\cdots,$$

we have $u_n(t)\to u^*(t)$ as $n\to\infty$, where $G(t,s)$ is given as (3).

Proof: To begin with, from Lemma 1, the fractional differential equation (1) has an integral formulation given by

$$\begin{aligned}u(t)&=\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s\\&=\int_0^1 G(t,s)[g(s,u(s))+h(s,u(s))]\mathrm{d}s\end{aligned}$$

where $G(t,s)$ is given as in Lemma 1.

Define two operators $A:P\to E$ and $B:P\to E$ by

$$\begin{aligned}Au(t)&=\int_0^1 G(t,s)g(s,u(s))\mathrm{d}s,\\Bu(t)&=\int_0^1 G(t,s)h(s,u(s))\mathrm{d}s.\end{aligned}$$

It is easy to prove that u is the solution of the fractional differential equation (1) if and only if $u=Au+Bu$. By assumption (H1) and Lemma 2, we know that $A:P\to P$ and $B:P\to P$. In the sequel we check that A, B satisfy all assumptions of Theorem 1.

Firstly, we prove that are two increasing operators.

In fact, from assumption (H1) and Lemma 2, for $u,v\in P$ with $u\ge v$, we know that $u(t)\ge v(t),\ t\in[0,1]$ and obtain

$$\begin{aligned}Au(t)&=\int_0^1 G(t,s)g(s,u(s))\mathrm{d}s\\&\ge\int_0^1 G(t,s)g(s,v(s))\mathrm{d}s\\&=Av(t)\end{aligned}$$

That is $Au\ge Av$. Similarly, $Bu\ge Bv$.

Next we show that A is a γ-concave operator and B is a sub-homogeneous operator.

In fact, for any $\lambda\in(0,1)$ and $u\in P$, from (H2) we know that

$$\begin{aligned}A(\lambda u)(t)&=\int_0^1 G(t,s)g(s,\lambda u(s))\mathrm{d}s\\&\ge\lambda^\gamma\int_0^1 G(t,s)g(s,u(s))\mathrm{d}s\\&=\lambda^\gamma Au(t)\end{aligned}$$

That is, $A(\lambda u)\ge\lambda^\gamma Au$ for $\lambda\in(0,1),u\in P$. So the operator A is a γ-concave operator. Also, for any $\mu\in(0,1)$ and $\mu\in P$, by (H2) we obtain

$$\begin{aligned}B(\mu u)(t)&=\int_0^1 G(t,s)h(s,\mu u(s))\mathrm{d}s\\&\ge\mu\int_0^1 G(t,s)h(s,u(s))\mathrm{d}s\\&=\mu Bu(t)\end{aligned}$$

That is, $B(\mu u)\ge\mu BAu$ for $\mu\in(0,1),u\in P$.So the operator B is a sub-homogeneous operator.

Now we show that $Aw\in P_w$ and $Bw\in P_w$, where $w(t)=t^{\alpha-2}(1-t)^2$.

By (H1) and Lemma 2,

$$\begin{aligned}\frac{\alpha-2}{\Gamma(\alpha)}w(t)\int_0^1 s^2(1-s)^{\alpha-2}g(s,0)\mathrm{d}s&\le Aw(t)\\&=\int_0^1 G(t,s)g(s,w(s))\mathrm{d}s\\&\le\frac{1}{\Gamma(\alpha)}w(t)\int_0^1 M_0 g(s,1)\mathrm{d}s\end{aligned}$$

From (H1) and (H3), we have

$$g(s,1) \geq g(s,0) \geq \delta_0 h(s,0) \geq 0$$

Since $h(t,0) \not\equiv 0$, we can get

$$\int_0^1 g(s,1)\mathrm{d}s \geq \int_0^1 g(s,0)\mathrm{d}s \geq \delta_0 \int_0^1 h(s,0)\mathrm{d}s > 0,$$

and in consequence,

$$l_1 := \frac{\alpha-2}{\Gamma(\alpha)} \int_0^1 s^2(1-s)^{\alpha-2} g(s,0)\mathrm{d}s > 0,$$
$$l_2 := \frac{1}{\Gamma(\alpha)} \int_0^1 M_0 g(s,1)\mathrm{d}s > 0.$$

So $l_1 w(t) \leq Aw(t) \leq l_2 w(t), t \in [0,1]$; and hence we have $Aw \in P_w$.Similarly,

$$\frac{\alpha-2}{\Gamma(\alpha)} w(t) \int_0^1 s^2(1-s)^{\alpha-2} h(s,0)\mathrm{d}s \leq Bw(t) \leq \frac{1}{\Gamma(\alpha)} w(t) \int_0^1 M_0 h(s,1)\mathrm{d}s$$

from $h(t,0) \not\equiv 0$, we easily prove $Bw \in P_w$.Hence the condition (i) of Theorem 1 is satisfied.

In the following we show that the condition (ii) of Theorem 1 is satisfied.

For $u \in P$, by (H3),

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) g(s,u(s))\mathrm{d}s \\ &\geq \delta_0 \int_0^1 G(t,s) h(s,u(s))\mathrm{d}s \\ &= \delta_0 Bu(t) \end{aligned}$$

Then we get $Au \geq \delta_0 Bu,\ u \in P$.

Finally, by means of Theorem 1, the operator equation $Au+Bu = u$ has a unique positive solution u^* in P_w. Moreover, constructing successively the iterative scheme

$$u_n = Au_{n-1} + Bu_{n-1}, n = 1, 2, \cdots$$

for any initial value $u_0 \in P_w$, we have $u_n \to u^*$ as $n \to \infty$.That is, the fractional differential equation (1) has a unique positive solution u^* in P_w. For any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t) = \int_0^1 G(t,s) f(s, u_n(s))\mathrm{d}s, \quad n = 0, 1, 2, \cdots,$$

we have $u_n \to u^*$ as $n \to \infty$.

Corollary 1 When $f(t,u) = q(t)[\varphi(u) + \psi(u)]$ satisfy the conditions of theorem 2, the nonsingular boundary value problem for fractional differential equation (1) has a unique positive solution u^* in P_w, where $w(t) = t^{\alpha-2}(1-t)^2, t \in [0,1]$. Moreover, for any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t) = \int_0^1 G(t,s) f(s, u_n(s)\mathrm{d}s, \quad n = 0, 1, 2, \cdots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$,where $G(t,s)$ is given as (3).

Remark 1 By Theorem 2, Corollary 1 is obvious. Moreover, the unique positive solution u^* we obtain satisfies:

(i) there exist $\lambda > \mu > 0$ such that$\mu t^{\alpha-2}(1-t)^2 \leq u^* \leq \lambda t^{\alpha-2}(1-t)^2, t \in [0,1]$,

(ii) we can take any initial value in P_w and then construct an iterative scheme which can approximate the unique solution.

IV. Example

We present one example to illustrate Theorem 2.

Example 1 Consider the following fractional differential equation:

$$\begin{cases} D_{0+}^{\frac{10}{3}} u(t) = u^{\frac{1}{5}}(t) + \arctan u(t) + t^3 + t^2 + \frac{\pi}{2}, \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \quad (5)$$

In this example, we have $\alpha = \frac{10}{3}$. Let

$$g(t,u) = u^{\frac{1}{5}}(t) + t^2 + \frac{\pi}{2},$$
$$h(t,u) = \arctan u(t) + t^3,$$
$$\gamma = \frac{1}{5}.$$

Obviously, $g, h : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous and increasing with respect to the second argument, $h(t,0) = t^3 \not\equiv 0$.Besides, for $t \in [0,1], \lambda \in (0,1), x \in [0,\infty)$, we have

$$\begin{aligned} g(t,\lambda u) &= \lambda^{\frac{1}{5}} u^{\frac{1}{5}}(t) + t^2 + \frac{\pi}{2} \\ &\geq \lambda^{\frac{1}{5}} u^{\frac{1}{5}}(t) + \lambda^{\frac{1}{5}}(t^2 + \frac{\pi}{2}) \\ &= \lambda^{\frac{1}{5}}(u^{\frac{1}{5}}(t) + t^2 + \frac{\pi}{2}) \\ &= \lambda^{\gamma} g(t,u) \end{aligned}$$

and for $t \in [0,1], \mu \in (0,1), x \in [0,\infty)$, we have $\arctan(\mu u) \geq \mu \arctan u$, and thus $h(t,\mu u) \geq \mu h(t,u)$. Moreover, if we take $\delta_0 \in (0,1]$, then we obtain

$$\begin{aligned} g(t,u) &= u^{\frac{1}{5}}(t) + t^2 + \frac{\pi}{2} \\ &\geq t^2 + \frac{\pi}{2} \\ &\geq t^3 + \arctan u \\ &\geq \delta_0(t^3 + \arctan u) \\ &= \delta_0 h(t,u) \end{aligned}$$

Hence all the conditions of Theorem 2 are satisfied. An application of Theorem 2 implies that the fractional differential equation (5) has a unique positive solution in P_w, where$w(t) = t^{\frac{4}{3}}(1-t)^2, t \in [0,1]$.

Remark 2 The nonlinearity f in example 1 can not be denoted by $f(t,u) = q(t)[\varphi(u) + \psi(u)]$, so the positive solution of example 1 can not been obtained by virtue of [16].Theorem 2 generalizes the results in [16].

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